

SECOND EDITION

MATHEMATICS

for Elementary School Teachers

BASSAREAR

CUSTOM EDITION

Summary of Changes in Content and Emphasis in K–4 Mathematics*

Increased Attention

Number

- Number sense
- Place-value concepts
- Meaning of fractions and decimals
- Estimation of quantities

Operations and Computation

- Meaning of operations
- Operation sense
- Mental computation
- Estimation and the reasonableness of answers
- Selection of an appropriate computational method
- Use of calculators for complex computation
- Thinking strategies for basic facts

Geometry and Measurement

- Properties of geometric figures
- Geometric relationships
- Spatial sense
- Process of measuring
- Concepts related to units of measurement
- Actual measuring
- Estimation of measurements
- Use of measurement and geometry ideas throughout the curriculum

Probability and Statistics

- Collection and organization of data
- Exploration of chance

Patterns and Relationships

- Pattern recognition and description
- Use of variables to express relationships

Problem Solving

- Word problems with a variety of structures
- Use of everyday problems
- Applications
- Study of patterns and relationships
- Problem-solving strategies

Instructional Practices

- Use of manipulative materials
- Cooperative work
- Discussion of mathematics
- Questioning
- Justification of thinking
- Writing about mathematics
- Problem-solving approach to instruction
- Content integration
- Use of calculators and computers

Decreased Attention

Number

- Early attention to reading, writing, and ordering numbers symbolically

Operations and Computation

- Complex paper-and-pencil computations
- Isolated treatment of paper-and-pencil computations
- Addition and subtraction without renaming
- Isolated treatment of division facts
- Long division
- Long division without remainders
- Paper-and-pencil fraction computation
- Use of rounding to estimate

Geometry and Measurement

- Primary focus on naming geometric figures
- Memorization of equivalencies between units of measurement

Problem Solving

- Use of clue words to determine which operation to use

Instructional Practices

- Rote practice
- Rote memorization of rules
- One answer and one method
- Use of worksheets
- Written practice
- Teaching by telling

The current NCTM *Principles and Standards for School Mathematics* can be found in Appendix A of this text or at www.nctm.org.

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MATHEMATICS

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CUSTOM EDITION

TOM BASSAREAR

KEENE STATE COLLEGE, NEW HAMPSHIRE

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by Tom Bassarear

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by Daniel C. Alexander and GERALYN M. KOEBERLEIN

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PREFACE

Note: If you haven't read the Explorations Preface, you might want to do that now before reading on. The Explorations Preface sets the overall theme of the course and explains how the two volumes work with each other. This Preface more specifically describes the goals of the course and the features of the textbook.

GOALS OF THE BOOK

Richard Skemp, a noted British mathematics educator, has powerfully articulated the difference between relational understanding (understanding the mathematical structures and connectedness of ideas within and between concepts) and instrumental understanding (simply being able to manipulate formulas). A primary goal of this course is to develop relational understanding. Relational understanding, by definition, can't be "shown." Relational understanding comes out of student exploration and investigation, out of attempts to connect new knowledge to old knowledge. I have seen so many conscientious but frustrated students have tremendous breakthroughs as they began to invest less energy into trying to get "it" and more energy into trying to *make sense* of the ideas and problems.

Conceptual Focus

This relates to the two volumes that comprise *Mathematics for Elementary School Teachers* in that I believe that understanding of mathematical procedures and the ability to solve complex problems emerge naturally from focusing on relational understanding. To me this is the heart of the changed focus in the mathematics reform movement, which is partly enunciated in the NCTM Standard on Problem Solving: "Use problem-solving approaches to investigate and understand mathematical content." Thus, in one sense, the book is backwards from many other mathematics books in which concepts are defined and presented and then the examples help the student to better understand the concepts. In the Explorations, a rich question is asked and the concepts emerge from the students' active involvement and from group discussions. In the text, definitions and theorems are often presented after an Investigation. From another perspective, I am seeking to invite and honor all three types of reasoning: deductive, inductive, and intuitive.

For example, rather than presenting the structure for the multiplication procedure, Exploration 3.8, has the students grapple with increasingly large multiplication problems. Invariably, the students will invent the distributive principle and be surprised when I give this name to what they have done, for many of them have negative memories of the distributive principle from Algebra 1. In Exploration 5.6, having to order fractions without finding LCM or converting to decimals forces the students to look at relationships between numerator and denominator. In Exploration 9.7, students apply their understanding of transformations and symmetry as they predict which figures will

tessellate. They also have to determine what variables—e.g., number of sides, size of angle, regular vs. irregular, concave vs. convex—are relevant. They come to their conclusions in different ways, some more inductively, some more deductively, and some more intuitively.

Integrated Big Content Ideas Throughout the Course

So many people think of elementary mathematics as chapter titles, and for a variety of reasons, I have chosen to live within a more traditional partitioning of mathematical content: fractions, decimals, percent, statistics, etc. However, a critically important outcome of this course has to be the students' realization of the underlying connectedness of mathematics. (There are many other big ideas, but if you write down too many, then most people lose focus.) My job as a writer is to try to keep many big ideas alive in my head while simultaneously articulating a small enough subset for you to keep track of as you learn the ideas in these books. Below are several big ideas that I have tried consciously to weave into the whole textbook.

Functions Functional relationships (the idea of a unique and predictable relationship between two sets or two variables) permeate mathematics and thus this book. Students will see that all mathematical operations are functions. In the operator construct of fractions, they use the fraction as a function machine, and I use function language in Investigation 5.5 (Determining an Appropriate Representation) to solve a fraction problem. Geometric transformations are functions, and geometric relationships—the number of sides and sum of interior angles, the relationship between the number of faces, the edges and vertices in polyhedra—embody functions, too. Students will explore the lack of a functional relationship between perimeter and area, and they will find that the act of measuring establishes a functional relationship between an attribute of an object and a number.

Composition-decomposition The ability to understand most computation algorithms and to do mental arithmetic and estimation relies on the ability to decompose and recompose numbers. For example, 15×12 can be seen as 30×6 or as $15 \times 10 + 15 \times 2$. However, the processes of composition and decomposition do not stop with numbers. When transforming or measuring geometric figures we decompose and recompose them. Similarly, the problem-solving heuristic of breaking a problem into parts manifests this idea.

Conventions Many students have no idea that some of the mathematics that they “learn” is invariant and some is convention. Making this distinction is an important idea that encourages students to look more carefully at what we ask them to learn and humanizes mathematics. For example, adding from right to left is a convention rather than mathematically required. On the other hand, in our numeration system, the order of numerals in a number is not a convention— $43 \neq 34$. Similarly, there are conventions in graphing and then there are no-nos. For example, in bar graphs, even if a frequency of a value of the independent variable is zero, you cannot just skip it. This has to do with proportional reasoning, as noted below.

Proportional reasoning Like functions, proportional reasoning shows up in almost every chapter. Proportional reasoning is critical to a deep understanding of fractions, of slope, of probability, of percent, of similarity, and so many other concepts. This idea shows up every year in the Exploration 7.3, Typical Person. As students discuss the graphs they have constructed, they grapple with why some graphs are valid and others are invalid. For example, the width

of the bars and the height of the bars is a convention; however, squishing the bars together when the frequency for a value is zero is not allowable.

Unit We refer to units when developing an understanding of numeration. One of the reasons that multiplication and division problems are generally more difficult than addition and subtraction problems is that in multiplication and division, the units are not the same. For example, if someone bicycles for 3 hours at 15 miles/hour, that person will travel 45 miles—the unit for the multiplicand is hours; for the multiplier, it is miles/hour; and for the product, it is miles. Three different units for each of the three numbers in the problem! We refer to unit fractions, and we need to distinguish between units and wholes when working with fractions. When we make a scale for a graph, we are selecting a unit. We need units when making things—for example, we refer to a unit in quilt designs and tessellation patterns. And yes, of course, we talk about units in measurement.

Multiplicity This is the antithesis of the notion of mathematics as black and white, with most problems having one right answer, one best way to get to the answer, and one way to represent mathematical ideas. By the end of the course, if your instructor has you construct a portfolio, you should be able to write coherently about each of the following:

- Virtually all non-trivial mathematical problems can be solved in more than one way.
- Virtually every mathematical idea has multiple connections to other ideas.
- Most mathematical situations can be represented in more than one way.

Develop Mathematical Thinking

Whole books have been written on this and while I do not want to oversimplify it, I don't want to avoid it either. I believe mathematical thinking is as much a mindset as it is a constellation of related "skills" and thus, like relational understanding, cannot be directly taught as much as encouraged to develop. When one can think mathematically, one can appropriately use a variety of problem-solving strategies; one can justify one's thinking, which involves being able to communicate; one looks for patterns and can (intuitively, inductively, or deductively) arrive at hypotheses that one can check and revise and then ultimately make generalizations. For instance, in Exploration 4.4 (Star Patterns), as the students make stars they look for patterns, make and check hypotheses, and come to realize that number theory ideas help to explain why different stars have different shapes.

Algebraic thinking I have chosen to de-emphasize the use of algebraic procedures—but *not* algebraic thinking—for two fundamental reasons. First, like it or not, so many of our students in the course are algebra phobic. I take heart from a statement in *Revitalizing Undergraduate Mathematics Education*: "Teach the students we have, not the ones we wish we had." I clearly think algebra is an important tool, but this course is not the course to pursue this and all the other goals of the course. Second, I try to de-emphasize algebraic solutions to problems in this course and emphasize other solution paths—e.g., guess-check-revise and draw a diagram—because solving equations algebraically is not a tool that elementary school students have, and we clearly want children to develop a wide repertoire of problem-solving tools. I need to note that I do believe the course encourages the development of algebraic thinking, which is different from emphasizing the use of formulas to solve problems.

Problem-solving and communication Problem-solving and communication are such important aspects of mathematical thinking that I want to discuss each of them in a bit more detail. As stated before, I believe that problem-solving is not something we do after the students learn the concepts and procedures. Rather, problem-solving should permeate all of the Investigations and Explorations. Developing a repertoire of problem-solving tools is an important goal in the course, with an emphasis on being able to solve nonroutine and multi-step problems. Therefore, this book has fewer exercises that are “just like” the examples. I have come to believe that spending more time on fewer but richer problems results in students developing more mathematical power than doing lots of problems. Alfred North Whitehead’s book *The Aims of Education* states this principle more eloquently than I ever could.

One of the most exciting pieces of feedback from instructors who have used this book is from one who said that for the first time while he had been teaching the course, students were actually talking about mathematical ideas. As a result of the repeated emphasis on active learning, discussing, justifying, and the like, many students come to see the crucial role of definitions and notation in mathematics. Many students find explaining and justifying their work and “explaining as if to a friend on the phone” to be very new and frustrating at first, but a consistent focus on communication bears rich fruit by the end of the course. As students struggle to explain an idea, a word like “prime” or “vertex” suddenly makes it much easier to describe the relationship they are trying to understand.

In the *Instructor’s Resource Manual*, I discuss how learning logs and other devices help students to develop the ability to communicate their ideas and, more importantly, to see the value of being able to communicate mathematically. I have borrowed the notion of “first draft” from my English teacher colleagues. In class I will ask for someone’s first draft of a definition or a concept, and throughout the text I encourage students to write and revise as a path to understanding.

ANNOTATED OVERVIEW OF THE COURSE

Let us take a walk through the book to help you get a more tangible (concrete) sense of the book and how the two volumes work together. Because of space limitations, I will highlight aspects of each chapter rather than discuss every section in detail.

Chapter 1: Foundations for Learning Mathematics

This chapter sets the tone for the whole book. The Explorations offer different kinds of problems to grapple with. In the Text, I take some time to explore important strands of mathematical thinking—problem-solving, communication, reasoning, patterns, representation, and connections. I present and discuss the NCTM Standards in more detail.

Chapter 2: Fundamental Concepts

When I first began teaching this course and was still trying to “cover” as much of the book as I could, one day I suddenly realized that what I really wanted from the material on sets and functions was for students to have an appreciation of the need for these ideas. This chapter lays that foundation. Section 2.1 gives students tools to be able to talk of sets and subsets, and to be able to use a Venn diagram when the need arises in other chapters, for example, to under-

stand the relationship between different sets of numbers. Similarly, the notion of function being a certain kind of relationship between two sets or two variables arises throughout the book. Any of the Explorations in Section 2.2 not only will have the students come to an understanding of the notion of functions, but also will get them started on developing mathematical thinking—looking for patterns, making and testing predictions, and making generalizations. The text’s Investigations elaborate and refine this idea.

Numeration, discussed in Section 2.3, is a huge concept in elementary school mathematics. Exploration 2.7, Alphabitia, is one of the most powerful I have used. Most of my students report this to be the most significant learning of the semester and/or a turning point in the semester. The Exploration unlocks powerful understandings related to numeration, which the text supports by discussing the evolution of numeration systems over time.

Chapter 3: The Four Fundamental Operations of Arithmetic

These four operations are crucial for elementary school mathematics. Students come to realize that addition, subtraction, multiplication, and division are more than just plus, minus, times, and “gazinta.” They see how the concepts of the operations, coupled with an understanding of base 10, enable them to understand how and why the procedures they have done by rote for years actually work. In Exploration 3.9, Egyptian duplation piques their interest. The Explorations and Investigations help them to make sense of both standard and non-standard algorithms.

In addition to making sense of standard algorithms, I have extended the treatment of alternative algorithms to four Explorations in this chapter. My students have found these explorations to be both enlightening and fascinating.

Chapter 4: Number Theory

Number theory can be one of the most enjoyable chapters in the book. The Star Patterns exploration is one of my favorites because students find so many different patterns and can actually make and check hypotheses. Each of the Explorations in Chapter 4 has students grapple with core concepts of number theory in a connected way, and the text unpacks these concepts by investigating distinct aspects of them.

Chapters 5 and 6: Extending the Number System and Proportional Reasoning

I have chosen an unorthodox organization for Chapters 5 and 6 for conceptual and pedagogical reasons. The sets of integers, fractions, and decimals represent three historically significant extensions to the set of whole numbers. Given the limited time we have with the students, I find it makes more sense to emphasize rational numbers over integers and to connect the development of fraction concepts and operations in Chapter 5. Although the ratio construct is one of four major constructs of rational numbers, I find it to be more powerful to discuss ratio, proportion, and percents together in the context of Proportional Reasoning in Chapter 6.

In Exploration 5.5, students construct fraction manipulatives and then look for rules when ordering fractions, a critically important first step in seeing fractions as more than numerator and denominator. In Exploration 5.11 (Meanings of Operations with Fractions), having students represent problem situations with diagrams requires them to adapt their understanding of the four

operations, developed in Chapter 3, to fraction situations. Having first constructed this concept through exploration, the student can approach an Investigation like 5.7 (Ordering Rational Numbers), with a richer understanding of what it really means to say that one rational number is bigger than another.

Exploration 6.5 (Which Ramp Is Steeper?), although challenging, requires students to examine the relationship between length and height to predict steepness; from this they come to see slope as a measure of steepness. They also construct an understanding of proportions that enables them to work much more confidently with percents and pays dividends in most other areas. And, as noted above, the text's focus on the "big idea" of proportional reasoning reaps rewards throughout the book.

Chapter 7: Uncertainty: Data and Chance

This chapter was originally called "Probability and Statistics." However, not only did students find that title intimidating, but I believe "Uncertainty: Data and Chance" more accurately describes what the chapter is actually about. My construction of this chapter was strongly influenced by another book I highly recommend: *On the Shoulders of Giants: New Approaches to Numeracy*, edited by Lynn Arthur Steen. In Exploration 7.1 (Population Growth, Population Density, and Area of Residence) students make sense of real-life data that has relevance. Exploration 7.3 (Typical Person) is a popular and powerful exploration in which students decide how to collect and then represent data and discuss do's and don'ts of graphing. In Exploration 7.3 they also grapple with centers and spreads in data, two big ideas in statistics. The text focuses on critically interpreting representations others have come up with, and on introducing standard tools of statistical analysis and counting that help us make sense of the world. I am particularly excited that the Investigations with the concepts of mean and standard deviation have been so successful with students — as a result they can express these ideas conceptually instead of simply reporting the procedure.

Chapters 8 and 9: Geometry as Shape and Geometry as Transforming Shapes

I will confess that geometry is my favorite field in mathematics and it pains me to find that so many of my students so dislike it. I have chosen to have three geometry chapters, each focusing on geometry from a different perspective. In Chapter 8, you have the option of introducing geometry through Explorations with Tangrams, geoboards, or pentominoes. This less formal, more concrete introduction allows students with bad memories of geometry to let down their guard, but at the same time do good mathematical explorations. The geometric transformations we explore in Chapter 9 can be one of the most interesting and exciting topics of the course. Quilts and tessellations both spark lots of interest and provoke good mathematical thinking. The text develops concepts and introduces terms that help students to refine understandings that emerge from the Explorations.

Chapter 10: Geometry as Measurement

This chapter addresses measurement from a conceptual framework (i.e., identify the attribute, determine a unit, and determine the amount in terms of the unit) and historical perspective. Both the Explorations and Investigations get the student to make sense of measurement procedures and to grapple with fundamental measurement ideas. Exploration 10.2 (How Tall?) generates lots of different solution paths and ideas, and lots of discussion about indirect

measurement and precision. Exploration 10.9 (What Does π Mean?) has demystified π in the minds of many of my students and is a wonderful exercise in communication. Exploration 10.13 (Irregular Areas) requires students to apply notions of measuring area to a novel situation; every year students will hypothesize many different strategies, some of which are valid and some of which are not. The text looks at the larger notion of measurement, presents the major formulas in a helpful way, and illustrates different problem-solving paths.

Overall Features

Worthwhile mathematical tasks In developing and selecting Explorations and Investigations, I have searched for ones that will make all students feel challenged and successful. In developing the book, if I found that only some of the students engaged in an Investigation or Exploration, I either modified it or deleted it. One criteria, for me, of a good Investigation or Exploration is that there is more than one way to answer the question or problem that is posed.

Readability Reviewers of the text and students who have used the book have praised its readability. I have tried to be informal and engaging and yet not water down the ideas.

Connections One of the ways to nurture this aspect of mathematics is to integrate it into the textbook as a way of highlighting the interconnected nature of concepts. Brief, boxed inserts called *Connections* appear frequently throughout every text chapter. They have been a popular feature with reviewers and students alike. There are five kinds of connections, all identified with an icon. Browse through the book to get a feel for each kind of connection:

- History—considerations of interesting topics that are related to concepts developed in the text;
- Mathematics—connections between the concept being addressed and other mathematical concepts;
- Language—etymology of terms and/or nuances of terms;
- Beyond the Classroom—applications and uses of mathematical concepts and procedures in the business world, science, and everyday life;
- Children—stories and anecdotes from children’s lives that connect to the concept being addressed.

At the same time, connections are interwoven into the chapters in addition to the big ideas mentioned earlier. For example, in Chapter 6, after the ratio model of fractions is developed, we go back and revisit “fraction” problems for which this model provides a new solution path. In Section 7.4, after combinations and permutations are developed, we go back and revisit problems in Section 7.3 for which these concepts may or may not apply. More efficient ways to determine the mean are connected to the idea of repeated addition.

Active learning As Piaget once said, “To know is to invent.” The students have to actively read the text and actively participate in the Explorations. I regularly invite them to “stop and think before reading on.” The pencil icon is a visual reminder to do so. In Investigation 6.14, I pose the dilemma of a “fair raise” to introduce the idea of percent increase—the student is encouraged to grapple with the merits of two different “equal” raises. In Investigation 7.21, rather than just presenting the answer to how many combinations of boy/girl

are possible among 4 children, the discussion explores different hypotheses that students consistently generate. In Chapter 7, rather than simply giving the definition of a fair game, I encourage it to develop over the course of several Investigations. In Chapter 8, instead of just defining a polyhedron, I ask students to try to extend their knowledge of polygons to see if they can define a polyhedron and the various other terms.

Exercises To paraphrase a colleague of mine, Neil Davidson, I look for problems that will *require* thinking. In the exercise sets that follow each section of the text chapters, while I do provide some traditional practice exercises (for which the answers are generally in the back of the book), I have come to believe that grappling with nonroutine, multistep problems produces deeper understanding of mathematical ideas and greater problem-solving skills. I have also sought to include many exercises that are more like problems encountered in everyday life and work settings. You will find a number of problems that friends have reported struggling with or that have come up in my own life.

I need to explicitly mention something this book does not do that many other texts do: identify exercises by type. For example, denoting certain problems as “thinking critically” implies critical thinking is not necessary elsewhere. Labeling certain problems as communication problems implies that communication is only salient sometimes and marginalizes this aspect of mathematical thinking. Denoting certain problems as calculator problems reinforces the traditional role of the teacher deciding when to use and not to use calculators. Finally, “Just for Fun” implies that “real math” is hard work and not fun.

NCTM references Rather than simply quote from the NCTM Curriculum Standards at the beginning of each chapter, I have sought to integrate the ideas from the Curriculum and Teaching Standards into the fabric of the course. Thus, there is some direct quoting, some paraphrasing, and some illustrating of my use of ideas from the Standards. In the *Instructor’s Resource Manual*, I suggest various assessment practices that I believe embody the Assessment Standards more than the traditional daily homework, weekly quiz, and chapter test format.

Website There is a website for this book at <http://college.hmco.com>. As I write this preface, the website is still under construction. The planned features of the website include:

1. Links to other websites.
2. Extra chapters, including constructions, LOGO, coordinate geometry, and topology; and Explorations that were deleted from the first edition.
3. New Explorations and Investigations.
4. An annotated bibliography of journal articles, books, and other resources.
5. A forum for getting feedback from students and instructors; possibly interactive.
6. Errata—a place for instructors and students to report and find any errors that may be in the second edition.

Features Within Each Chapter

What do you think? These provocative questions may be discussed in class, assigned as journal writings, or given as homework.

Chapter and section introductions These set the context for the work the students are to begin and connect the new ideas to previous ideas. I try to consistently offer a rationale for each of the topics. See, for example, how the introduction to Chapter 7 focuses on broader concepts of what it means to talk about data and chance.

Investigations Questions are posed and then the students are asked to “stop, think, and then read on.” The pencil icon is a visual reminder to do so. Then there is a Discussion of the investigation, which generally involves several different solution paths. In some cases, the answers are given right there. In some cases, the answer is not given, thus giving you the option of giving the answer after the students have investigated the question, or to use the question as a journal entry or as a homework problem.

Section summaries I have sought to make these more than simply a restating of what students have learned but rather a looking back at and weaving together of the ideas of the section and putting them in a broader context.

Chapter summary with key terms Each Chapter Summary begins by summarizing the big ideas of the chapter—those ideas that connect the different sections or connect ideas from this chapter with ideas from earlier chapters. The Chapter Summary also includes a complete list of key terms and basic concepts introduced in the chapter, with page references. Many students find this to be a useful reference.

End-of-Text Features

Four-color insert This contains actual pages from the Houghton Mifflin K–6 series and enables students to see children interacting with the same concepts that they are now studying.

Answers to selected exercises These are provided for student reference. The *Instructor’s Resource Manual* contains complete solutions to all end-of-section exercises.

Changes From the First Edition

- Chapter 1 and the *NCTM Standards* appendix have been rewritten to include the framework of the new NCTM Standards, including a standard on Representation.
- Chapter 1 has an expanded treatment of Logic.
- Chapter 1 has been divided into two parts, with an exercise set for each part.
- The title of Chapter 2, Section 2 has been changed from *Functions* to *Algebraic Thinking*. This section has been substantively rewritten to reflect the four major algebraic ideas we want students to bring from elementary school: algebra as a set of rules and procedures; algebra as the study of structures; algebra as the study of relationships among quantities; algebra as generalized arithmetic.
- The sourced data in Chapter 7 has been updated or replaced in order to provide the most current data available at the time of publication.
- Chapter 9 has an expanded treatment of symmetry.

- Appendices A (*Constructions*) and B (*LOGO*) in the last edition have been removed from the text and placed on the website.
- Throughout, there have been some deletions of old, as well as insertions of new, Investigations, Connections, and Exercises.
- The exercise sets now include identification (the numerals are in blue) of those exercises that have answers given at the back of this text.

Supplements

Instructor's Resource Manual This teacher-to-teacher resource manual serves as a guide to building an active, constructivist course. Part 1 includes five brief chapters. Chapter A is an elaboration of the goals of the course and how the Explorations work. Chapter B focuses on creating a positive learning environment, including a discussion of the physical environment, expectations and norms. Chapter C presents some basics on cooperative learning. Chapter D examines some important differences among students that an instructor needs to think about, e.g., attitudes and beliefs, personal traits, gender, and ethnicity. Chapter E discusses assessment, a critical issue in a more innovative text such as this one.

Part 2 includes instructor's notes for each chapter of the Explorations. These notes include a brief chapter overview for the instructor, Intended Outcomes for each Exploration, an indication of time and materials needed, key Issues and Considerations which cover common difficulties/problems students may have, effective opening questions and other tips gleaned from my experience with the Explorations, and examples of students' work.

The *Instructor's Resource Manual* also contains complete solutions to all the end-of-section exercises from the text. As much as possible, I have sought to include different ways that students might solve the problems.

Computerized Test Generator The Computerized Test Generator contains more than 1000 test items. This ancillary can also be thought of as a database of additional exercises. These items were carefully written to reflect the philosophy of the text. The emphasis is on understanding concepts rather than symbolic manipulation. Therefore, you will find almost none of the typical multiple-choice type questions. Instead, there are many essay, open-ended, and free-response questions. The first part of the test bank includes questions referenced for each section of the text. The second part includes comprehensive chapter questions for each chapter. The Test Generator is available for the IBM PC and compatible computers and for the Macintosh. Both versions provide **on-line testing** and **gradebook** functions.

Printed Test Bank The Printed Test Bank is a printout of all items in the Computerized Test Generator. Instructors using the Test Generator can use the test bank to select specific items from the database. Instructors who do not have access to a computer can use the test bank to select items to be included on a test being prepared by hand.

Student Solutions Manual This student supplement contains complete solutions to those text exercises that have answers given at the back of the text. There are no other student supplements. When I first began writing the book, I intended to write the standard "Student Resource Handbook." However, over the course of writing and revising the textbook, I have become convinced that such a supplement detracts from the main foci of the book and actually hinders rather than helps the students' development.

ACKNOWLEDGMENTS

It is impossible to completely acknowledge all the people and materials that have influenced the construction of this textbook. I find that several conversations with and writings by Marty Simon, Deborah Schifter, and Deborah Ball have felt, to use an analogy, like a tape loop—ideas or statements that have played over and over and over in my mind. For two years I worked on an NSF project called Mathematics for Tomorrow at Education Development Center. My work and conversations with Ellen Davidson and Jim Hammerman provoked a level of intellectual disequilibrium that often felt overwhelming but has been very helpful.

I highly recommend books and research by the following authors whose writings have enabled my practice to be more deeply connected to frameworks: John Clement, Paul Cobb, Robert Davis, Carol Dweck, Constance Kamii, Robert Karplus, Richard Lesh, Jack Lochhead, and Alan Schoenfeld.

There is not room to cite all the books that I read in preparing for this book. Certain of those books, though, were more influential than others and I particularly recommend them. Over the course of preparing for the book, I paged through the *NCTM Standards* documents many times and found many of the books in the NCTM Addenda series very helpful. Other notable books include: *American Indian Design and Decoration* by Le Roy H. Appleton; *Capitalism & Arithmetic: The New Math of the 15th Century* by Frank Swetz; *From One to Zero: A Universal History of Numbers* by Georges Ifrah; *Introduction to Tessellations* by Dale Seymour & Jill Britton; *Mathematics* by David Bergamini (Ed.); *Metamorphosis: A Source Book of Mathematical Discovery* by Lorraine Mottershead; *Number Words and Number Symbols: A Cultural History of Numbers* by Karl Menninger; *Numbers: Their History and Meaning* by Graham Flegg; *Symmetry: A Unifying Concept* by Istvan & Magdolna Hargittai; *The History of Arithmetic* by Louis Charles Karpinski; and *The Language of Functions and Graphs* by the Shell Centre for Mathematical Education.

I certainly cannot claim that I invented all of the explorations, investigations, and problems in these two volumes. In many cases, I have made specific acknowledgments; if I missed someone, I apologize in advance. It is my hope that, with the increased amount of collaboration in mathematics education spurred by the NCTM and by advances in technology, teaching and writing will be much less solitary occupations than they have been in the past.

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Finally, I would like to thank the many people at Houghton Mifflin who have supported my desire to present to the mathematics community a book that I feel breaks new ground in some important ways. It is not as radical a departure as I had wanted but I understand that a book that looks too different will likely not sell and thus won't effect much change! I want to thank Maureen O'Connor, Sponsoring Editor, for recognizing the potential of this book from a very rough prospectus seven years ago and supporting my desire to write a book that tries to break new ground. I also want to thank Dawn Nuttall, Development Editor, and Rachel D'Angelo Wimberly, Project Editor, for their hard work. Attention to detail is not one of my God-given virtues and this created more work for them, as well as my always wanting to change material as it changed in my head. I would especially like to thank my developmental editors, Karla Paschkis and Cindy Harvey. Over the four years it took to write the first edition, Karla learned a lot of math and a lot about the reform movement, and I learned to be a more careful thinker. Having to justify all my ideas was at times frustrating but very helpful. Cindy's ideas and suggestions for the second edition were extremely valuable.

T.B.

Foundations for Learning Mathematics

- ◆ 1.1 Getting Comfortable with Mathematics
- ◆ 1.2 Problem-Solving
- ◆ 1.3 Representation
- ◆ 1.4 Patterns
- ◆ 1.5 Reasoning and Proof
- ◆ 1.6 Communication
- ◆ 1.7 Connections

Knowing mathematics means being able to use it in purposeful ways. To learn mathematics, students must be engaged in exploring, conjecturing, and thinking rather than only in rote learning of rules and procedures. Mathematics learning is not a spectator sport. When students construct personal knowledge derived from meaningful experiences, they are much more likely to retain and use what they have learned. This fact underlies [the] teacher's new role in providing experiences that help students make sense of mathematics, to view and use it as a tool for reasoning and problem solving.¹

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

SECTION

1.1

GETTING COMFORTABLE WITH MATHEMATICS

Were you excited when you signed up for this course? Were you worried? Do you fondly remember your math lessons in elementary school, or have you been trying to forget them for all these years? You are at the beginning of a course in which you will reexamine elementary school mathematics to understand the underlying concepts better and to learn *why* mathematical procedures and formulas actually work. Your approach to this course depends on the attitudes and beliefs you bring to the classroom as a student; in subtle or not-so-subtle ways, you may pass these beliefs along when you enter the classroom as a teacher. So before we start working with mathematical concepts, let's look at our current conceptions of mathematics.

¹ NCTM, *Curriculum and Evaluation Standards for School Mathematics: Executive Summary* (Reston, VA: NCTM, 1989), p. 5.

Beliefs About and Attitudes Toward Mathematics

This preliminary exercise is designed with two purposes in mind. First, it will help you examine and reflect on your beliefs and attitudes at the beginning of the course. Second, it will help you see a practical use of mathematics.

Rate your attitudes Six pairs of statements concerning attitudes toward and beliefs about mathematics are given in Table 1.1. Score your beliefs in the following manner:

- If you strongly agree with the statement in column 1, record a 1.
- If you agree with the statement in column 1 more than with the statement in column 2, record a 2.
- If you agree with the statement in column 2 more than with the statement in column 1, record a 3.
- If you strongly agree with the statement in column 2, record a 4.

TABLE 1.1

Column 1	Column 2
1. There will be many problems in this book that I won't be able to solve, even if I try really hard.	1. I believe that if I try really hard, I can solve virtually every problem in this book.
2. There is only one way to solve most "word" problems.	2. There are usually many different ways to solve most "word" problems.
3. The best way to learn is to memorize the different kinds of problems—rate problems, mixture problems, coin problems, etc.—and how to solve them.	3. The best way to learn is to make sure that I understand each step.
4. Some people have mathematical minds and some don't. Nothing they do can <i>really</i> make a difference.	4. Some students may have more aptitude for mathematics than others, but everyone can become competent in mathematics.
5. The teacher's job is to show us how to do problems and then give us similar problems to practice.	5. The teacher's job is more like that of a coach or guide—to help us develop the problem-solving tools we need.
6. A good test consists of problems that are just like the ones we have done in class.	6. A good test has problems at a variety of levels of difficulty, including some that are not like the ones in the book.
7. I don't need to know all the ideas covered in this book because I'm going to teach younger children.	7. Even teachers of younger children need to have a good understanding of the ideas in this book.

Adaptive and maladaptive beliefs We will return to your responses to Table 1.1 soon. Meanwhile, let me say that in over twenty years of teaching mathematics in elementary school, middle school, high school, and college, I have worked with thousands of students. One thing (of many) these students have in common is that their beliefs about mathematics influence how they learn. With respect to their beliefs and attitudes, students generally fall into four basic categories:

- Some students work hard, confident in their problem-solving abilities.

- Some students work hard at memorizing everything the teacher says because they are fearful of getting a wrong answer.
- Some students don't try very hard because they are convinced that mathematics is too difficult for them to understand.
- Some students don't try very hard because they don't think mathematics is relevant or useful.

The first group has what are called **adaptive beliefs** that help them approach math with a positive and confident attitude. The other three groups have different combinations of **maladaptive beliefs** that may keep them from thinking of learning as an evolving and enjoyable process. Many students find that they fit into more than one category or that the category they fall into depends upon the teacher and the material.



BEYOND THE CLASSROOM²

The pervasiveness of negative attitudes toward mathematics was powerfully illustrated in 1992 when Mattel introduced a new talking *Barbie* doll that said, "Math is tough." Now this may be true for many people, but having Barbie say it only reinforced that stereotypical perception of mathematics in the United States especially among females. Mattel was persuaded to change Barbie's statement.

Negative attitudes toward mathematics persist in many people. One article summarized a common attitude toward mathematics and mathematics teaching: "Mathematics—a set of facts, rules, and procedures—is a 'package' to be passively received. . . . Mathematics teachers are supposed to spend time explaining or 'covering' material from the textbook. . . . Teachers verify that

students have received knowledge by checking the students' answers to make sure they are correct" (Martha Frank, "Problem Solving and Mathematical Beliefs," *Arithmetic Teacher*, 35(5), January 1988, pp. 32–34).

Stop and take stock of your reaction to this statement. To what extent do you agree with some or all of this statement? 

That this view was held so widely by so many people was one of the reasons that the National Council of Teachers of Mathematics developed the many support materials that are available to teachers today, beginning with *The Curriculum and Evaluation Standards*, which were published in 1989.

In Table 1.1, the statements in column 1 indicate maladaptive beliefs and the statements in column 2 indicate the corresponding adaptive beliefs. If you take the arithmetic average, or *mean*, of your scores (by adding up your scores and dividing by 6), you will get a number that we could call your belief index. If your belief index is less than 2, I would say that your beliefs are probably more maladaptive than adaptive. If you encounter difficulties in this course, it may be because some of your beliefs are hindering your ability to learn the material. If you do find this course frustrating, try to discuss your beliefs with your professor, with someone at a math center (if your college has one), or with a friend who is doing well in the course.

Individual beliefs and class beliefs Compare your present beliefs with those of your classmates and then discuss how you could represent what these data tell you—about you as an individual and about the class as a whole.

²At appropriate places, you will see the arrow icon indicating a specific connection that I will be making—connections called CHILDREN, LEARNING, MATHEMATICS, HISTORY, BEYOND THE CLASSROOM, and LANGUAGE. These notes will help develop the notion of the connectedness of mathematics.

What did you come up with? Below are several possibilities. If you were to add the belief indexes of all students in your class and divide the total by the number of students, you would get the belief index for the whole class at the beginning of the semester. You could then compare that to the belief index at the end of the semester. You could also compare it to the belief index in other mathematics courses (for example, calculus) or to the belief index of high school freshmen, seventh graders, etc.

You could get the average score for the class for each question. You could represent that score either verbally (for example, the average in our class for the first pair of beliefs was 2.4) or visually (for example, on a bar graph). These are just two possibilities; you and your classmates may come up with others.

If my past experience is a reliable guide, many of you are entering this course with more maladaptive than adaptive beliefs.³ Yet you just tackled a multistep and nonroutine problem, more of which are appearing in today's elementary textbooks. You figured out how to represent abstract data concretely, and you figured out how to pull together a large group of data into a more easily understandable form. You probably also saw that the people in your class approached this problem in a lot of different ways.

What Is Mathematics?

Many of my students ask, "Why do I have to learn all this mathematics in order to become an elementary school teacher?" I hope that by the end of the course, this book will have helped you to give a good answer to that question. A question that needs to be addressed before that question is "What is mathematics?" Think about this question for a minute and then read on. . . . 

You may be surprised to learn that not all mathematicians give the same response to this question. *On the Shoulders of Giants: New Approaches to Numeracy*⁴ was written partly to help expand people's views of mathematics beyond the common stereotype of "mathematics is a bunch of formulas and rules for numbers." A group of mathematicians and mathematics educators brainstormed a number of possible themes for that book. In the end, it was agreed that the idea of *pattern* permeates all fields of mathematics. Five mathematicians were asked to write chapters on the following themes:

Dimension. In school, you have studied two- and three-dimensional shapes. Mathematicians have gone far beyond three dimensions for years. Recently, a field of mathematics has opened up the exploration of fractional dimensions. For example, the coastline of Britain (which can be modeled by a long, squiggly line) has been calculated to have a dimension of 1.26.

Quantity. This begins (with children) with the question "how many," for which the counting numbers (1, 2, 3, . . .) are appropriate; it moves in complexity to the question "how much," for which fractions and decimals were invented, and then to questions far more complex, for which other numbers and systems were invented.



MATHEMATICS

Keith Devlin has written three fascinating and readable books on this subject,⁵ one of which is a companion to a PBS series entitled *Life by the Numbers* (which your college or local library may have). The chapter titles for *Mathematics: The Science of Patterns* are "Counting," "Reasoning and Communicating," "Motion and Change," "Shape, Symmetry and Regularity," and "Position." He discusses (among many other things) how mathematicians helped us to understand why leopards have spots and tigers have stripes, how they helped American ice skaters learn how to perform triple axel jumps, and how we use mathematics to measure the heights of mountains.

³ It is important to note that I do not want to imply that if you have not done well in mathematics, this is because you have maladaptive beliefs. There are many factors that can affect a student's success in a course. Here we are focusing on changes in students' attitudes and beliefs.

⁴ Lynn Arthur Steen, ed., *On the Shoulders of Giants: New Approaches to Numeracy* (Washington, DC: National Academy Press, 1990).

⁵ *Mathematics: The Science of Patterns* (New York: W. H. Freeman, 1996); *The Language of Mathematics: Making the Invisible Visible* (New York: W. H. Freeman and Company, 1998); and *Life by the Numbers* (New York: John Wiley, 1998).

LEARNING

Deborah Meier, an award-winning principal of several elementary schools in the New York City area, wrote a book called *The Power of Their Ideas*.⁷ One aspect of her leadership style was to ask, “So what?”, which is the antithesis of the common anthem of “Just do it.” I encourage my students not just to do it but to ask this question of me and of themselves. For example, I think it is important for you to examine your attitudes and beliefs toward mathematics. Do you buy this or have the last several pages simply been a blur of words? There are many wonderful teachers in this country, but there are also many teachers who “just do it.” For them mathematics is just another subject to be “covered”; it is somewhat intimidating and not very interesting. Do you want your students to form this view of mathematics and carry it from your classroom? If not, then you need to ask yourself the “So what?” question frequently, and when you cannot give a satisfactory answer, talk with your instructor or go elsewhere—to the library, to the Web, to articles in *Teaching Children Mathematics*, to classrooms—and see for yourself the kinds of mathematical experiences that we want young children to have in school.

Uncertainty. Questions of uncertainty permeate everyday life: How long will I live? What are my chances of getting a job after I graduate? What are the chances that my baby will be “normal”?

Shape. I think that humans’ relationship with shape is a fascinating history—the shape of one’s environment (desert, forest, mountain), what shape is best for packaging, the shapes that artists make, and the shapes we manufacture for quilts, clothing, and so on.

Change. We live in a world that is constantly changing. The development of computers enables us better to understand and manage change, whether it be the changing weather, the change in epidemics (such as AIDS), the change in populations (human and animal), or changes in the economy.

I cannot overstate the importance of getting you, as future elementary teachers, to expand your view of what mathematics is. Mathematics is far more than titles of courses and chapters in textbooks—whole numbers, fractions, decimals, percents, algebra, geometry, etc. These topics represent tools that are needed in order to answer important questions about dimension, quantity, uncertainty, shape, and change. “The numbers, lines, angles, shapes, dimensions, averages, probabilities, ratios, operations, cycles, correlations, etc., that make up the world of mathematics enable people to make sense of a universe that otherwise might seem to be hopelessly complicated.”⁶ It is crucial that elementary teachers believe that mathematics is about fundamental ideas that are accessible to “regular” people and to understand these ideas so that *all* of your students will be at least competent and that *all* will see that mathematics is as important as other subjects.

The NCTM Standards

▶ The National Council of Teachers of Mathematics (NCTM) published *The Curriculum and Evaluation Standards for School Mathematics* in 1989. This landmark book articulated, for the first time, why the nature of mathematics instruction in the United States has to change and described a much larger vision of what students ought to be learning in school about mathematics. The book presented detailed standards for elementary, middle, and high school mathematics. Two other important books followed: *Professional Standards for Teaching Mathematics* in 1991 and *Assessment Standards for School Mathematics* in 1995. As I am writing the second edition of this textbook, the NCTM is finalizing a new book, *Principles and Standards for School Mathematics*, which will incorporate and update the three books just mentioned. This book presents 10 standards that run through mathematics from pre-kindergarten through 12th grade.

Standard 1: Number and Operation	Standard 6: Problem Solving
Standard 2: Patterns, Functions, and Algebra	Standard 7: Reasoning and Proof
Standard 3: Geometry and Spatial Sense	Standard 8: Communication
Standard 4: Measurement	Standard 9: Connections
Standard 5: Data Analysis, Statistics, and Probability	Standard 10: Representation

⁶American Association for the Advancement of Science, *Benchmarks for Scientific Literacy* (New York: Oxford University Press, 1993), p. 25.

⁷Boston: Beacon Press, 1995.

Chapters 2–10 in this textbook will engage you in the important ideas related to the first five standards for elementary school children. The last five standards permeate this textbook. Without them, all the concepts and formulas and procedures in the other areas are inert; that is, these *process standards*⁸ make the content come to life. The rest of the first chapter and the Explorations for the first chapter have been designed to give you a sense of these five standards and their importance in the mathematics classroom. In the following sections, I ask that you read the NCTM summary of each of the process standards carefully. Assess the extent to which you understand what you are reading and the extent to which you “buy it.” At the end of the section, go back and reread this summary to see whether my examples and elaborations have helped it to make more sense. None of the excellent elementary teachers I know applies these five standards “by the book.” Rather, they look at their lesson plans (daily plans, unit plans, and yearly plans) and they make sure that these standards are alive in their lesson plans.

Using Mathematics

Let us now turn our attention to how people use mathematics (in everyday situations and in work situations). Take a few minutes to jot down some instances in which you have used mathematics in your life and some instances in which you know that mathematics is used in different careers and work situations. Then read on. . . . 

We find that people generally use mathematics for a purpose, for example:

- To persuade a boss that our idea will make money
- To persuade a potential customer that our product will save money
- To predict—tomorrow’s weather or who will win the election
- To make a personal decision—whether we can afford to buy a house
- To make a business decision—how much to charge for a new product or whether a new medicine (for example, a cure for AIDS) really works

Solving problems In each of these examples, people are using mathematics as a tool for solving problems. To decide whether you can afford a new car, you have to collect data (on insurance, for example), add decimals, and work with percents (such as sales tax and interest on the loan). The mathematics you will use will help you solve the problem of whether to buy a car.

Think about this situation involving weather forecasters. In 1994 a hurricane brought severe rains to Georgia. Forecasters predicted that the Flint River would crest at 20 feet above flood level; the river actually crested at 13 feet above flood level, much to the relief of many residents. The forecasters probably used decimals and volume formulas or conversions to determine the maximum volume of water that would be flowing. They probably based their final results on computer models of flooding rivers, and the computer models were based on data collected on previous flooding.

Now that we have discussed mathematics in general, we are ready to focus on developing problem-solving skills that can be used in a wide variety of situations.

⁸ I will use the term *process standards* to refer to Standards 6–10. However, note that this is simply a phrase that enables me to communicate quickly. The longer version is “the standards that cut across all content areas in mathematics and that should be alive in virtually every lesson.”

1.2

In the following NCTM summary of the standard on problem-solving, I have added parenthetical comments that are meant to elaborate on the meaning and importance of each statement. As you read the summary, check the extent to which you understand each statement (both the NCTM's and mine).

- What problem-solving tools do you bring to this course?
- In addition to verifying your computation, how can you verify your answer to a problem?

▶ Mathematics instructional programs should focus on solving problems as part of understanding mathematics so that all students—

- NCTM *Principles and Standards for School Mathematics*: Discussion Draft
(Reston, VA: NCTM, 1988)

When you say **problem-solving** to most people, they think of an image something like Figure 1.1. Many of my students tell me that when they came into the course, their primary learning tool was memorization and their primary problem-solving tool was what they called “trial and error.”

But problems need not be a source of dread. If you have done some of the explorations in the *Explorations* volume, you have already discovered some new tools for solving problems. In this section, we will examine some of the tools that are essential for solving multistep and nonroutine problems. Think of problems beyond the walls of the classroom that require mathematics. Generally, they are not one-step problems (such as simply dividing a by b), nor are they usually *just like* a problem you have solved before. We do our students a disservice if we lead them to believe that problem-solving is simply memorizing formulas and procedures.

Before we do some problems, stop for a moment. What kinds of problem-solving tools do you bring to this course? Many people will find this exercise more productive if they think of actual problems they have had to solve, such as buying a car, saving for college, or deciding how much food and beverages to buy for a party. Stop and write down your thoughts before reading on. If possible, discuss your ideas with another student also. 



* Source: THE FAR SIDE copyright 1987 FARWORKS, INC. Used by permission. All Rights Reserved.

Our first problem, though silly, is well known because it nicely illustrates a number of important problem-solving strategies.

INVESTIGATION 1.1 Pigs and Chickens

A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, “I count 24 heads and 80 feet. How many pigs and how many chickens are out there?”

Before reading ahead, work on the problem yourself or, better yet, with someone else. Close the book or cover the solution paths while you work on the problem. 

When you come up with an answer, compare it to the solution paths below.

DISCUSSION

STRATEGY 1: Use trial and error

One way to solve the problem might look like what you see in Figure 1.2.

$\begin{array}{r} 12 \\ \times 4 \\ \hline 48 \end{array}$	$\begin{array}{r} 12 \\ \times 2 \\ \hline 24 \end{array}$	$\begin{array}{r} 5 \\ \times 4 \\ \hline 20 \end{array}$	$\begin{array}{r} 19 \\ \times 2 \\ \hline 38 \end{array}$	$\begin{array}{r} 19 \\ \times 4 \\ \hline 76 \end{array}$	$\begin{array}{r} 5 \\ \times 2 \\ \hline 10 \end{array}$	$\begin{array}{r} 18 \\ \times 4 \\ \hline 72 \end{array}$	$\begin{array}{r} 6 \\ \times 2 \\ \hline 12 \end{array}$	$\begin{array}{r} 16 \\ \times 4 \\ \hline 64 \end{array}$	$\begin{array}{r} 8 \\ \times 2 \\ \hline 16 \end{array}$
$\begin{array}{r} 48 \\ + 24 \\ \hline 72 \end{array}$	$\begin{array}{r} 20 \\ + 38 \\ \hline 58 \end{array}$	$\begin{array}{r} 76 \\ + 10 \\ \hline 86 \end{array}$	$\begin{array}{r} 72 \\ + 12 \\ \hline 84 \end{array}$	$\begin{array}{r} 64 \\ + 16 \\ \hline 80 \end{array}$					

FIGURE 1.2

Unfortunately, trial and error has a bad reputation in schools. Many students come to class believing that it does not yield a “real” solution to a problem. Also, the words *trial* and *error* do not sound very friendly. However, this strategy is often very appropriate. In fact, many advances in technology have been made by engineers and scientists who were guessing with the help of powerful computers using **what-if programs**. A what-if program is a logically structured guessing program. Informed trial and error, which I call **guess-check-revise**, is like a systematic what-if program. Random trial and error, which I call **grope-and-hope**, is what the student who wrote the solution in Figure 1.2 was doing. In this case, the student finally got the right answer. In many cases, though, grope-and-hope does not produce an answer, or if it does produce an answer, the student does not have much confidence that it is correct.

STRATEGY 2: Use guess-check-revise (with a table)

One major difference between this strategy and grope-and-hope is that we record our guesses (or hypotheses) in a table and look for patterns in that table. Such a strategy is a powerful new tool for many students because a table often reveals patterns. Look at Table 1.2. A key to “seeing” the patterns is to make a fourth column called “Difference.” Do you see how this column helps? We will



LANGUAGE

The full description of this strategy is “Think, then guess, then check, then think, then revise (if necessary), and repeat this process until you get an answer that makes sense.” A somewhat condensed description is think-guess-check-think-revise. I will refer to this strategy throughout the book simply as guess-check-revise, but I urge you not to let the strategy become mechanical.

TABLE 1.2

	Number of pigs	Number of chickens	Total number of feet	Difference	Thinking process
First guess	10	14	68		
Second guess	11	13	70	+2	Increasing the number of pigs by 1 adds 2 feet to the total. What if we add 2 more pigs?
Third guess	13	11	74	+4	Increasing the number of pigs by 2 adds 4 feet to the total. Because we need 6 more feet, let's increase the number of pigs by 3 in the next guess.
Fourth guess	16	8	80	+6	Yes!

explore the notion of how seeing patterns can enhance our problem-solving ability in Section 1.3.

It is important to note that the guesses shown in Table 1.2 represent one of many variations of a guess-check-revise strategy.

STRATEGY 3: Make a diagram

Some people think in words, others in numbers, and still others in pictures. Sometimes making a diagram can show one solution to a problem. I stumbled across this approach one day as I was walking around the classroom listening to students work on this problem in small groups. Figure 1.3 shows what one student had done. How do you think she had solved the problem? Write your thoughts before reading on. . . .

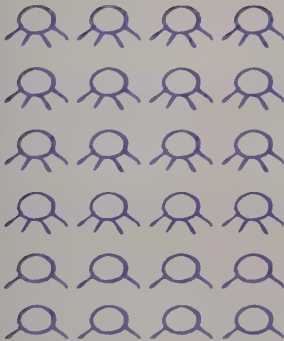


FIGURE 1.3

I asked her how she had solved the problem. She replied that she had made 24 chickens, which gave her 48 feet. Then she kept turning chickens into pigs (by adding 2 feet each time) until she had 80 feet! I was thrilled because she had represented the problem visually and had used reasoning instead of grope-and-hope. She was embarrassed because she felt she had not done it “mathematically.” However, she had engaged in what I call mathematical thinking.

Furthermore, this solution connects nicely to an algebraic solution, as we shall see shortly. I have come to believe that for *many* students, the best way to understand the more abstract mathematical tools is first to solve problems using more concrete tools and then to see the connections between the concrete approach (in this case a drawing or guess-check-revise) and the abstract approach (in this case two equations).

STRATEGY 4: Use algebra

Because the range of abilities present among students taking this course is generally wide, it is likely that some of you fully understand the following algebraic strategy and some of you do not. Furthermore, many students enter this and other college math courses believing that the algebraic strategy is the *right* strategy, or at least the *best* strategy. Let’s look at an algebraic solution and then see how it connects to other strategies and to the goals of this course.

Go back and review strategies 1 and 2. They both involved a total of 24 pigs and chickens. Can you explain in words why this is so? Think about this before reading on. . . . 

Most students say something like “Because the total number of animals is 24” or “Well, 24 animals will have 24 heads.” Therefore, if we say that

p = the number of pigs

c = the number of chickens

then the *number* of pigs plus the *number* of chickens will be 24. Hence, the first equation is

$$p + c = 24$$

Many students have difficulty coming up with the second equation. If this applies to you, look back at how we checked our guesses when using Strategy 2, guess–check–revise: We multiplied the number of pigs by 4 and the number of chickens by 2 and then added those two numbers to see how close that sum was to 80. In other words, we were doing the following:

$$\begin{array}{ccc} \text{(The guess for number of pigs)} \times 4 & + & \text{(the guess for number of chickens)} \times 2 \\ (p \times 4) & + & (c \times 2) \end{array}$$

More conventionally, this would be written as

$$4p + 2c$$

Using guess–check–revise, we had the right answer when this sum was 80. Thus, the second equation is

$$4p + 2c = 80$$

If you solve these two equations, using basic algebra, you will discover that $p = 16$ and $c = 8$.

The algebraic strategy in perspective Students who do not do well with equations generally understand this algebraic solution better if they see it *after* they have used guess–check–revise. The reason for this is simple and is grounded in learning theory: *The teacher’s explanation (or another student’s) is far more meaningful if you can connect it to something you already know.* This is such an important learning principle that I want to elaborate. If I show you how to create the equation before you have attempted the problem on your own, my explanation is not likely to stick. In this case, your knowledge is “Teflon knowledge.” The Teflon molecule was specially designed by chemists so that other materials tend not to stick to it. This is a wonderful property for frying pans to have, but not for human brains! To the extent that new ideas and concepts have **connections** to ideas and knowledge you already have, to that extent you are more likely to retain (that is, to **own** rather than **rent**) that new knowledge.

Algebra can be a very effective strategy for many problems. However, I offer the following cautionary notes with respect to the use of algebra in this course. First, the algebraic strategy is not more mathematical than the guess–check–revise strategy; it is simply more abstract. Second, it is better to use guess–check–revise and be confident of the process and of the answer than to use grope-and-hope to come up with an algebraic equation that you hope works out. Furthermore, guess–check–revise is a tool all of your future students should have, regardless of their age, whereas only a small percentage of your future students will have formal algebraic knowledge before grade 8 or 9.

Problem-Solving and Toolboxes

An image of problem-solving I would like to develop in your mind is that of a **toolbox**. Imagine that your car breaks down and is towed to the garage, where a novice mechanic, right out of training school, is the first to look at it. The novice will probably try a few standard procedures: Insert the key to see what happens, check the battery connections, look for a loose wire, and so on. If none of those strategies work, the novice mechanic will be stumped and will have to summon the senior mechanic. The senior mechanic may try the same basic procedures and may solve the problem by *interpreting* the results. If this does not solve the problem, the mechanic will have to go to two toolboxes. The first toolbox is a physical one. The second toolbox is a mental one.

At the beginning of this course, many students are like the novice mechanic: When they encounter a problem, they have a limited repertoire of strategies. However, as the course develops, their toolbox grows in two ways. First, the number of tools grows. This is like the novice mechanic's learning to use the garage's diagnostic equipment. Second, their ability to use each tool also grows. This is analogous to the novice mechanic's learning how to use a voltmeter more skillfully.

Looking back At this point, I want to introduce another process that most successful students have incorporated into their toolbox. They **reflect** on their work. That is, after solving a problem, they stop and examine their toolbox—what tools they used better and what new tools they used. Take a few minutes now to look at your notes on the strategies we discussed in Investigation 1.1 and then look at “4 Steps for Solving Problems” on the inside front cover of the *Explorations* volume. What tools were used in the various solutions of the pigs-and-chickens problem? What made those tools work better? Then read on. . . . 

My list includes:

1. Draw a diagram—a picture is often worth a thousand words.
2. Guess-check-revise—starting here will help some students to develop the two equations.
3. Make a table—this increases the possibility of seeing more patterns.
4. Look for patterns—extending the table enables certain patterns to become more visible.
5. Develop an equation—a powerful tool, but also surrounded by quicksand for many.

Two final notes about Investigation 1.1:

1. The strategies discussed in Investigation 1.1 do not represent all the different ways in which the pigs-and-chickens problem has been solved. Because of space restrictions, only four were discussed.
2. Many students think that there is only one tool per problem. An important adaptive belief is that there are often several different ways (using different tools) to solve a problem and that the tools are often used in combination instead of separately.

Polya's Four Steps

Now that we have examined the solving of one problem in detail, let us stop to reflect for a bit. In this textbook, we will examine problem-solving from the perspective of a toolbox—that is, from the perspective of developing tools to

help you solve problems. George Polya developed a framework for problem-solving that breaks down problem-solving into four distinguishable steps. In 1945, he outlined these steps in a now-classic book called *How to Solve It*.

When you approach a problem—a math problem, a writing assignment, even a personal problem—if you think that you have to come up with an answer immediately and that there is only one “right” way to reach that answer, a solution may seem to be beyond your grasp. But if you break the problem down and creatively and mindfully approach each *step* of the problem, it generally becomes more manageable. Polya suggests that you first need to make sure you **understand the problem**. Once you understand the problem and know what you need to find out, you **devise a plan** for solving the problem. Then you **monitor your plan**; you check frequently to see whether it is productive or is going down a dead-end street. Finally, you **look back at your work**. This last step involves more than just checking your computation; for example, it includes making sure that your answer makes sense. For each of these four steps, there are specific strategies that we will explore in this chapter and that you will refine throughout this course.

Owning versus renting Instead of just listing Polya’s strategies, we are going to discover them by putting them into action. You will notice that I often ask you to stop, think, and write some notes. I really mean it! I have come to distinguish between those students who *own* what they learn and those who simply *rent* what they learn. Many students rent what they have learned just long enough to pass the test. However, within days or weeks of the final exam, it’s gone, just like a video that has been returned to the store. One of the important differences between owners and renters is that those who own the knowledge tend to be *active readers*.

Think and then read on . . . Throughout the book, I will often pose a question and ask you to “think and then read on. . . .” Rather than just look to the next paragraph and see the “answer,” you will learn much more if you immediately cover up the next paragraph or close the book . . . stop . . . think . . . write down your thoughts . . . and then read on. The phrase “think and read on . . .” is there to remind you to read the book actively rather than passively. An **active reader** stops and thinks about the material just read and asks questions: Does this make sense? Have I had experiences like this? The active reader does the examples with pen or pencil, rather than just reading the author’s description. I recommend that you keep a journal (your instructor may give you specific instructions). Keeping a journal has many benefits. Many people find that they learn more by writing down their responses to “think and read on . . .” rather than just stopping for a moment to reflect. Many students find that reading over their journals every week or at the end of each chapter provides new insights into the mathematics and into their own beliefs and attitudes about mathematics.

Using Polya’s four steps On the inside front cover of the *Explorations* volume is an outline of Polya’s four steps for successful problem-solving. I encourage you to use that outline in all of the following ways:

1. Use it as a guide when you get stuck.
2. Don’t rent it, buy it. Buying it involves paraphrasing my language and adding new strategies that you and your classmates discover. For example, many of my students have added one whole step to help reduce anxiety: First take a deep breath and remind yourself to slow down.
3. Just as we did in Investigation 1.1, after you have successfully solved a problem, stop for a moment and reflect on the tools you used. As the course pro-

gresses, you should find that most problems involve the use of several strategies, and you should find that you use the tools more skillfully. For example, using “Make a diagram” skillfully involves deciding how precise your diagram needs to be, checking to see that the diagram illustrates the relevant given information, seeing whether the diagram can help you to paraphrase the problem or see it from a new perspective, and so on.



LEARNING

A colleague of mine was working through a word problem with her class one day and encouraging the students to think about what they were doing. Suddenly one of the students said, “But you don’t need to do all this stuff you are teaching us; you just know the answer.” She was stunned, and the ensuing discussion was informative. It turns out that many students believe that the difference between a student and a teacher is that the teacher just knows the answer or automatically knows how to get the answer. That is, teachers don’t need such strategies as guess-check-revise, make a table,

draw a diagram, look for patterns, etc. The truth is that we do! Virtually all of the most brilliant workers—whether they be engineers, scientists, businesspeople, carpenters, researchers, or entrepreneurs—approach complex problems by using the very tools that are being stressed in this text. Furthermore, even the top people in a field often make hypotheses that seem reasonable (to them and to their colleagues) but that turn out not to be true. So when you are working on these problems, please realize that the tools being discussed in this book are used by people in all kinds of situations.

Why Emphasize Problem-Solving?

Although Polya described his problem-solving strategies back in 1945, it was quite some time before they had a significant impact on the way mathematics was taught. One of the reasons is that the desired outcomes of mathematics instruction were defined too narrowly. If the stated goal of mathematics classes was to learn and practice the “right” techniques to answer textbook problems, chances are that students rarely saw how math applied to situations outside the classroom.

Another reason for the limited impact of Polya’s work is that until recently, “problems” were generally defined too narrowly. For example, many of you learned how to do different kinds of problems—mixture problems, distance problems, percent problems, age problems, coin problems—separately but never realized that they have many principles in common. To use language from the NCTM, there has been too great a focus on single-step problems and routine problems. Consider the examples from the National Assessment of Educational Progress shown in Table 1.3.

TABLE 1.3

Problem	Percent correct Grade 11
1. Here are the ages of six children: 13, 10, 8, 5, 3, 3 What is the average age of these children?	72
2. Edith has an average (mean) score of 80 on five tests. What score does she need on the next test to raise her average to 81?	24

To solve the first problem, one only has to remember the procedure for finding an average and then use it:

$$\frac{13 + 10 + 8 + 5 + 3 + 3}{6}$$

However, there is no simple formula for solving the second problem. Try to solve it on your own and then read on. . . . 

To solve this one, you have to have a better understanding of what an average means. One approach is to see that if her average for 5 tests is 80, then her total score for the 5 tests is 400. If her average for the 6 tests is to be 81, then her total score for the 6 tests must be 486 (that is, 81×6). Because she had a total of 400 points after 5 tests and she needs a total of 486 points after 6 tests, she needs to get an 86 on the sixth test to raise her overall average to 81.

Many students still consider the second question to be a “trick” question unless the teacher has explicitly taught them how to solve that kind of problem. However, many employers note that problems that occur in work situations are rarely *just* like the ones in the book. What employers desperately need is more people who can solve the “trick” problems, because, as one wag put it, “life is a trick problem!”

The difference between traditional word problems and many real-life problems Table 1.4 contrasts the difference between the word problems generally found in textbooks and real-life problems.

TABLE 1.4

Textbook word problems	Real-life problems
1. The problem is given.	1. Often, you have to figure out what the problem really is.
2. All the information you need to solve the problem is given.	2. You have to determine the information needed to solve the problem.
3. There is always enough information to solve the problem.	3. Sometimes you will find that there is not enough information to solve the problem.
4. There is no extraneous information.	4. Sometimes there is too much information and you have to decide what information you need and what you don't.
5. The answer is in the back of the book, or the teacher tells you if your answer is correct.	5. You, or your team, decides whether your answer is valid. Your job may depend on how well you can “check” your answer.
6. There is usually a right or best way to solve the problem.	6. There are usually many different ways to solve the problem.

When students undertake what some authors call more authentic problems, they come to realize that mathematics is more than just memorizing and using formulas, and they come to value their own thinking.

With respect to problem-solving, the NCTM has urged a turnabout. Traditionally, the teacher “taught” a new concept or skill, and *then* students did *problem-solving*. The NCTM has rejected this separation of teaching and problem-solving. That is, problem-solving involves more than just doing word problems to apply the concepts. In this book, we will investigate a wide variety of problems. By solving these problems, you will discover the meaning of the concepts and how to apply them and come to a much deeper understanding of and appreciation for the formulas or procedures you learn.

INVESTIGATION 1.2 How Much Will the Patio Cost?

Let's say you are building a patio in your back yard. You have decided to make the patio 12 feet by 8 feet. The local lumber store sells premade patio blocks, which are 18 inches by 12 inches, for 75¢ each. How much will the patio cost?

Solve this problem on your own, taking time to understand the problem and consider how you might plan to solve it. Then compare your solution and strategies to the ones discussed below. . . . 

DISCUSSION

STRATEGY 1: Make a diagram

Let us examine two kinds of diagrams that students often come up with. Because the blocks are $1\frac{1}{2}$ feet long and 1 foot wide, some students find a piece of graph paper and let each square represent $1/2$ foot, as shown in Figure 1.4. If you make one x on the grid for each patio block, you can determine the total number. What do you get? 

Other students simply sketch the problem on a blank piece of paper as shown in Figure 1.5.

In either case, we can see that we will need 64 blocks, and 64 blocks times 75¢ per block is 4800¢ or \$48.

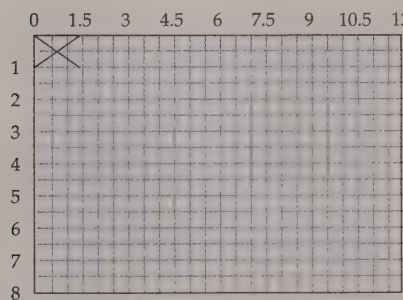


FIGURE 1.4

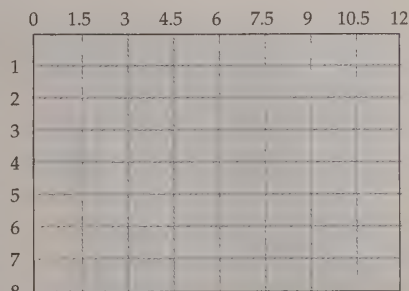


FIGURE 1.5

STRATEGY 2: Divide

There is another way to solve the problem that is quicker but also requires more thinking at the beginning of the problem. If we divide the total area of the patio by the area of one block, the quotient will tell us how many blocks we need.

The total area of the patio is $12 \text{ feet} \times 8 \text{ feet} = 96 \text{ square feet}$.

The area of each block is $1.5 \text{ feet} \times 1 \text{ foot} = 1.5 \text{ square feet}$.

When we divide 96 square feet by 1.5 square feet, we get 64 blocks.

Some readers may be kicking themselves for not thinking of this strategy. In Chapter 3, you will find that one of the meanings of division is repeated subtraction. From one perspective, this problem is asking how many times we can subtract 1.5 from 96.

STRATEGY 3: Use dimensional analysis

We can treat the “labels” as algebraic terms. Just as numbers can “cancel,” so too can labels. We will investigate canceling further in Chapter 5. Using *dimensional analysis*, we have

$$96 \text{ square feet} \div \frac{1.5 \text{ square feet}}{1 \text{ block}}$$

We can treat this as a division-of-fractions problem; thus, we need to invert and multiply.

$$= 96 \text{ square feet} \times \frac{1 \text{ block}}{1.5 \text{ square feet}}$$

$$= 64 \text{ blocks}$$

Because square feet “cancel,” the meaning of 64 is 64 blocks.

After Investigation 1.1, and also in the *Explorations*, you were asked to take some time to stop and reflect on what you learned from the activity. I have found that most of my best students intuitively do this after every problem that requires thinking on their part. I encourage you to do this on your own throughout the book. What problem-solving tools did you learn or refine in this investigation? Write them down before doing Investigation 1.3 in the next section. If possible, share your insights with another student. 

SECTION

1.3

REPRESENTATION

WHAT DO YOU THINK?

- Representation was not one of the standards in the 1989 NCTM Curriculum Standards. Why do you think it was made a separate standard in the 2000 Standards?
- Why do you think I chose to put the Representation standard (Standard 10) next to the Problem Solving standard (Standard 6) instead of at the end of the chapter?

▶ **Standard 10: Representation**
Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics so that all students—

- create and use representations to organize, record, and communicate mathematical ideas;
[When you can represent a situation or idea in multiple ways—tables, graphs, symbols, diagrams, words—your understanding of that situation or idea is much stronger than when you can only represent the situation/idea in only one way.]
- develop a repertoire of mathematical representations that can be used purposefully, flexibly, and appropriately;
[This relates to the toolbox metaphor and also to the notion of mindfulness as opposed to the mechanistic application of rote procedures.]
- use representations to model and interpret physical, social, and mathematical phenomena.
[We will talk more about modeling later.]

(*Principles and Standards*, p. 94)

At the simplest level, a **representation** is a picture—a picture of your image of a problem. Representations can take many forms: diagrams, graphs, tables, sketches, equations, words, etc. One of the key ideas about representations is that most problems and most mathematical concepts can be represented in different ways. For example, there were several ways to represent the pigs-and-chickens problem from Investigation 1.1. Students commonly ask me which representation is “best,” but this is just not a useful question. A more useful question is whether this representation fits the purposes before us. Representing the problem with 24 circles (representing 24 bodies) and then putting legs on the bodies is a very appropriate representation for this problem with children. This notion of multiple representations is also related to the standard on Connections. For example, some of my students *do* represent the problem with 24 circles. In this case, the next step is to *connect* this representation to other representations that are more useful with larger numbers, such as the table and the formula.

Just as many problems can be represented in multiple ways, this is also true for most important mathematical ideas. Let me use language to illustrate the idea of multiple representations. Think of the word *hot*. What does it mean? Write down the different meanings of *hot* before reading on. . . . 

When I looked up this word in the dictionary, I found 14 different meanings. Following are phrases that illustrate some of these meanings: a hot forehead, hot peppers, a hot temper, hot for travel, a hot topic, a hot suspect, hot on the trail, I’m not so hot at math, a hot sports car.⁹

Now consider the word *fraction*. Let me be more specific: In how many different ways can you represent $\frac{3}{4}$? Do this and then turn to page 48 in the textbook for a sneak preview of this important idea. 

Just as a literate person knows the multiple meanings of many words, a mathematically literate person knows multiple representations of important mathematical ideas.

I chose the next investigation because it illustrates two important aspects of representations.

INVESTIGATION 1.3 Making Paper Dinosaurs

 A first-grade teacher is going to have her students make a number of paper dinosaurs for a counting project. She needs one hundred 2 inch by 3 inch rectangles, on which the students will trace and then cut out dinosaurs. How many sheets of paper will she need for this project? Think and then read on. . . . 

DISCUSSION

The first step is to stop and make sure you understand the problem. Did you realize that we do not have enough information to solve the problem? What else is needed? 

We need to know the dimensions of the paper. Let’s say she uses standard paper; that is, each sheet is $8\frac{1}{2}$ inches by 11 inches.

⁹ *American Heritage Dictionary of the English Language* (Boston, Houghton Mifflin, 1992), p. 874.

STRATEGY 1: Divide

This problem may feel similar to the one in Investigation 1.2. Let's see if we can use the approach we learned in that problem. The total area of one sheet of paper is 8.5 inches $\times$ 11 inches, or 93.5 square inches. If we divide the total area by the area of one rectangle (6 square inches), we get 15.58 rectangles. According to this strategy, we can get 15.58 rectangles from one sheet of paper. That said, how many sheets will the teacher need in order to make 100 rectangles? Figure it out yourself and read on. . . . 

Many students ponder for a while here. Do you divide 100 by 15.58 or by 15? What do you think?

$$100 \div 15.58 = 6.4$$

$$100 \div 15 = 6.7$$

Does this mean that the teacher needs 6 sheets or 7 sheets? If you are wondering about this yourself, one option is to continue to reason it out until it makes more sense. Another option is to try a different strategy. This also can serve as a way of checking your answer. If you get the same answer with a different strategy, the odds are that you are on the right track.

STRATEGY 2: Make a diagram

When we begin to make a diagram, we quickly realize that there are two different ways to lay out the 2 inch by 3 inch rectangles. These are illustrated in the diagrams in Figure 1.6. (Make sure the diagrams make sense to you before reading on.) 

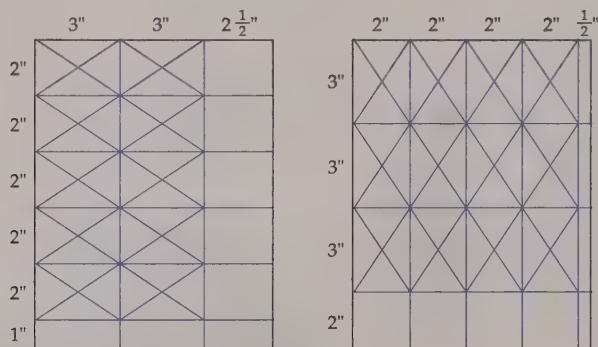


FIGURE 1.6

Cutting out the rectangles according to the diagram on the left produces ten 2 inch by 3 inch rectangles; cutting out the rectangles according to the diagram on the right produces twelve 2 inch by 3 inch rectangles.

Actually, we can get more 2 inch $\times$ 3 inch cutouts. Do you see how? If not, experiment for a while and then read on. . . . 

I must confess that the first draft of this chapter did not contain the layouts shown in Figure 1.7, but only those in Figure 1.6. A colleague of mine caught my error. Even professors sometimes get stuck in ruts! This also happens in the business world and is one of the reasons why people work together on teams and have "outside" reviews of their solutions.

We now have a discrepancy between the two methods. When we used division, we got an answer of 15 rectangles. Using the "make a diagram" strategy,

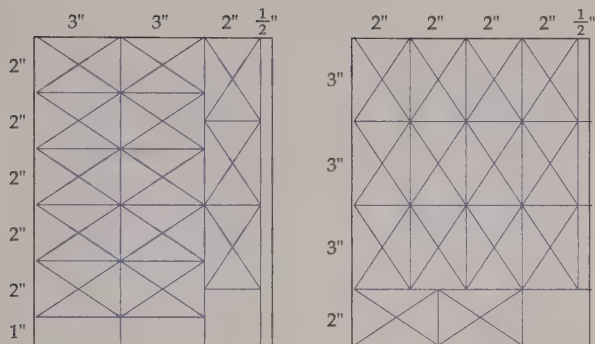


FIGURE 1.7

we can get at most 14 rectangles. If this were a homework assignment or a quiz, which answer would you turn in? Think and read on. . . .

In this case, the fifteenth rectangle would have to be pieced together from scraps, and so it is more realistic to use 14 rectangles per sheet. If you can get 14 rectangles per page, and you want 100 rectangles, how many sheets will you need? Think before reading on. . . .

If you divide 100 by 14, you get approximately 7.1, and so you will need a bit more than 7 sheets; that is, you will need 8 sheets.



LEARNING

"I know it but I can't explain it." Most of the time this statement indicates that the student's understanding is partial; that is, the student understands *what* but not *why*. One of the major goals of this course is to help future teachers across this bridge, from simply knowing what (to do) to also knowing why and being able to explain why.

Why didn't the first strategy work here? It is important to note that the "division" strategy was very appropriate for Investigation 1.2 but was inappropriate here. This highlights a belief that many students have about how to go about learning mathematics. Let me explain. If I need to find a file in my computer, I can press the "Find" button and give the name of the file; the computer will search through its memory until it finds that file. Many students use a similar approach when solving problems: Instead of trying to understand the problem, they search their memory for a similar problem. If they find one, they solve the present problem in just the same way. Too often the students do not ask themselves whether that strategy is applicable to the new problem. Can you explain *why* the division strategy worked on the patio problem and not on this one? Think and then read on. . . .

In the patio problem, the dimensions of the brick divided the total length and width with no remainder. That is, $12 \div 1.5 = 8$ and $8 \div 1 = 8$. In this investigation, we do have a remainder:

$$11 \div 3 = 3\frac{2}{3} \quad \text{and} \quad 8\frac{1}{2} \div 2 = 4\frac{1}{4}$$

Thus, the division strategy works only in some cases. This is true beyond the classroom, too. In businesses and laboratories, workers frequently have to decide whether solutions to old problems are applicable to new ones. In some cases, the solution is directly applicable. In other cases, it is partially applicable. In yet other cases, it is not applicable at all, because there are fundamental differences between the two situations.

Refining the problem In many everyday and work situations, we would still not be finished with the problem. Although we have solved the question that was asked, a teacher who is frugal or environmentally sensitive would realize

that she can get more rectangles if they don't have to be exactly 3 inches by exactly 2 inches. If the teacher makes the rectangles "a little" smaller, she should be able to get more than 14 per sheet. What would you recommend? Think and then read on. . . . 

If you make the length of each rectangle in the left-hand diagram of Figure 1.6 or $1.7\frac{5}{6}$ inches, you will be able to make 15 rectangles whose dimensions are $2\frac{5}{6}$ inches by 2 inches. If you make the length of each rectangle in the right-hand diagram $2\frac{3}{4}$ inches, you will be able to make 16 rectangles whose dimensions are $2\frac{3}{4}$ inches by 2 inches.

Reflecting on the problem Take a minute to reflect on this problem. What did you learn about problem-solving? What tools did we use on this problem? You may want to refer to the "4 Steps for Solving Problems" on the inside front cover of the *Explorations* volume.

Two very important points about problem-solving come out in this investigation. The first is that you need to take care when adapting a strategy from one situation to another. Second, a different kind of thinking happens when we bring real life into problem-solving situations. The computations must be interpreted; we are not finished when we have a number.

There are two aspects of "representations" that are important to discuss in this problem. From the perspective of "representation as picture," you saw that different ways of representing the 2 inch by 3 inch rectangles on the paper yielded different answers. From the perspective of "representation as a model of the problem," representing the problem as dividing the total area of the construction paper by the area of each rectangle did not work. Having to check to see that your representation of the problem is valid is crucial, but it is something many students forget to do. Whenever scientists and businesspeople make a model of a problem, they frequently ask themselves whether the model validly represents the essence of the problem.

Let me summarize some of the important mathematical and pedagogical aspects of representations. Most mathematical concepts and most problems can be represented in multiple ways. Depending on the situation, some representations are more appropriate or more useful than others. From the perspective of learning styles, different students are likely to find different representations more meaningful. At the same time, students need to appreciate that most representations have various strengths and weaknesses. Thus, students need to be able to create accurate and complete representations of important mathematical ideas, and students need to investigate why some representations are more useful than others in certain situations. We will uncover other important aspects of representations at appropriate times in this book.

SECTION

1.4

PATTERNS

WHAT DO YOU THINK?

- What do patterns have to do with learning mathematics?
- How does the ability to recognize and analyze patterns add to your problem-solving toolbox?

In the 1989 *Curriculum and Evaluation Standards*, Patterns and Relationships was one of the 13 standards for elementary school mathematics. In *Principles and Standards for School Mathematics*, patterns appear in Standard 2: Patterns, Functions, and Algebra. Although I understand the reasons for including patterns with functions and algebra (you will see this connection in Section 2.2), I still think it is important to discuss patterns on its own in an introductory chapter.



BEYOND THE CLASSROOM

In one sense, all patterns can be represented mathematically. This is not as radical a statement as you might think. Look at your television set. Most television sets still consist of 512 lines. The image from the transmitting station is coded by establishing a number (symbol) for each point on each of the 512 lines. Your television set then decodes that message and quickly puts the appropriate color in the appropriate spot, creating a picture.



CHILDREN

A common exploration in preschool is to give the children a bunch of buttons and ask them what they see. The children will observe different colors, different shapes, different numbers of holes, different patterns with buttons, etc. Using this natural tendency to see in multiple ways and to look for patterns, we can help children develop the ability to form generalizations.



LEARNING

“As an important step in their initial work with patterns, students often describe the underlying regularity verbally rather than with mathematical symbols. One goal of school mathematics instruction is to build on this verbalization and provide students with sufficient experience to become comfortable with, and fluent in, using mathematical symbols to represent generalizations.” (Principles and Standards, p. 57)

Patterns exist virtually everywhere in the world. Look at the floor, the walls, and the ceiling of most rooms, and you are sure to see patterns. Look at a wallpaper display in a store and see patterns. Virtually all clothing contains patterns. Look at an article of clothing that seems to have no patterns—a plain white blouse, a pair of pants. But now look at it through a magnifying glass; you will see patterns in the way the cloth was woven. Look at Islamic architecture and you will see more patterns than in probably any other kind of architecture. Mohammed, the founder of Islam, decreed that there be no statues or paintings of people in that religion. There are also patterns in weather that help us to make predictions.

Patterns are also an important part of how children organize the world: Nap comes after lunch; Monday we have art, Tuesday we have music; all dogs have tails. One of the goals of school mathematics is to build on children’s curiosity to help them see patterns that underlie mathematical structure. For example, when we count, there is a repeating pattern in the ones place (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, . . .); when we count by 5, the ones place alternates between 5 and 0; take any multiple of 9 and the sum of its digits will also be a multiple of nine (try 5986×9). In 1978, Mary Baratta-Lorton published a landmark book for elementary teachers called *Mathematics Their Way* (Menlo Park, CA: Addison-Wesley, 1976). She saw patterns as one of the unifying themes in elementary school mathematics. She said that “[L]ooking for patterns trains the mind to search out and discover the similarities that bind seemingly unrelated information together in a whole. . . . A child who *expects* things to “make sense” looks for the *sense* in things and from this sense develops understanding. A child who does not see patterns often does not expect things to make sense and sees all events as discrete, separate, and unrelated” (italics added).

Patterns are a tool to help you learn mathematics, and mathematics often reveals seemingly hidden patterns that help us understand the world about us. Mathematics has been described by several writers as the science of pattern and order. Many important mathematical ideas have arisen from the recognition of patterns. To use a metaphor, patterns are not the pot of gold at the end of the rainbow; rather, patterns are the rainbow that leads to the pot of gold, which is the mathematical structure.

Now that we have discussed the prevalence of patterns in the natural world and the human-made world and the importance of patterns in the mathematics classroom, let me ask you to try to define the term *pattern*. That is, think of patterns and then consider what all these patterns have in common. What words, ideas, and images come to mind? Write down your thoughts and then read on. . . . 

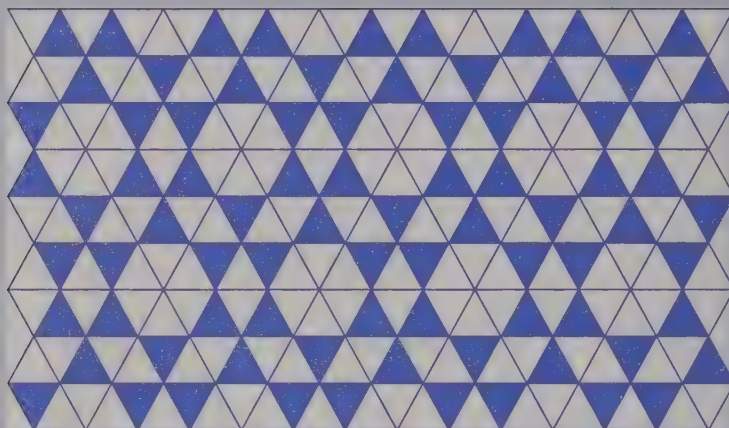
One aspect common to all patterns, whether geometric or numerical, is that there is repetition. When we examine this repetition, we find an underlying structure or organization. Understanding this structure enables us to make generalizations, which in turn enables the users of this knowledge to do more powerful mathematics, to design more powerful machines and equipment, or to make manufactured items more cheaply.

Let us examine a geometric and a numerical example of patterns.

INVESTIGATION 1.4 Recognizing the Mathematical Structure of Patterns

- A. Look at the geometric pattern following. How would you describe what it is that repeats?

Then read on. . . 



B. Now look at the numerical pattern below. How would you describe what it is that repeats? Then read on. . . 

2, 5, 8, 11, 14, . . .

DISCUSSION

A. There are many ways to look at the geometric pattern, which is replicated from tiles from a Roman palace. From one perspective, the entire pattern consists of equilateral triangles that are either white or black. However, the pattern in which they are shaded creates a number of six-pointed stars with hexagons in their centers. You can see other shapes in the figure too: trapezoids, rhombuses, and other shapes for which there are no names—for example, the shape in Figure 1.8.



FIGURE 1.8

B. The repetition in the number pattern is 3. That is, the difference between each term in this arithmetic sequence is 3. In the next section, we will investigate this pattern further.

Recognizing patterns Being able to use patterns makes the guess–check–revise strategy much more powerful, as you saw in Investigation 1.1 (Pigs and Chickens). However, as many of you discovered, this is easier said than done. In order to use a pattern, you have to *recognize* it. As we found, making a table increases the likelihood of our recognizing a pattern. Recall also that we did not have just two columns in the pigs-and-chickens problem. *Extending* the table, we created a new column called “Difference” that helped to make the pattern more visible. We then *analyzed* the pattern to see that each time we changed a chicken into a pig, we added two feet. This enabled us to get the solution on the third guess.

Let us now investigate some problems that illustrate the pervasiveness and importance of patterns in mathematics.

INVESTIGATION 1.5 The Sum of the First 100 Numbers

Legend has it that when Carl Gauss (1777–1855), one of the greatest mathematicians of all time, was in first grade, his teacher gave the class the following problem: Find the sum of the first 100 numbers. The teacher then sat down to grade papers, knowing that it would take most of the students a good half-hour to do this problem. Gauss came to the teacher's desk in 5 minutes with the answer. He had found a “large” pattern. Work on this problem and then read on. . . . 

DISCUSSION

The goal of this investigation isn't to solve the problem as quickly as Gauss did but to understand why patterns are such a remarkable problem-solving tool and to realize that there are many different patterns, depending on how you approach the problem; it is much like seeing an interesting sculpture from different perspectives.

STRATEGY 1: Look for patterns that might make the problem smaller

Suppose we find the sum of the first 10 numbers:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

What is the sum of the second 10 numbers?

$$11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 = 155$$

What do you think will be the sum of the next 10 numbers? 

If you continue this pattern, you will get the answer relatively quickly.

TABLE 1.5

The sequence	Sum
1 + 2	3
1 + 2 + 3	6
1 + 2 + 3 + 4	10
1 + 2 + 3 + 4 + 5	15

STRATEGY 2: Start with a simpler problem and look for patterns

What if we look at the sums of $1 + 2$, $1 + 2 + 3$, $1 + 2 + 3 + 4$, etc., as shown in Table 1.5, and see what patterns emerge?

Do you see any patterns here? Think and then read on. . . . 

Actually, there are many patterns, and a crucial point about patterns and problem-solving is that not all patterns lead directly to a solution. An analogy to paths on a mountain is relevant: All paths lead somewhere; however, some paths are more helpful than others. Let us look at some of the patterns and then analyze one that helps us find the answer to the question.

Looking at differences When we compare the sums of these sequences, we see that the difference between successive sums increases by 1 each time. For example, the difference between 6 and 3 is 3, the difference between 10 and 6 is 4, the difference between 15 and 10 is 5, and so on. Although this is a pattern, it's not very useful at this point because we can't use it to determine the sum of the first 100 numbers without knowing the sums of the first 98 and 99 numbers.

Comparing the number of terms and the sums Another pattern emerges when, instead of looking at the relationship between numbers in the second column, we look at the problem from a different perspective. For example,

what if we compare the number of terms in each sequence to the sum of the sequence (see Table 1.6). What do you see now?

TABLE 1.6

The sequence	Number of terms in the sequence	Sum
$1 + 2$	2	3
$1 + 2 + 3$	3	6
$1 + 2 + 3 + 4$	4	10
$1 + 2 + 3 + 4 + 5$	5	15

Some students see that when there are 3 terms, the sum is 6, and 6 is $3 \cdot 2$; and when there are 5 terms, the sum is 15, and 15 is $5 \cdot 3$. One guess (or hypothesis) that can be made from this pattern is that when there are 7 terms in the sequence, the sum will be $7 \cdot 4$. Indeed, the sum of the first 7 numbers is 28! But how can we use this pattern to answer the question? Play with this for a bit before reading on. . . . 

There are actually a couple of ways to use this pattern. One way is to make a new table and look at what is going on; Table 1.7 illustrates this.

TABLE 1.7

The sequence	Number of terms in the sequence	Sum	What we multiply the second column by to get the sum
$1 + 2$	2	3	$1\frac{1}{2}$
$1 + 2 + 3$	3	6	2
$1 + 2 + 3 + 4$	4	10	$2\frac{1}{2}$
$1 + 2 + 3 + 4 + 5$	5	15	3
$1 + 2 + 3 + 4 + 5 + 6$	6	21	$3\frac{1}{2}$
$1 + 2 + 3 + 4 + 5 + 6 + 7$	7	28	4

LEARNING

When looking at Table 1.7, some students ask, "How was I supposed to know to do that?" Researchers who looked at differences between "expert" problem-solvers and "novice" problem-solvers found that the experts were more likely to play around with a problem for a while if they didn't feel they had a good problem-solving strategy. There are many instances in which mathematical discoveries and solutions to problems have come from playing around with a problem. This idea of playfulness in mathematics is bizarre to many students. However, I seriously encourage you to think of it as a problem-solving tool. Furthermore, this adds a dash of creativity to problem-solving, which, in turn, makes it more enjoyable for many students.

Some people can leap from this table to the first 100 numbers and say that we multiply the number of terms in the sequence, 100, by $50\frac{1}{2}$ because when the number of terms is an even number, what you multiply by is simply half of that number plus $1/2$.

Some people don't like fractions and find that if they double the numbers in the second column, they get a table that is easier to work with (see Table 1.8). Some students can make the leap from looking at this table to realizing that the sum of the first 100 numbers will simply be $1/2$ of $100 \cdot 101$. Do you see why this is so? 

At the most general level, we can now say that the sum of the first n numbers is $\frac{1}{2}n(n + 1)$.

STRATEGY 3: Look for patterns in the whole string of numbers

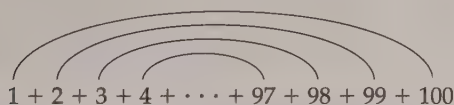
What happens if you add the first and last numbers? What does this lead to? Think and read on. . . . 

TABLE 1.8

The sequence	Number of terms in the sequence	Double the sum	The third column decomposed or broken down
$1 + 2$	2	6	$2 \cdot 3$
$1 + 2 + 3$	3	12	$3 \cdot 4$
$1 + 2 + 3 + 4$	4	20	$4 \cdot 5$
$1 + 2 + 3 + 4 + 5$	5	30	$5 \cdot 6$
$1 + 2 + 3 + 4 + 5 + 6$	6	42	$6 \cdot 7$
$1 + 2 + 3 + 4 + 5 + 6 + 7$	7	56	$7 \cdot 8$

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots + 95 + 96 + 97 + 98 + 99 + 100$$

If we add $1 + 100$ and then $2 + 99$ and then $3 + 98$ and so on, we always get 101. The question now is how many pairs adding up to 101 we have. Once we know that, we can multiply the number of pairs by 101. This is the large pattern that Gauss discovered! The diagram below, in which we connect the pairs of numbers, can help you to understand this strategy better.



STRATEGY 4: Use a geometric representation

Some students remember using manipulatives in elementary school, and manipulatives provide concrete models for solving problems. In this case, manipulatives in the form of square tiles can actually change the perspective from which one views the problem. Figure 1.9 shows the sum of the first five numbers represented with tiles. In this case, the student places the tiles on top of each other. If you see this as “half a rectangle,” then you can see the connection between this figure and the formula $\frac{1}{2}n(n + 1)$. If you don’t see the connection, copy this figure twice (either on paper or with tiles). Put the two copies together to make a rectangle. Because the area of a rectangle is found by multiplying the length times the width, and because each of these “triangles” has half the area of the rectangle, this is another way to solve the problem.

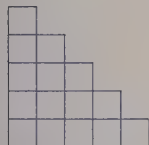


FIGURE 1.9



LEARNING

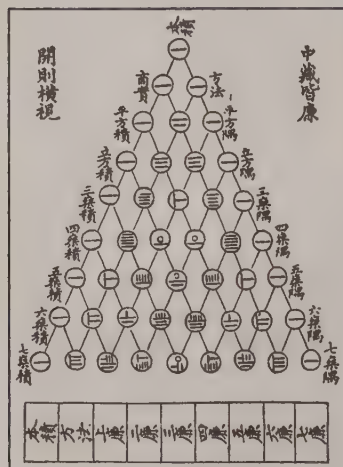
A point that I feel cannot be overemphasized is that for many problems from everyday life and business life, there is more than one way to solve the problem, just as there is usually more than one way to get from here to there—whether we are talking about paths up a mountain or routes from New York to Los Angeles. Many students have told me that before they took this course, they thought

there was just *one right way* to do a problem, and so they never looked for patterns but instead looked for formulas or procedures. Once the students started looking for patterns, they found them everywhere, and over time they learned how to use their awareness of patterns more powerfully. We can refer to different ways to do a problem as different **solution paths**

INVESTIGATION 1.6 Pascal's Triangle

HISTORY

As you may have already discovered in other areas, a discovery is not necessarily named after the first person who discovered it. This is true of Pascal's triangle. Figure 1.10 comes from *The Precious Mirror of the Four Elements*, which was written in 1303 by Chu-Shih-chieh, and he himself acknowledged that he had taken the triangle from a book written several hundred years earlier!





MATHEMATICS

When I ask students at the end of my courses what they have learned about mathematics, many of them say that they have come to realize that mathematics is much more connected than they had thought. They saw more connections both among concepts (such as fractions, decimals, and percents) and among problems that they had solved. Do you see connections between Pascal's triangle and other problems in this chapter? Think and read on. . . . 

If you didn't, look back at Investigation 1.5 and Exploration 1.1. Those sequences can be found within Pascal's triangle. Can you see them?

In other words, the sum of the numbers in the n th row can be obtained by raising 2 to the $(n - 1)$ power. If you have a calculator with an x^y function, you can determine 2^9 by doing the following: $2 \ x^y \ 9 =$; 512 will appear.

The Power of Patterns

Some of the greatest mathematicians in history made their mark by discovering and articulating patterns. The wonderful thing about patterns, though, is that they make mathematics more accessible to everyone. This was recognized by the authors of *Everybody Counts*,¹⁰ who wrote,

Virtually all young children like mathematics. They do mathematics naturally, discovering patterns and making conjectures based on observation. Natural curiosity is a powerful teacher, especially for mathematics. Unfortunately, as children become socialized by school and society, they begin to view mathematics as a rigid system of externally dictated rules governed by standards of accuracy, speed, and memory. Their view of mathematics shifts gradually from enthusiasm to apprehension, from confidence to fear. Eventually, most students leave mathematics under duress, convinced that only geniuses can learn it.

We explored several famous, powerful patterns in this section. You will see many of these patterns again, both in this book and beyond it. But more important, I hope you will see that stepping back from a problem can reveal patterns within it. Sometimes these patterns can help you devise a plan for solving the problem; at other times they emerge as you look back on your work.

Summary

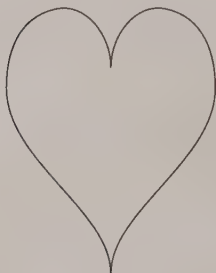
In the first half of Chapter 1, I have asked you to examine your beliefs and attitudes toward mathematics, I have introduced the NCTM Curriculum Standards, and we have investigated the standards on problem solving, representation, and patterns. Before moving on, many instructors find it helpful for students to do some thinking and some problems that involve these themes. In the second half of Chapter 1, we will explore the remaining process standards.

¹⁰ National Research Council, *Everybody Counts: A Report to the Nation on the Future of Mathematics Education* (Washington, DC: National Academy Press, 1989), pp. 53–54.

EXERCISES

1. A farmer looks out into the barnyard and sees the pigs and the chickens. He says to his daughter, "I count 169 heads and 398 feet. How many pigs and how many chickens are out there?"
2. A Martian farmer looks out into the barnyard and sees tribbles (which have four legs) and chalkas (which have seven legs). She says to her son, "I count 97 heads and 436 feet. How many tribbles and how many chalkas are out there?"
3. At a benefit concert, 600 tickets were sold and \$1500 was raised. If there were \$2 and \$5 tickets, how many of each were sold?
4. For a certain event, 812 tickets were sold, for a total of \$1912. If students paid \$2 per ticket and nonstudents paid \$3 per ticket, how many student tickets were sold?
5. At a bicycle store, there were a bunch of bicycles and tricycles. If there were 32 seats and 72 wheels, how many bicycles and how many tricycles were there?
6. If you have a bunch of 10¢ and 5¢ stamps, and you know that there are 20 stamps and their total value is \$1.50, how many of each do you have?
7. Make up your own problem like the ones in Exercises 1–6.
8. Let's say you are building a patio in your back yard. You have decided to make the patio 12 feet by 9 feet. The local building supply store sells bricks that are 6 inches by 4 inches for 25¢ each. How much will the patio cost?

9. Let's say you have decided to make a number of hearts for an activity in your class. Ideally, the hearts would be the size shown. How many sheets of construction paper would you need in order to make 96 hearts that are exactly this size? If it were your class, would it be worth your while to make them slightly larger or slightly smaller? Explain why or why not.



10. At the beginning of the month, Jack and Jill had \$642 in a checking account. During the month, they wrote checks for \$22, \$53, and \$55. They made a withdrawal of \$50 from the automatic teller machine, and they made deposits of \$142 and \$100. How much money did they have at the end of the month? Solve this problem two different ways.
11. A soft drink manufacturer produces 3240 cans in an 8 hour day. Cans are packed 24 to a case. How many cases are produced between Monday and Friday?
12. Sarah's diet allows 1500 calories per day. Thus far, she has had a glass of milk (90 calories), pancakes (150 calories), an apple (75 calories), a salad (150 calories), salad dressing (200 calories), and a piece of cake (350 calories). Can she have a steak (250 calories) and a salad with salad dressing for dinner? If not, what can she have? Solve the problem two different ways.
13. A sewing machine representative earns \$125 a week plus \$8 for each deluxe model she sells and \$5 for each economy model. How many economy models did she sell in a week in which she made \$184 and sold 3 deluxe models?
14. Sally works 40 hours a week and makes \$6.85 an hour, but her kids are in child care for 32 hours a week and the day care center charges her \$15 per day. If you deduct her child care expenses, how many dollars per hour does she actually make?
15. Martha planted 14 rows of apple trees, and each row had 21 trees. If each tree yielded an average of 250 apples, how many apples grew in her orchard?
16. A farmer needs to fence a rectangular piece of land. She wants the length of the field to be 80 feet longer than the width. If she has 1080 feet of fencing material, what should be the length and the width of the field?
17. Bill is being paid \$12 to rake the yard. He raked $\frac{2}{3}$ of the yard, and then Jim came by and helped him finish. How much should Bill pay Jim?
18. Joni is the owner of Red Oak Furniture, which is doing so well that the present staff is overworked. She has two choices: continue to pay overtime or hire a new person. She now has three employees, and they earn time-and-a-half for all time over 40 hours worked in a week. In the past three weeks, they have worked the number of hours shown. Should she hire a new employee?

	Week 1	Week 2	Week 3
Gerald	47	55	42
Nancy	43	51	48
Jose	56	51	42

19. To prepare the second edition of this textbook, I had to trim each of the almost 1000 pages (of the text and the accompanying *Explorations*) and then tape each page to a blank sheet of $8\frac{1}{2}$ inch by 11 inch paper. The reason for this (and believe me, I asked) was that the manuscript goes through many hands and many copy machines, and the pages of the book are much thinner than 20 pound paper (used in copying).
- How many inches of tape did I use?
 - How many rolls of tape did I need to buy for this project?
20. McDonald's has sold "over 100 billion hamburgers." If we were to stack these hamburgers, how high would the stack be?
21. You are going to visit your grandmother in Buffalo. You had told her that you would be there for dinner, but you are behind schedule. It is now 4:35 p.m. and you are 210 miles from Buffalo. You pull off the freeway and call her to tell her you will be late. What time do you tell her to expect you?
22. Find the two-digit number that, when added to its reverse, is the closest to 130. (Note: The reverse of 79, for example, is 97.) Show your work. If you got it on the first try, briefly explain your thinking. Otherwise, your successive guesses will suffice.
23. Two 2-digit numbers satisfy the following conditions:
- The sum of the digits in each number is 10.
 - All four digits are different.
 - The sum of the numbers is 155.
- Determine the two numbers.
24. Lisa was born on July 4, 1976. How many days old was she on January 1, 2000?
25. On which day of the year does the millionth second occur?
26. A special rubber ball is dropped from the top of a wall that is 16 meters high. Each time the ball bounces, it rises half as high as the distance it fell. The ball is caught when it bounces 1 meter high. How many times did the ball bounce?

27. a. In how many different ways can you make change for a quarter?
 b. In how many different ways can you make change for 50 cents?
 c. In how many different ways can you make change for one dollar without using pennies?
28. List the next term in each of the sequences below and verbally describe the pattern that enabled you to predict the next term.
- $\square, \triangle, \square, \square, \triangle$
 - $\Gamma, \perp, \perp$
 - 1, 4, 7, 10,
 - 2, 6, 18, 54,
 - 1, 4, 9, 16,
 - 1, 2, 4, 7, 11,
 - 5, 1, 4, 2, 3, 3,
29. Continue the pattern in the three number sentences below, and then describe as many patterns as you can.
- $$3 \times 3 - 1 = 8$$
- $$5 \times 5 - 1 = 24$$
- $$7 \times 7 - 1 = 48$$
30. Determine the following products: 67×67 , 667×667 .
- Describe the pattern you see.
 - Predict the product of 6667×6667 .
 - This is an example of "what if" and "I wonder." What if the two numbers didn't have the same number of 6s? For example, is there any connection between what we have seen above and, for example, 667×67 ? Explore this question and report your findings.
31. Determine the following products: 1×1089 , 2×1089 , 3×1089 .
- Describe the pattern(s) you see.
 - Predict the next several products.
 - Along the "what if" theme, will this pattern continue when you reach 10×1089 and further?
32. Determine the following products: 1×142857 , 2×142857 , 3×142857 .
- Describe the pattern(s) you see.
 - Do more computations until you can predict the next product.
 - Along the "what if" theme, this pattern seems to break at 7×142857 . Continue to find and write products of 142857 and 8, 9, 10, etc. What patterns do you see now?
33. Palindromes are numbers whose value is the same backwards as forwards. For example, 1331 is a palindrome.

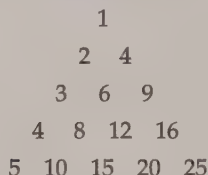
Most children love explorations with palindromes because they find the patterns fun and the investigations are so mathematically rich. How many palindromes can you find between 100 and 999?

34. Making palindromes from nonpalindromes is another interesting exploration. For example, take any number, reverse the digits, and add the two numbers together. Then determine how many steps it takes until a palindrome is reached. For example, $38 + 83 = 121$. Thus, we can say that 38 is a 1-step palindrome. On the other hand, 87 is a 4-step palindrome. Explore different 2-digit numbers. Can you describe those numbers that will be 2-step palindromes? 3-step palindromes? etc.?
35. One of the most famous patterns in mathematics was discovered by the Italian mathematician Leonardo of Pisa (1170–1250), known to us as Fibonacci. He discovered this sequence, which we today call the Fibonacci sequence, while studying the birth rates of rabbits. He posed the following question: Suppose that a pair of rabbits produce a pair of baby rabbits every month, and that rabbits cannot reproduce until they are two months of age. How many pairs of rabbits will you have after one year (assuming, of course, that no rabbits die)? He found that if he listed the number of pairs of rabbits he had after each month, he had a very interesting sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...
- Add the first seven Fibonacci numbers. What do you notice?
 Add the first eight Fibonacci numbers. What do you notice?
 Try to describe this relationship both in words and using notation.
 - Pick three consecutive terms. Multiply the first term and the third term. How does this compare to the square of the middle term? Describe this relationship using notation.
36. There is a connection between the Fibonacci sequence and Pascal's triangle. Can you find it? Let me give you a hint: The Fibonacci sequence can be found in Pascal's triangle.

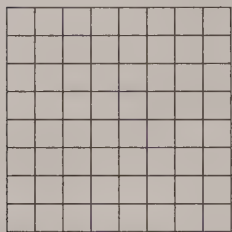
							1					
							1		1			
						1	2		1			
					1	3	3		1			
				1	4	6	4		1			
			1	5	10	10	5		1			
		1	6	15	20	15	6		1			
	1	7	21	35	35	21	7		1			
1	8	28	56	70	56	28	8		1			

37. a. Describe as many patterns as you can find in the triangle below.

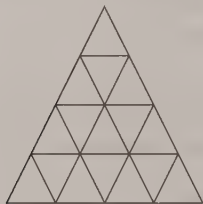
- b. Come up with a name for this triangle that connects to what the triangle is about. Briefly explain the reasoning behind your choice for the name.



38. Examine the following phrase made famous by Fred Flintstone: y a b b a d a b b a d o o. If this pattern is repeated over and over, what letter will be in the 275th position?
39. Using four 4s and any operations, make the numbers 1–10. For example, $4 + 4 + 4 - 4 = 8$
40. How many 3×2 rectangles are contained within a 8×8 square?



41. How many triangles can you find in the figure below?



42. a. On a digital clock, how many times in one day will the numbers be consecutive digits, as they are, for example, in 1:23?
- b. On a digital clock, which digit will appear most often in one day?
43. a. How many different ways can you insert four Xs into a 4 by 4 array so that no row, no column, and no diagonal has more than one X? One solution is given below.

	X		
			X
X			
		X	

- b. Find a solution for a 5 by 5 array using 5 Xs.
44. a. Using each of the numbers 1–9 exactly once, fill in the blanks below:

$$\begin{array}{r}
 \square \square \square \\
 + \square \square \square \\
 \hline
 \square \square \square
 \end{array}$$

- b. Find all solutions.

SECTION

1.5

REASONING AND PROOF

WHAT DO YOU THINK?

- Do you think young children can do proofs?
- How can the classroom nurture intuitive reasoning?

The development of powerful methods of reasoning has taken many thousands of years and is closely linked to the development of science. A primary purpose of this section is to develop reasoning tools to help us understand mathematics. From another perspective, the development of the ability to make logical arguments is part of a child's natural development. In this section, we will examine two classical kinds of reasoning used in developing and teaching mathematics: inductive and deductive reasoning. We will also discuss intuitive reasoning. Figure 1.11 humorously illustrates the need for reasoning.

I have seen countless similar examples in classrooms. Let us investigate one such example.

CROCK RECHIN & WILDER

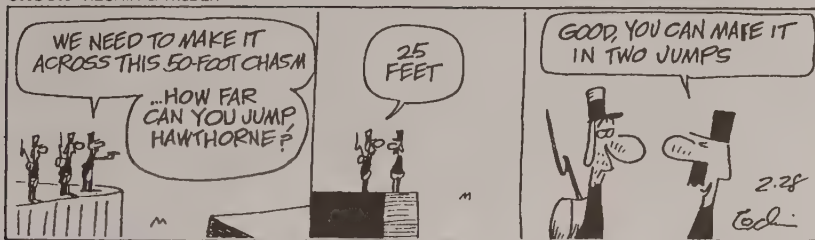


FIGURE 1.11*

INVESTIGATION 1.7 The 100-Meter Women's Breaststroke

Once I asked my students to take the world-record time for the 100-meter women's breaststroke and determine how fast this was in miles per hour. Without even dealing with the numbers, what does your common sense tell you? That is, what is a realistic range for how fast that person was swimming? Visualize yourself walking/jogging/running along the side of the pool to match the speed of the swimmer. Write down this range—that is, your guesstimate for the answer—and then read on. . . .

DISCUSSION

The answer was that the person was averaging 4.07 miles per hour. However, several students got the decimal point wrong, for various reasons, and I have received answers of .407 miles per hour and 40.7 miles per hour. It is not that the students were stupid; they just weren't using their common sense. I often tell my students that reasoning is super-charged common sense.

With these examples in mind, read what the NCTM has to say about reasoning. As with the other standards, check to see the extent to which these words make sense and the extent to which you buy them, and then read on. . . .

Standard 7: Reasoning and Proof

Mathematics instructional programs should focus on learning to reason and construct proofs as part of understanding mathematics so that all students—

- recognize reasoning and proof as essential and powerful parts of mathematics;
[This is crucial. You will see how children can and do “prove” conjectures, and you will come to understand the role of proof in the elementary classroom.]
- make and investigate mathematical conjectures;
[This is something children do naturally about the world. Much of this develops from looking for and making sense of patterns.]

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- develop and evaluate mathematical arguments and proofs;
[Formal proofs don't happen until high school, but children can construct informal proofs.]
- select and use various types of reasoning and methods of proof as appropriate.
[We will examine several examples in this section.]

(*Principles and Standards*, p. 80)

At the heart of this standard of Reasoning and Proof must be the belief that all mathematical ideas make sense. I realize that many students come to college believing that making sense of mathematics is reserved only for the “smart” students; for the rest of the students, the best advice is “just do it.” To me that is one of the maladaptive beliefs about mathematics. Here is an example I have used in workshops all over the country with elementary teachers.

INVESTIGATION 1.8 Circles and Mathematical Relationships

Probably all of you know the value of pi (π) to two decimal places—3.14. Let me ask you what pi means. Think about how you would answer that question to a grade school student who says she or he learned that $\pi = 3.14$ and learned some formulas but wants to know what pi means. How would you answer that child? Think before reading on. . . .



DISCUSSION

Figure 1.12 shows a perfect circle. Now imagine a piece of sticky string that is exactly the same length as the diameter. If you were to use that string like a ruler to measure the distance around the circle, how many lengths of that string would it take to go around the circle?

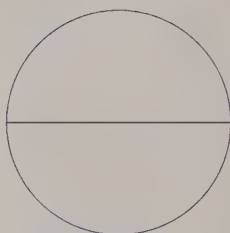


FIGURE 1.12

At this point, most people “understand” what pi means; they can “see” that it will take three times and a “little bit more.” Thus, what pi means is that the circumference of every circle and its diameter have this relationship—the diameter can wrap around 3.14 times. The reaction of many of the teachers and students is “Wow, that makes sense” and “Why didn’t somebody show me that before?” My response to the first reaction is “Great, I’m glad it makes sense,” and my response to the second reaction is that I think it is because many teachers don’t understand or like mathematics. I hope you will be the kind of teacher who knows mathematics well enough so that your students have the tools for it to make sense.

Inductive Reasoning

Inductive reasoning is a process of coming to a general conclusion from seeing patterns in specific examples and looking for the regularity in those patterns. This kind of reasoning is crucial in the child’s construction of the world, and it involves making generalizations from seeing patterns in specific examples. Think of a chair in the library, a chair at a restaurant, a folding chair, and a recliner. At some point, early in your life, when you asked “What is that?” with respect to many different kinds of chairs, the answer was always “A chair.” From these many specific examples, you probably induced that a chair

is something that you sit on, that has four legs (if it has three legs, we generally call it a stool), and that has a back to lean against (if it doesn't, we generally call it a bench).

Inductive reasoning can cause problems too. For example, when I was a child, my dad occasionally went away on business trips. Whenever he returned, he always brought us children a small present (candy, a toy, etc.). I continued this tradition with my two children. On one trip, things were so hectic that I forgot. When I came home with no gift, my two young children were crushed. Peace was restored only when I promised that I would substitute ice cream as their treat. In this case, the children had observed (and come to rely on) a pattern of behavior: Whenever I went away (wasn't home at night to put them to bed), I would give them a present when I returned.

Let's examine a pattern you may not have seen in this context.

INVESTIGATION 1.9 Inductive Thinking with Fractions

Let's say you don't know how to add fractions with different denominators. Look at the three examples below. Can you see a pattern that would enable you to add other fractions? Think about this before reading on. . . .

$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$\frac{1}{5} + \frac{1}{7} = \frac{12}{35}$$

$$\frac{1}{2} + \frac{1}{5} = \frac{7}{10}$$

DISCUSSION

We can describe the pattern in words by saying that whenever you have two fractions with a 1 in the numerator, the numerator of the sum is found by adding the two denominators, and the denominator of the sum is determined by multiplying the two denominators. I wrote this description in everyday English. More formally, we would say that when you add two unit fractions whose denominators are relatively prime, the numerator of the sum is equal to the sum of the denominators of the two fractions, and the denominator of the sum is equal to the product of the denominators of the two fractions. Unit fractions are fractions whose numerators are 1, and two numbers that are relatively prime have no common factors other than 1. The pattern also works when the denominators are not relatively prime, but then the answer is not in simplest form; an example is

$$\frac{1}{4} + \frac{1}{6} = \frac{10}{24} = \frac{5}{12}.$$

The result can be expressed more succinctly with notation:

$$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}.$$

LEARNING

When I wrote this book, one of the criteria that I used when selecting problems for investigations and explorations was that the problems should be extendible. This gets to the "what if" and "I wonder" questions that young children naturally ask often. Unfortunately, this frequency seems to decline steadily as we get older. One of my goals in this course is that you resolve to reawaken this curiosity in mathematics. In this case, a "what if" or "I wonder" question is "What if the numerators are not 1 but are the same, as in $\frac{2}{3} + \frac{2}{5} = ?$ This will be left as an exercise.

INVESTIGATION 1.10 Regions Formed by Polygons in a Circle

**MATHEMATICS**

In Figure 1.13, I directed the art people at Houghton Mifflin to make the “any three points” in such a way that an equilateral triangle was formed, and the four points to make a square, the five points to make a regular pentagon, and the six points to make a regular hexagon. I did this because these shapes are more aesthetically pleasing than if the points were placed randomly, but the results would be the same. If you don’t believe me, check it out by drawing your own circles and “any” three points, “any” four points, “any” five points, etc.

Look at Figure 1.13.

If we draw any two points on a circle and connect them, we create two regions.

If we draw any three points on a circle and connect each pair of points, we create four regions.

If we draw any four points on a circle and connect each pair of points, we create eight regions.

If we draw any five points on a circle and connect each pair of points, we create . . .

Well, you do it! What do you conclude? Do this and then read on. . .

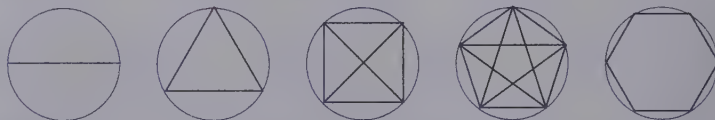


FIGURE 1.13

**MATHEMATICS**

This apparently innate tendency to draw generalizations from specific examples needs careful attention in schools and is one of the many reasons why elementary teachers need a strong background in elementary mathematics. For example, consider explaining what an isosceles triangle is to young children. If all isosceles triangles shown to them have the base parallel to the bottom of the page, as do the first three triangles to the right, children will mistakenly not consider the triangle at the far right to be isosceles, although it is.

DISCUSSION

Most reasonable people conclude that the number of regions doubles each time. That is, 2 points will create 2 regions, 3 points will create 4 regions, 4 points will create 8 regions, 5 points will create 16 regions, and so on. . . . Alas, not so. This pattern breaks down with 6 points. Try as you might (and believe me, mathematicians have tried), no matter where those 6 points are placed, the maximum number of regions formed is 31. Thus, there is an important caution that needs to be noted with inductive reasoning: Be careful. Some patterns do not always hold.



INVESTIGATION 1.11 Sequences, Patterns, Reasoning, and Proof

Do you recall the numerical sequence that we examined in Investigation 1.4? Young children can explore many similar sequences and then can determine the pattern and continue the sequence. Older students in elementary school can explore and figure out even more complex patterns. Let us examine three sequences here to understand how reasoning can help us go beyond simply noticing a pattern to understanding the organization (structure) of the sequence. For each of the sequences below, see whether you can determine the next number in the sequence,

the 20th number (term) in the sequence, and then the n th term. Then read on. . . . 

Sequence 1: 2, 5, 8, 11, 14, . . .

Sequence 2: 3, 6, 12, 24, 48, . . .

Sequence 3: 4, 9, 19, 39, 79, . . .

DISCUSSION

SEQUENCE 1 Let us examine the first sequence: 2, 5, 8, 11, 14, ... It is not terribly difficult to determine that the 6th term will be 17. One way to determine the 20th term is simply to continue the sequence until you get to the 20th term. This is a bit tedious, though you could program a spreadsheet to do the tedious part for you. With a bit of analysis, though, you could realize that you need to add 15 more 3s after the 5th term, and thus, the value of the 20th term is $14 + 45$. If you saw this, great; if you didn't, look at the second row of Table 1.10, where I have not written the actual number but have broken down each term in the sequence so you can see how it came to be. Now, using the fact that multiplication is repeated addition, we can represent the terms more economically. This not only saves time but also begins to reveal the mathematical structure of the sequence. Do you see this? Do you understand the n th term now? If not, the last row might help. Here I have shown in bold the number that tells how many 3s. Notice that this number is always 1 less than the number representing the position of the term in the sequence. That is, the 4th term has 3 3s, the 5th term has 4 3s, etc. Thus, the n th term must have $(n - 1)$ 3s.

TABLE 1.10

1st	2nd	3rd	4th	5th	n th
2	5	8	11	14	
2	$2 + 3$	$2 + 3 + 3$	$2 + 3 + 3 + 3$	$2 + 3 + 3 + 3 + 3$	$(n - 1)$ times $2 + \overbrace{3 + 3 + \cdots + 3}^{(n-1) \text{ times}}$
2	$2 + 3$	$2 + 2 \cdot 3$	$2 + 3 \cdot 3$	$2 + 4 \cdot 3$	$2 + (n - 1) \cdot 3$
2	$2 + 3$	$2 + 2 \cdot 3$	$2 + 3 \cdot 3$	$2 + 4 \cdot 3$	$2 + (n - 1) \cdot 3$

Sequences like the one above, where the difference between each pair of consecutive terms is constant, are called **arithmetic sequences**. What is different is the starting number and the common difference. If we represent the first term by a and the common difference by d , then we can state a rule for finding the n th term of any arithmetic sequence, as shown in the last column of Table 1.11.

TABLE 1.11

1st	2nd	3rd	4th	5th	n th
a	$a + d$	$a + 2 \cdot d$	$a + 3 \cdot d$	$a + 4 \cdot d$	$a + (n - 1) \cdot d$

SEQUENCE 2 Now let's examine the second sequence: 3, 6, 12, 24, 48, ... You probably realized that this sequence is not "just like" the preceding one. If so, you are right. They are similar in that there is a pattern and there is a



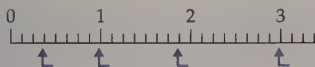
LANGUAGE

We can describe arithmetic sequences in different ways. For example, we can say that the relationship between each term and the following term is that we add the same number each time. On the other hand, we can say that the difference between each pair of consecutive terms is constant.



MATHEMATICS

When I was writing this book on my computer, each of the terms in Table 1.10 took more space than the preceding term (2, $2 + 3$, $2 + 3 + 3$, $2 + 3 + 3 + 3$, $2 + 3 + 3 + 3 + 3$, $2 + 3 + 3 + 3 + 3 + 3$). I had a choice. I could just write each term and then manually push the space bar to make some space between the terms. However, this would make it hard for each of the columns to line up correctly. Or I could use the ruler menu and the tab button on the computer. Here is what I chose. Do you see a pattern here? The distance between each pair of consecutive arrows (imagining an arrow at 0) is 3, 5, 7, 9, 11, etc.



relationship between each term and the next. In this case, however, the relationship is that each term is twice, or double, the preceding term.

As before, in the second row of Table 1.12, I have broken down each term in the sequence so you can see how it came to be. Now, using the fact that exponentiation is repeated multiplication, we can represent the terms more economically. This not only saves time, but it also begins to reveal the mathematical structure of the sequence. Do you see this? Do you understand the n th term now?

TABLE 1.12

1st	2nd	3rd	4th	5th	n th
3	6	12	24	48	
3	$3 \cdot 2$	$3 \cdot 2 \cdot 2$	$3 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot \overbrace{2 \cdot 2 \cdot \dots \cdot 2}^{(n-1) \text{ times}}$
3	$3 \cdot 2$	$3 \cdot 2^2$	$3 \cdot 2^3$	$3 \cdot 2^4$	$3 \cdot 2^{(n-1)}$

Sequences like the one above, where the relationship between each term and the following term is that they always have the same ratio (in this case 2), are called **geometric sequences**. What is different is the starting number and the common ratio, also called the unit (rate) of increase. If we represent the first term by a and the common ratio by r , then we can state a rule for finding the n th term of any geometric sequence, as shown in the last column of Table 1.13.

TABLE 1.13

1st	2nd	3rd	4th	5th	n th
a	$a \cdot r$	$a \cdot r^2$	$a \cdot r^3$	$a \cdot r^4$	$a \cdot r^{(n-1)}$

SEQUENCE 3 Finally, let us examine the third sequence: 4, 9, 19, 39, 79, ... Some students call this a hybrid sequence in that it is not “just like” either of the ones above. That is, the “bad news” is that we can’t use either formula just developed. The “good news” is that we can use problem-solving tools and reasoning to determine the n th term. Try it yourself before reading on. . . . 

One way of describing what repeats is to look at the relationship between each term and the following term. That is, what do you have to do to each term to produce the next term? One way of describing the operation is to say that you double each term and then add 1. We can represent this relationship as $2n + 1$. This is nice, but it won’t help us to determine the value of the 20th term or the n th term. Thus, we have to look further. Here is where number sense and intuition come into play in mathematics. You might have realized that, after the first term, all of the terms end in 9. What if we wrote out a similar sequence—one in which each term is 1 more than the terms of our sequence. This has been done in the third row of Table 1.14. What do you see?

TABLE 1.14

1st	2nd	3rd	4th	5th	n th
4	9	19	39	79	
4	$4 \cdot 2 + 1$	$9 \cdot 2 + 1$	$19 \cdot 2 + 1$	$39 \cdot 2 + 1$	
5	10	20	40	80	

Either by breaking each number apart ($5, 5 \cdot 2, 5 \cdot 2 \cdot 2, 5 \cdot 2 \cdot 2 \cdot 2, 5 \cdot 2 \cdot 2 \cdot 2 \cdot 2$) or by realizing that this is a geometric sequence where $a = 5$ and $r = 2$, we can determine that the n th term of this sequence is $5 \cdot 2^{(n-1)}$. Then we can see that the n th term of the desired sequence ($4, 9, 19, 39, \dots$) is simply 1 less; that is, $5 \cdot 2^{(n-1)} - 1$.

Deductive Reasoning

Let us begin our study of deductive reasoning with an amusing but true story. When Julia was three, she was taking a ride in the country with her mother. She looked out the window and saw a cow, but she had never seen a cow before. Thus, she exclaimed to her mother, "Look, mom, a big dog!" Do you understand Julia's mistake? Can you explain it? By the end of this section, you will be able to do this.

Julia had generalized, from her limited experience in the city, that all animals that have four legs and a tail are dogs. We can translate this sentence into if-then language as follows: If it has four legs and a tail, then it is a dog.

Statements of the form "if p , then q " are called **conditional statements**. The "if" part of a conditional is called the **hypothesis** of the implication, and the "then" part is called the **conclusion**.

All if-then statements have three interesting "relatives." In Julia's case, if we switched the if and then clauses, we would have a true statement: If it is a dog, then it has four legs and a tail.¹¹ These two statements (Julia's and the new one) are said to be *converses* of each other. A statement and its converse are often not both true. Consider some examples.

Statement	Converse
If it is a fruit, then it contains sugar.	If it contains sugar, then it is fruit.
If it is a newborn baby, then it cries at night.	If it cries at night, then it is a newborn baby.
If it is cotton candy, then it has no protein.	If it has no protein, then it is cotton candy.

There are two other statements that have an interesting relationship to the original statement. Let us consider the original statement here to be the true one: If it is a dog, then it has four legs and a tail. We can write the **converse**, **inverse**, and **contrapositive** of any if-then statement. The second column of Table 1.15 shows the converse, inverse, and contrapositive in everyday English;

TABLE 1.15

Statement	If it is a dog, then it has four legs and a tail.	If p , then q .	$p \rightarrow q$
Converse	If it has four legs and a tail, then it is a dog.	If q , then p .	$q \rightarrow p$
Inverse	If it is not a dog, then it doesn't have four legs and a tail.	If not p , then not q .	$\sim p \rightarrow \sim q$
Contrapositive	If it doesn't have four legs and a tail, then it is not a dog.	If not q , then not p .	$\sim q \rightarrow \sim p$

¹¹ Here, we are assuming "typical" dogs. That is, we are ignoring three-legged dogs, which have lost a leg as a result of injury or disease.



LANGUAGE

It is a convention among mathematicians to represent statements with lower-case letters.

the third column shows the statements in shorthand, where the hypothesis is denoted by p and the conclusion by q ; and the fourth column shows the statements in their most succinct form. I hope you see that this last form enables us to grasp more easily the relationships among the four statements.

As we just saw, the converse of a true statement is not always true. What about the inverse and the contrapositive? What do you think? 

It turns out that a statement and its contrapositive are *logically equivalent*. That is, if a statement is true, the contrapositive of that statement is also true. Try this with the three examples described earlier. If you are reading this section actively, you will cover the right-hand column and see whether you can make the sentences yourself.

Statement	Contrapositive
If it is fruit, then it contains sugar.	If it doesn't contain sugar, then it is not fruit.
If it is a newborn baby, then it cries at night	If it doesn't cry at night, then it is not a newborn baby.
If it is cotton candy, then it has no protein.	If it has protein, then it is not cotton candy.

Deductive reasoning is a process of reaching a conclusion from one or more statements, called the hypotheses. An **argument** is a set of statements in which the last statement is called the conclusion and there is one or more hypotheses. The laws of deductive reasoning enable us to determine whether the reasoning in an argument is **valid** or **invalid**. Below are the four possibilities, stated in their general form at the left and with a mathematical example at the right. Case 1 and Case 2 show examples of valid arguments.

Case 1 In this case, we have a general statement that has been determined to be true, and we have a specific instance (example) of the first part of the statement. In this case, the conclusion is true. This law of logic is called the **Law of Detachment**. It is also called *affirming the hypothesis*.

If p , then q .	If a figure is a square, then it has four sides.
p is true.	This figure is a square.
Then q is true.	Therefore, this figure must have four sides.

Case 2 This law of logic is called **Modus Tollens**. It is also called *denying the conclusion*. Essentially, this is a restatement of what we saw earlier: The contrapositive of a true statement is also true. That is, if we know that $p \rightarrow q$, then we know that $\sim q \rightarrow \sim p$.

If p , then q .	If a figure is a square, then it has four sides.
q is not true.	This figure does not have four sides.
Then p is not true.	Therefore, this figure is not a square.

Case 3 and Case 4 show examples of two common mistakes (equating a statement and its converse or inverse) that lead to invalid arguments. In each of the cases, we can demonstrate the falsity of the conclusion with a counterexample.

HISTORY

The origin of formal logic can be traced back to the Greeks. Aristotle (384–322 B.C.) wrote a book that summarized principles of reasoning and laws of logic. Modern mathematics is built on laws of reasoning. As you will discover throughout this book, much of the mathematics that we use today and often take for granted is relatively recent. For example, the notation that we use dates back to a German mathematician Gottlob Frege (1848–1925).

Case 3If p , then q . q is true.Then p is true.

If a figure is a square, then it has four sides.

This figure has four sides.

Therefore, this figure is a square.

One counterexample is a trapezoid, which has four sides but is not a square.

Case 4If p , then q . p is not true.Then q is not true.

If a figure is a square, then it has four sides.

This figure is not a square.

Therefore, this figure does not have four sides.

One counterexample is a trapezoid, which is not a square but does have four sides.

Venn diagrams Now let's examine conditional statements from another perspective. Recall our discussion about representations in Section 1.3. We have just seen different representations of conditional statements (if p , then q and $p \rightarrow q$). We can also represent conditional statements (for example, the four statements concerning dogs and four legs and a tail) with **Venn diagrams** (which we will investigate in more detail in Chapter 2). In the Venn diagram in Figure 1.14, all dogs are inside the small circle. All animals that have four legs and a tail that are not dogs are inside the large circle but outside the dog circle. All other animals are inside the square but outside the circles. This seems rather formal and technical, but it is a good representation of what is going on inside Julia's head and the heads of other children as they are developing more and more sophisticated understandings of how animals are related. Later, other categories are developed, such as mammal, reptile, invertebrate, and so on.

The four Venn diagrams in Figure 1.15 illustrate the four statements in Table 1.15.

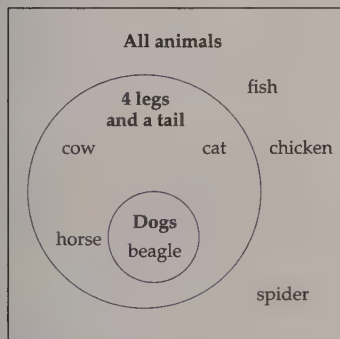
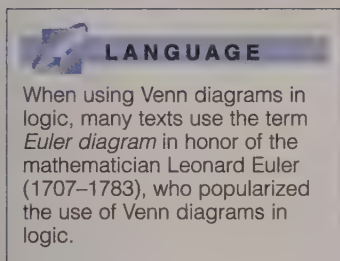


FIGURE 1.14

Statement	Converse	Inverse	Contrapositive
All animals 4 legs and a tail Dogs	All animals 4 legs and a tail Dogs	All animals 4 legs and a tail Dogs	All animals 4 legs and a tail Dogs
If p, then q. $p \rightarrow q$ If it is a dog, it has four legs.	If q, then p. $q \rightarrow p$ If it has four legs, and a tail, it is a dog.	If not p, then not q. $\sim p \rightarrow \sim q$ If it is not a dog, it doesn't have four legs and a tail.	If not q, then not p. $\sim q \rightarrow \sim p$ If it doesn't have four legs and a tail, it is not a dog.

FIGURE 1.15

The first Venn diagram represents the true statement: If it is a dog, then it has four legs and a tail. Because all dogs have four legs and a tail, the small circle is entirely within the larger circle.

The second Venn diagram illustrates why the converse (If it has four legs and a tail, then it is a dog) is not necessarily true. There are two regions

(denoted by the x 's) that satisfy the statement "animals having four legs and a tail": inside the smaller circle and inside the larger circle but outside the smaller circle. This latter region is where the counterexamples come from: sheep, cows, horses, etc.

The third Venn diagram illustrates why the inverse (If it is not a dog, then it doesn't have four legs and a tail) is not necessarily true. There are two regions (denoted by the x 's) that satisfy the statement "animals that are not dogs": inside the larger circle but outside the smaller circle and outside both circles. This former region is where the counterexamples come from. That is, sheep, cows, and horses are not dogs, but they do have four legs and a tail.

The fourth Venn diagram illustrates why the contrapositive (If it doesn't have four legs and a tail, then it is not a dog) is true. There is only one region for "doesn't have four legs and a tail"; outside both circles. And all animals in this region (such as chickens and fish) are definitely not dogs.

Biconditional statements There is one other concept from deductive reasoning that we will discuss in this chapter: the *biconditional* statement. A **biconditional statement** occurs when a statement and its converse are both true. This is especially relevant in mathematics: Most mathematical definitions are biconditional statements.

Consider the following two statements:

If two lines intersect to form right angles, then they are perpendicular.

If two lines are perpendicular, then they intersect to form right angles.

In this case, the statement and its converse are both true. We can use "if and only if" language to combine them into the following biconditional statement, which is also the definition of perpendicular:

Two lines are perpendicular iff they intersect to form right angles.

The shorthand for "if and only if" is **iff**. So when you see iff in this book, realize that this is not a typographical error!

Intuitive Reasoning

There is one other type of reasoning that occurs a lot in mathematics. **Intuitive reasoning** is not well understood; and we know very little about how to "teach" it. Let's look at a couple of examples.

INVESTIGATION 1.12 The Nine Dots Problem



FIGURE 1.16

Not all problem-solving involves computation and formulas, as this investigation shows.

Without lifting your pencil, can you go through all nine dots in Figure 1.16 with only four lines?

DISCUSSION

This is a very famous problem, which some of you may have already encountered because of its moral: This problem is impossible to solve as long as you "stay inside the box." That is, there is no possible way to go through all nine dots with four lines if you stay within the box. In order to solve the problem, you need to go "outside the box." If you haven't solved the problem yet, try to work with this hint. . . . 

The solution to the problem can be seen on page 56. This idea of not getting stuck inside the box is crucial to good problem-solving. In many real-life problems, the solution to a problem requires that people think about the problem differently.

In this case, inductive reasoning is not very relevant. We don't have other examples before us from which we can generalize. Classic deductive reasoning is not applicable here. At some point, either out of desperation or after concluding that no solution within the square is possible, some people solve this puzzle by trying solutions that go outside the boundaries of the invisible square surrounding this set of dots. This is similar to figuring out the n th term of the 4, 9, 19, 39, 79, . . . sequence by "feeling" its connectedness to the 5, 10, 20, 40, 80, . . . sequence.

Let me give another example. Believe it or not, even though I am a mathematics teacher, I hated algebra in high school, mostly because I had poor teachers. The following kind of problem almost killed me in Algebra 2 because I couldn't solve it "the teacher's way," which then was "the right way" and the only way you could get credit. The actual problem (I have saved it!) was even more complex, but I have simplified the numbers here to illustrate the point.

The problem: A grocer has a barrel of peanuts that sell for \$2 a pound and a barrel of cashews that sell for \$5 a pound. If he wants to make 60 pounds of a mixture of the two that he will sell for \$3 a pound, how much of each should he mix?

Now the classical way to approach this problem is to set up two equations in two unknowns and then solve. I will let p stand for the number of pounds of peanuts and c stand for the number of pounds of cashews (in my day we had to use x and y , even though p and c make it easier to remember which is which).

$$\text{Equation 1: } p + c = 60$$

$$\text{Equation 2: } (2p + 5c) = 3 \cdot 60$$

Try as I might, I just couldn't remember how to get Equation 2 for this kind of problem. However, I could get the answer using a way that "just came to me" and that I could not explain. I intuitively saw the problem in terms of ratios. What stood out for me was that \$2 is only \$1 away from \$3 and that \$5 is \$2 away from \$3. That is, 5 is twice as far from 3 as 2 is. Figure 1.17 represents this relationship visually.



FIGURE 1.17

Therefore, the correct answer must have twice as many pounds of peanuts as pounds of cashews. It only took a few rounds of "guess-check-revise" to determine that 40 pounds of peanuts and 20 pounds of cashews was the answer.

With practice, I was able to refine my method for even more complex problems. However, because I couldn't do it "the right way," I received no credit. I remember getting a 37 on one test. The only reason I passed the course is that the teacher graded on the curve. The starting guard on the basketball team was in the class; I almost always beat him, and whatever he got was a C. [Double standards for athletes have been and continue to be a common practice in many high schools and colleges.] The only reason why I stayed in mathematics was that we moved, and my teacher in the new school was fascinated by my way and helped me to understand it and prove it. In fact, he encouraged me to submit it as an original piece of work in the science fair, which I won. Had my family not moved in the middle of that school year, I would probably not be a college mathematics teacher!

Even though intuition is not well understood, I believe it deserves mention in this course because many, if not most, mathematicians have experienced the power of intuition in their own mathematical work, and many important mathematical (and scientific) discoveries have been the result of what we call

intuition. At the same time, as Blaise Pascal, a member of the Mathematics Hall of Fame, once said, “Chance favors the prepared mind.” This statement has been interpreted by many mathematics educators to mean that intuition generally does not spring from a void. It generally comes because the person did have a lot of relevant information but wasn’t making the connections at a conscious level. I find the analogy of islands in the ocean to be useful here. If you look at islands in an ocean from one perspective (aerial view), they are separate—that is, not connected. However, if you look at them from another perspective (topographical or relief map), they are like peaks of a single mountain range.

SECTION

1.6

COMMUNICATION

WHAT DO YOU THINK?

- Why does the NCTM encourage mathematics classrooms to provide numerous opportunities for communication?

Read the standard on Communication and consciously ask yourself whether the statements make sense. Are they meaningful? To what extent do you agree with them? Then read on. . . . 

 Standard 8: Communication
Mathematics instructional programs should use communication to foster understanding of mathematics so that all students—

- organize and consolidate their mathematical thinking to communicate with others;
[Being able to communicate doesn’t just come from thin air!]
- express mathematical ideas coherently and clearly to peers, teachers, and others;
[Most mathematical notation and vocabulary has been invented to make communication simpler.]
- extend their mathematical knowledge by considering the thinking and strategies of others;
[Many of my students say they learn as much from their peers as they do from me.]
- use the language of mathematics as a precise means of mathematical expression.
[The precision of mathematics ultimately makes it easier, not more difficult, to say what you see.]

(*Principles and Standards*, p. 85)

Albert Einstein is said to have remarked that “a description in plain language is a criterion of the degree of understanding that has been reached.” Communication and understanding go hand in hand. “When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others verbally or in writing, they are faced with the task of stating their ideas clearly and convincingly to an audience” (*Principles and Standards*, p. 85). Implicit in the previous sentence is that there are two distinct kinds of communication that you need to be aware of as you work in this course.

1. Communicating with yourself about the problem—that is, being able to make sense of your own strategies and solutions.

If you go back (and I urge you do to so) and read “4 Steps for Solving Problems” on the inside front cover of *Explorations*, you will realize that many of these steps are tools to help you better communicate with yourself about the problem.

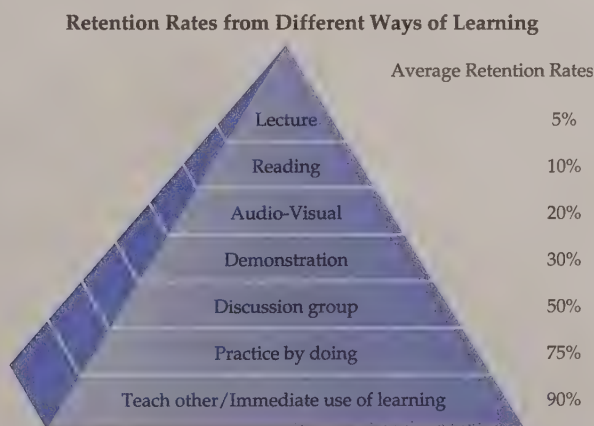
2. Communicating with other people—sharing your observations and solutions and being able to understand others' observations and solutions.

Look back on the investigations we have already done in this chapter and on the explorations you have done in *Explorations*. You should find that there were times when the words you used to communicate were not clear to others, or vice versa, and that better use of mathematical language could help facilitate communication, whether in describing your strategies for the handshakes problem, explaining assumptions you made about the coffee filter problem, or choosing language to describe patterns you saw in magic squares.

As we move to learning specific content in the following chapters, you will come to realize that one of the reasons for all the mathematical notation and language that sometimes seems to get thrown at you is that if you understand the notation and the terms (digit, place value, row, column, and so on), then two things are more likely. First, you can reduce ambiguity and confusion in mathematical conversations. Second, you are more likely to understand fully the mathematical structures being presented.

Learning by Communicating

There is another reason for emphasizing communication at the beginning of this course. Few people learn best in isolation. The National Training Laboratories in Bethel, Maine, have created the pyramid in Figure 1.18 to represent what we know about learning.



Source: Reprinted with permission from NTL Institute from "Retention Rates from Different Ways of Learning."

FIGURE 1.18

When we bring communication into the learning process—by having students work in small groups, by having students explain how they solved a problem, by having students justify the steps in their solution—the students are more likely to “own” the knowledge they gain.

Let us now examine an aspect of using patterns that was mentioned earlier: being able to describe patterns you see in such a way that others can understand what you are saying.

INVESTIGATION 1.13 Describing Shapes

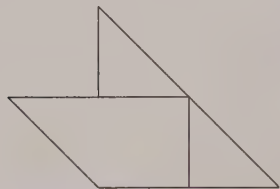


FIGURE 1.19

Suppose a friend of yours misses class and calls you up on the phone to find out what the homework was. You tell her that the teacher drew a shape and asked you to write a verbal description of it. "What was the shape?" your friend asks. How would you describe the shape in Figure 1.19? Write down your description, and then read on. . .

DISCUSSION

Following are three possible descriptions. You may want to sketch the shape following the instructions in each of these, just to see which one seems clearest to you.

DESCRIPTION 1

Draw a horizontal line. Beginning at the left endpoint of that line, draw a vertical line going up that is the same length as the horizontal line. Erase the bottom half of the vertical line. Now draw a (diagonal) line that connects the top of the vertical line to the right endpoint of the horizontal line. Find the middle of the horizontal line and draw a vertical line until you hit the diagonal line. Without lifting your pen, now draw a horizontal line to the left. Keep going after you cross the original vertical line and make the part after you hit the vertical line as long as the part before you hit it. Now connect this point to the bottom of the first vertical line you drew.

DESCRIPTION 2

Make a right isosceles triangle with the right angle at the bottom left corner of the triangle. Draw a square inside that triangle; the length of the sides of the square is one-half the length of the equal sides of the triangle. Erase the left side of the square. Now construct a trapezoid. The bottom base of the trapezoid is also the bottom of the square. To make the top base, begin at the midpoint of the hypotenuse of the original triangle and draw a line that is twice the length of the square.

DESCRIPTION 3

Construct a square. Now construct three right isosceles triangles connected to the east, north, and west sides of the square. First, the east side: The hypotenuse starts at the top of the right side of the square and runs in a south-east direction. Second, the north side: The hypotenuse starts at the same spot (the top of the right side of the square) but runs in a northwest direction. Third, the west side: The hypotenuse starts at the bottom of the left side of the square and runs in a northwest direction. Now you need to erase the left side of the square.

Which description did you like best? The first description involved very little formal language. The second description used several mathematical terms: right isosceles triangle, square, hypotenuse, trapezoid, and midpoint. The third description involved both mathematical terms and lay terms. If everyone in your class were to read all of these descriptions and select the "best" one, it is likely that each of the three would be seen as the best by some students.

INVESTIGATION 1.14 Darts, Proof, and Communication

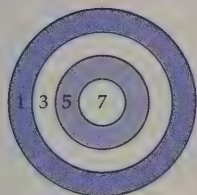


FIGURE 1.20

If you did not do Exploration 1.3 in *Explorations*, let me briefly set the problem up.

Suppose you have a dart board like the one in Figure 1.20. You throw four darts, all of which land on the dart board. One of the questions I asked (in the exploration and of the fifth graders I taught in the 1998–1999 school year) was what kinds of scores would be possible and what kinds of scores would be impossible.

DISCUSSION

After a few minutes, one of the students, Erika, suddenly said, “Only even numbers are possible.” I asked her how she came to that conclusion, and she said, “Well I know that an odd plus an odd is even and an odd plus an even is odd. [At this point, she held up four fingers to represent the four darts.] So, the first two darts are odd and so when you add them, you have an even number. [She joined two of her fingers together to indicate the combined score from two darts.] Now this number (even) plus the next dart (odd) will make an odd number. [She now joined three of her fingers together to indicate the combined score from the first three darts.] Now this number (odd) plus the last dart (odd) will make an even number. So the only possible scores you can get are even numbers.”

We can represent Erika’s proof as shown in Figure 1.21.

$$\begin{array}{l}
 (\text{odd} + \text{odd}) + \text{odd} + \text{odd} \\
 \quad \swarrow \\
 (\text{even} + \text{odd}) + \text{odd} \\
 \quad \swarrow \\
 \text{odd} + \text{odd} \\
 \quad \swarrow \\
 \text{even}
 \end{array}$$

FIGURE 1.21

Reflect on this problem for a moment, along with the standards we have discussed: problem-solving, patterns, representation, reasoning, and communication. Do you see them working interactively as opposed to separately? What aspects of each standard do you see in operation? 

Erika chose to solve the whole problem in her head because she intuitively grasped the solution to the problem. I have presented her communication of her solution (verbally), but because not everyone learns and understands the same way, I have presented another way of representing the problem (Figure 1.21). Other students solved the problem more inductively. That is, they saw patterns in their table that helped them to realize that odd + odd is even, and so on.



Standard 9: Connections

Mathematics instructional programs should emphasize connections to foster understanding of mathematics so that all students—

WHAT DO YOU THINK?

- How does making connections add to your toolbox?
- What connections among concepts and problem-solving strategies have you seen thus far in the chapter?

- recognize and use connections among different mathematical ideas; [Over the course of this book, you will come to see that there are rich relationships between most mathematical ideas and that their separateness (such as geometry being totally different from algebra, like English and Vietnamese) is an illusion.]
- understand how mathematical ideas build on one another to produce a coherent whole; [When you buy this building-block metaphor, you take care to make sure that you understand the “big” ideas, which are like building blocks.]
- recognize, use, and learn about mathematics in contexts outside of mathematics. [There are many readable books, such as the ones mentioned before by Keith Devlin, that will amaze you!]

(*Principles and Standards*, p. 90)

Making *connections* is at the heart of the NCTM Standards. We have already noted connections between Pascal’s triangle and Exploration 1.1 (The Handshakes Problem) and Investigation 1.5 (The Sum of the First 100 Numbers). We have also noted that the idea of connections is new to many students. This is addressed in the NCTM Curriculum Standards:

The mathematics curriculum is generally viewed as consisting of several discrete strands. As a result, computation, geometry, measurement, and problem solving tend to be taught in isolation. It is important that children connect ideas both among and within areas of mathematics. Without such connections, children must learn and remember too many isolated concepts and skills rather than recognizing general principles relevant to several areas.

(*Curriculum Standards*, p. 32)

Let us examine some of the kinds of connections that we would like you to make in this course.

Making Connections Between the Problem and What Is Inside Your Head

Look back at the problems in this chapter that you solved on your own *and* that gave you some difficulty—that is, problems you had to do some thinking in order to solve as opposed to those for which you immediately knew what to do and then did it. In these cases, you literally have to find some way to connect the words of the problem to a representation of the problem that will enable you then to solve the problem. Many of my students report that before this course, the teacher did the connecting for them. Because you are to become teachers, this is where you develop the ability to make such connections.

Recall the example from the National Assessment of Educational Progress in Table 1.3. To solve the first problem, you simply had to add the numbers and divide by 6. However, to solve the second problem, you had to think about the information. In other words, you had to create in your head a **model** of the problem that made sense and that connected the relevant information to your mathematical knowledge. Developing the ability to connect the given information to your mathematical knowledge and to your problem-solving toolbox is one of the central objectives of this course. Some students have already developed this ability. However, every year, most of my students enter the course

without these tools. Seeing those students' toolboxes develop and seeing their confidence grow is one of the many reasons why I cannot imagine being in any other profession!

Connecting New Concepts to Old Concepts

One important kind of connection is the connection between new ideas and something that is familiar to you. As we discussed in Section 1.2, if you are not able to make these connections, you may end up with "Teflon knowledge." Consider another example from the NAEP. Students were asked to select the decimal equivalent to 12 percent and then to select the decimal equivalent to .9 percent. The results are given in Table 1.16.

TABLE 1.16

	Percent correct	
	Grade 7	Grade 11
Which decimal is equivalent to 12 percent?	71	90
Which decimal is equivalent to .9 percent?	25	56

Under any circumstances, we would expect a smaller percentage of correct responses for the second question. However, the drop-off is enormous. For seventh graders, the percentage who got the second question correct was barely one-third that for the first question. What connections could have helped these students? Try to answer this question yourself before reading on. . . . 

First, they could have connected percents to rational numbers, reasoning that 12 percent means $12/100$. Then they could have connected the fraction to a decimal: $12/100 = 0.12$. Following this reasoning for the second question, .9 percent means $0.9/100$, which converts to 0.009. Alternatively, they could have applied the algorithm "Move the decimal point two places." In other words, .9 becomes .009. Of course, they had to realize that they had to put zeros to the left of .9 in order to be able to move two decimal places! We examine decimals and percents in more detail in Chapters 5 and 6.

Making Connections Among Different Concepts

Many, if not most, students have come to view mathematics as a collection of separate topics. Most mathematicians, however, see mathematics as a network of *interconnected* concepts, similar to a map of the highways in the United States, the freeways being equivalent to major connections and surface roads to smaller connections.

Looking at the four basic operations—addition, subtraction, multiplication, and division—we can make the following connections: Addition and subtraction are inverse operations, and multiplication can be seen as repeated addition. Likewise, multiplication and division are inverse operations, and division can be seen as repeated subtraction. We will investigate these concepts in Chapter 3. In Chapter 6, you will discover or rediscover that percents have close connections to ratios and to equivalent fractions. Many students see algebra and geometry as completely separate, whereas mathematicians see them as very connected.

Connecting Different Models for the Same Concept

In mathematics, many concepts can be represented in different ways. For example, consider the ways of representing $3/4$ shown in Figure 1.22. One of them is not a valid representation of what we mean by $3/4$. Do you understand why? Can you express that understanding in words? 

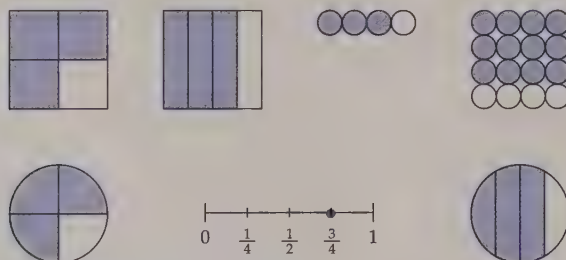


FIGURE 1.22

When you can connect many different representations to the concept underlying all of them, you are well on your way to the kind of mathematical thinking encouraged by the NCTM. Many “good” problem-solvers are good not because they know more than others but because their knowledge is better connected.

Connecting Conceptual and Procedural Knowledge

Most of you have probably learned a variety of standard procedures, also called **algorithms**. However, fewer of you are likely to know why they work. For example, although you can probably convert a mixed number to an improper fraction—for example, $4\frac{2}{3} = 14/3$ —do you know *why* we multiply the whole number by the denominator, add the numerator, and then put the whole thing over the denominator? 

Similarly, do you know *why* we “move over” in whole-number multiplication?

$$\begin{array}{r} 47 \\ 35 \\ \hline 235 \\ 141 \\ \hline 1645 \end{array}$$

Coming to understand why these and many other algorithms work will make you a more powerful problem-solver and a stronger teacher.

Making Connections Between Mathematics and “Real Life” and Between Mathematics and Other Disciplines

Mathematics enters into some of the most important decisions we make, both as individuals and as a society. Recall the explorations dealing with money. Think about buying tires, buying a house, investing your savings, comparison shopping, deciding how much wallpaper or paint to buy, building a fence, making a budget, figuring out a tip, choosing among credit cards—the list could go on for pages!



BEYOND THE CLASSROOM

Dorothy Wallace, a mathematics professor at Dartmouth College, developed a number of mathematics courses that she thought would be more interesting than the traditional courses to liberal arts students not majoring in math and science. One of these courses was Pattern, an interdisciplinary course on art and mathematics, which she co-taught with a local artist, Pippa Drew. In 1997, college professors from all over the country were invited to Dartmouth to get a sense of these courses. I was invited to join as a teacher, educator, and facilitator. The group that I worked with consisted of 25 mathematics and art professors. They quickly came to see that they had far more in common than most of them had realized—both art and mathematics professors got excited about patterns. Finding similar patterns in different areas was exciting, and finding breaks in patterns was exciting as well.

Think of how many political issues are connected to mathematics:

- The current and future effects of the federal deficit—How many people realize what a really big number 4 trillion is?
- The issue of income inequality—women earn an average of 70¢ for every dollar men earn, for doing the same job!
- The cost of illiteracy and innumeracy to the United States—In 1990 it was estimated that American employers spent about \$40 billion to train their employees in the basic reading, writing, and arithmetic skills they should have learned in school.

Unfortunately, all too many students do not see mathematics as connected. My greatest hope for this course is that it leads you to see that part of your job as a student is to make connections—to make sure you see how the new concepts and ideas connect to what you already know, to make sure you see connections within and between different concepts, and to make sure you see the connection between the mathematics in this course and mathematics beyond the classroom. If that happens, then our society will begin to be more mathematically literate rather than rank near the bottom internationally in mathematics performance.

The following investigation problem illustrates the importance of making relevant connections when solving problems.

INVESTIGATION 1.15 How Many Pieces of Wire?

A jewelry artisan is making earring hoops. Each hoop requires a piece of wire that is $3\frac{3}{4}$ inches long. If the wire comes in 50 inch coils, how many $3\frac{3}{4}$ inch pieces can be made from one coil, and how much wire is wasted? Solve this problem on your own and then read on. . . . 

But before you do, ask yourself, “Do I understand the problem? Does the problem’s wording help me devise a plan for solving it? Once I have a solution, can I check it?” Perhaps most important, ask yourself, “What did thinking about *this* problem reveal to me about problem-solving in general?”

DISCUSSION

STRATEGY 1: Divide

Some people quickly realize that you can divide “to get the answer.” If you use a calculator, it shows 13.333333. . . . If you use fractions, you get $13\frac{1}{3}$. Many people interpret these numbers to mean that you can get 13 pieces and you will have $\frac{1}{3}$ inch wasted. Unfortunately, that is not correct. If you also got

1/3 inch as wasted, stop! Go back and see if you can figure out why 1/3 inch is not the correct answer and what the correct answer is. Then read on. . . . 

One of the reasons why I, like many other math teachers, stress the importance of labels is that they illustrate the meaning of what we are doing. The *meaning* of the quotient ($13\frac{3}{4}$) is 13 whole hoops and 1/3 of a hoop. That is, $50 \text{ inches} \div 3\frac{3}{4} \text{ inches/hoop} = 13\frac{3}{4}$ hoops. The *meaning* of the fraction (1/3) is that what we have left would make (1/3) of a hoop. Because one whole piece is $3\frac{3}{4}$ inches long, 1/3 of a piece is 1/3 of $3\frac{3}{4}$, that is, $1\frac{1}{4}$ inches is wasted.

How would we check this answer? Think and then read on. . . . 

One way to check would be to multiply $3\frac{3}{4} \times 13$. Do you see why? This would tell us the length of the 13 whole pieces. If this number plus $1\frac{1}{4}$ equals 50, then our answers are correct. In fact, $3\frac{3}{4} \times 13 = 48\frac{3}{4}$ and $48\frac{3}{4} + 1\frac{1}{4} = 50$.

STRATEGY 2: “Act it out”

Some people understand the problem better when they draw a line to represent the 50 inch piece of wire. Draw a picture like that in Figure 1.23 to represent what happens when the artisan cuts off pieces $3\frac{3}{4}$ inches long until the wire is used up.

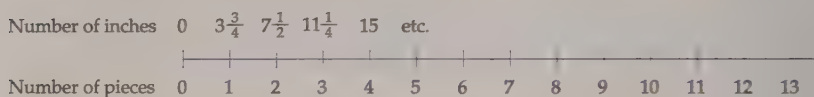


FIGURE 1.23

STRATEGY 3: Make a table

Refer to Table 1.17.

TABLE 1.17

Number of pieces	Number of inches	Thinking process
1	$3\frac{3}{4}$	Using the concept of ratio and proportion, we can reason that if 4 pieces make 15 inches, then 12 pieces would make 45 inches. So 1 more piece works.
2	$7\frac{1}{2}$	
3	$11\frac{1}{4}$	
4	15	
12	45	
13	$48\frac{3}{4}$	

The answer to the second part of the question is simply

$$50 - 48\frac{3}{4} = 1\frac{1}{4}$$

What kinds of connections do you see in this problem? Think and then read on. . . . 

Connections Several of the kinds of connections mentioned at the beginning of Section 1.7 are evident in this problem.

- Connecting new concepts to old concepts: If the concept of division of whole numbers (investigated in Chapter 3, Section 2) is well understood,

the student can immediately see this as a division problem, but one that involves fractions.

- Connecting different models for the same concept: In Chapter 3, Section 2, you will find that one model for division is repeated subtraction. Drawing the diagram of the wire and marking off $3\frac{3}{4}$ inch segments often triggers the connection between this problem and repeated subtraction, which then produces the insight “Oh, now I get it! I just have to divide 50 by $3\frac{3}{4}$!”
- Connecting conceptual and procedural knowledge: The procedure for dividing by fractions or decimals is not terribly complex. However, the result of the computation ($13\frac{1}{3}$) is not quite the answer to the question.

Summary

We have finished our first pass over the NCTM’s five process standards, and we have examined several specific aspects of those standards. I will argue that without them, the rules and formulas and procedures of mathematics are inert and lifeless, and that without them, *the mathematics classroom is not very interesting for most students*. As you carry out the various investigations and explorations in this course, I strongly encourage you to stop and reread the NCTM standards on the web at <http://www.nctm.org>. Are they making sense? Are you developing the abilities described in the standards?

Many students find it helpful to keep a reflection notebook, also called a **journal** or a **learning log**. In fact, your instructor may make this part of the course. If you do take this time to examine your learning, you will own more of the ideas studied in this course and will be more likely to bring those ideas to the children you teach after you graduate.

▶ With good instruction, where the five process standards are alive and well, students develop what the NCTM calls **mathematical power**.

Mathematical power includes the ability to explore, conjecture, and reason logically; to solve nonroutine problems; to communicate about and through mathematics; and to connect ideas within mathematics and between mathematics and other intellectual activity. Mathematical power also involves the development of personal self-confidence and a disposition to seek, evaluate, and use quantitative and spatial information in solving problems and making decisions. Students’ flexibility, perseverance, interest, curiosity, and inventiveness also affect the realization of mathematical power.¹²

The last sentence brings us back to students’ attitudes and beliefs—about mathematics, about learning, about teaching, and about themselves. In my doctoral dissertation (a study of how attitudes and beliefs influence the learning of mathematics), I wrote that affect (beliefs, attitudes, and emotions) influences both the *quantity* and the *quality* of the thinking you bring to the problem—that is, both how hard you try and how smart you try.

I offer two metaphors. Each has been useful to a number of my students. Imagine driving in a car on a highway and suddenly encountering fog “thick as pea soup.” What would you do? Of course, you would slow down. Now imagine you are reading a textbook and the words feel like a blur. Why not slow down? Or imagine working on a problem at the end of a chapter and it just doesn’t make sense, but you plug along anyway. Why not slow down? Just as you can see much better, in fog, at 30 miles per hour than you can at 70 miles

¹² NCTM, *Professional Standards for Teaching Mathematics* (Reston, VA: NCTM, 1991), p. 1.

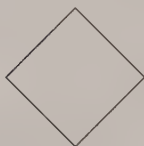
per hour, most people can make more sense of ideas and problems when they *slow down*.

Imagine you are going to visit someone in another city. The directions are not terribly clear, and you're not sure you made the right turn. What do you do? The reasonable course is to stop at the first opportunity (gas station or convenience store) and *ask for directions*. You might also stop and consult a map and your directions. Similarly, if you are reading an Investigation in the text or working on a problem and you feel lost, what do you do?

These metaphors deal with an underlying belief that mathematics is more than just "getting it" but rather about "making sense." This practice of pausing frequently to ask whether the ideas or problems are making sense is crucial in order for you to own what you learn, as opposed to simply renting it until the test or the end of the course.

EXERCISES

- Describe the mistake in inductive reasoning made by the child who says that the figure below is a diamond and "not" a square.

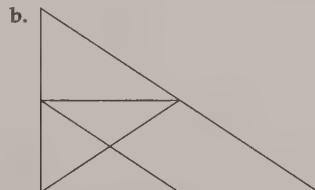
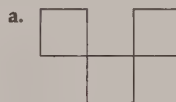


- Along the lines of Investigation 1.9, make up a number of fraction addition problems where the two numerators are 2 (such as $\frac{2}{3} + \frac{2}{5}$). Describe the generalization you come up with in words and in mathematical notation.
 - Make up a number of fraction addition problems where the two numerators are equal (such as $\frac{3}{4} + \frac{3}{5}$ and $\frac{5}{8} + \frac{5}{9}$). Describe the generalization you come up with in words and in mathematical notation.
- Represent the statements below with Venn diagrams.
 - All dogs like to be petted.
 - Some Americans did not vote in the last election.
 - Some students like mathematics.
 - No cotton candy has protein.
- Translate the following statements into if-then form.
 - All fruit contains sugar.
 - All dogs bark when the mail carrier comes to the door.
 - No cats bark when the mail carrier comes to the door.
 - All babies cry at night.
 - In order to drive a car, you must pass a written test.
- Write the inverse, converse, and contrapositive of each of these statements.
 - If you exercise three times a week, then you will not get sick.
 - If a number is a multiple of 10, then it is a multiple of 5.
 - If Washington had not crossed the Delaware, then we would not have won independence from Britain.
 - If you are a good elementary school teacher, then you know elementary mathematics well.
- Determine whether the arguments below are valid or invalid.
 - All polygons have angles.
A circle has no angles.
A circle is not a polygon.
 - If you don't work hard, then you won't succeed.
You work hard.
Therefore, you will succeed.
 - If you make an A on the midterm, you won't have to take the final.
Jose did not take the final.
Therefore, Jose made an A on the midterm.
 - All blippies are orgs.
All blippies are magas.
Therefore, all orgs are magas.
 - If I were rich, I would buy a cabin.
I am not rich.
Therefore, I have not bought a cabin.
 - If a number is divisible by 8, then it is divisible by 4.
 x is not divisible by 8.
Therefore, x is not divisible by 4.
 - All comedians are funny.
Shelly is not funny.
Therefore, Shelly is not a comedian.
- Using Venn diagrams, determine whether the arguments below are valid or invalid.
 - All professional athletes are rich.
All rock stars are rich.
Therefore, some rock stars are professional athletes.
 - All bunnies are skittish.
Some skittish animals have fur.
All cows have fur.
Therefore, cows are skittish animals.
 - All squares are rectangles.
Some quadrilaterals are squares.
Therefore, some quadrilaterals are rectangles.

- d. All rectangles are quadrilaterals.
All quadrilaterals are polygons.
Therefore, all rectangles are polygons.
- e. Michael Jordan drinks Gatorade.
Michael Jordan has a lot of stamina.
Therefore, if you drink Gatorade, then you will have a lot of stamina.
8. A boatman is to transport a fox, a goose, and a sack of corn across the river. There is room in his boat for only one of the three at a time. Furthermore, if the fox and the goose are left together, the fox will eat the goose. If the goose and the corn are left together, the goose will eat the corn. How can the boatman do the job?
9. The following two jar problems have been around for many years. They are contrived problems to promote reasoning. In these problems, a jar must be filled *completely* and emptied *completely*. For example, you cannot say, "Fill the big jar half full and fill the second jar half full" to get 4 gallons.
 - a. You are given a 5 gallon and a 3 gallon pail, both of which are unmarked. You are asked to fetch 4 gallons of water from the well in one trip. How?
 - b. You have three jugs of capacities 8, 5, and 3 gallons. The largest jug is full of water. The other two are empty. Your task is to redistribute the water so that you wind up with 2 gallons of water in the large jug and 3 gallons of water in each of the other jugs.
10. What information would you need in order to estimate the cost of having a car on campus?
11. When Joe wakes up in the morning, the clock is blinking and says 3:30. His watch says 6:45. When did the power go off? What assumption(s) did you make in order to solve the problem?
12. If Sue can eat a small pizza in 6 minutes and Sally can eat one and a half small pizzas in 15 minutes, how long will it take the two of them to eat three small pizzas (assuming they don't get full)?
13.
 - a. Imagine a jar with a lid that is 6 centimeters (cm) across. The jar is 8 cm high. At the bottom of the jar, there is a caterpillar. Each day the caterpillar crawls up 4 cm. Each night she falls down 2 cm. How long will it take her to touch the lid of the jar?
 - b. What if the jar is 20 cm high, and each day the caterpillar crawls up 5 cm and each night falls down 2 cm? How long will it take the caterpillar to touch the lid of the jar?
 - c. What if the problem is now about a person at the bottom of a 40 foot well. The person can climb 6 feet in $1/2$ hour but must then rest for 15 minutes, during which time he slips 2 meters? How long until the person reaches the top?
 - d. Describe patterns you see in the three problems.
14. At the writing of this book, many different companies were making television commercials to induce people to call collect using their company. One company adver-

tised that you could call "anywhere anytime for just 10 cents a minute." Sounds pretty good. I pay an average of 10 cents a minute for long-distance calls placed from my home. How do they do it? Well, in small print at the bottom of the picture, the reader learns that there is a \$1.95 surcharge on every call. Describe the deal now.

15. When Goodyear Tire Company introduced the Aquatread tire, one of the television commercials stated that the tire displaces over 1 gallon of water per second (at highway speeds); the announcer further stated that this amounts to over 396 gallons per mile. Critique this commercial.
16. Write instructions for drawing a square, as though to someone who doesn't know the word *square* and has no formal training in mathematics.
17. Assume that someone is visiting you and has never used a phone. You are going to work, and this person is going to be home alone. Write directions to help this person make phone calls. Include all the different kinds of phone calls one might make.
18. Describe the following figures, as though to someone you were talking to on the phone.



19. Let's say there are 145 students in your class. If each student greeted and shook hands with every other student in the class, how many handshakes would there be?
20. Suppose a family of 2 parents and their 10 children decides that each person will buy a gift for each member of the family for Christmas. How many gifts will be under the Christmas tree? Now suppose that the family decides that each parent and child will exchange gifts and that the 10 children will have a drawing in which each child chooses the name of one sibling to whom to give a gift. How many gifts will now be under the Christmas tree?
21. Alice, Betty, and Carla traded hats and gloves. Alice wore Betty's hat and Carla's gloves. Write a paragraph explaining how you know whose hat and gloves Betty and Carla wore.
22.
 - a. How many $3\frac{2}{3}$ oz bottles of perfume can be filled from a jug containing 64 oz?
 - b. How many ounces of perfume are left over per jug?
 - c. Determine a jug size that is a whole number and that would produce no waste.
23. Make up a 3×3 magic square with $5\frac{1}{2}$ in the middle square.
24. One of the more famous magic squares was created by Albrecht Dürer in his 1514 engraving *Melancholia*. He

created a square in which the year appeared in the bottom row of the square.

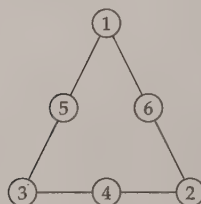
- Describe all the patterns you see in the magic square below.
- Describe one of the patterns in words, as though you were talking to someone on the phone.

13	3	2	16
8	10	11	5
12	6	7	9
1	15	14	4

- Make up a 4×4 magic square.
- The following magic figure originated in West Africa.
 - Describe the patterns you see in this magic figure.
 - Describe the relationship between this figure and the first magic square in Exploration 1.5.

		3		
	2		6	
1		5		9
	4		8	
		7		

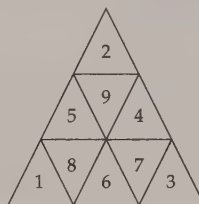
- Examine the magic triangle below.



- Describe all the patterns you notice.
- Make a new magic triangle with different numbers.
- What generalizations can you make about magic triangles of order 3—that is, those in which there are three numbers on each side of the triangle?
- Make a magic triangle of order 4. Briefly describe the strategies and knowledge that enable you to make the three sums equal.

- Below is a different kind of magic triangle.

- Describe the patterns you see in this magic triangle.
- Create a new magic triangle in which the nine numbers are not consecutive numbers.



- Leonard Euler, whom many consider to be the greatest mathematician of all time, created this magic square. Describe three patterns that you find in this magic square.

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

- Benjamin Franklin created this magic square. Describe three patterns that you find in this magic square.

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

CHAPTER SUMMARY

1. One's beliefs about mathematics can powerfully affect one's ability to do mathematics. Many students find that the big ideas of this chapter involve a major shift in the beliefs that they brought into the course.
2. George Polya's description of the four basic steps in solving problems is a useful framework, but it is the starting point, not the ending point, of your using it.
3. Many problems can be solved in a variety of ways. Some of the more common problem-solving strategies are guess-check-revise, make a diagram, develop an equation, make a table, look for patterns, connect the problem to a similar problem, break the problem into smaller pieces and act it out.
4. Solving a problem involves more than simply computation. The number is not the answer, and you are not finished unless the answer makes sense.
5. There is generally more than one way to represent a problem. Thus, it is important to construct a repertoire of representations in order to develop mathematical power and in order to understand other students' representations of problems.
6. Patterns permeate our lives as well as permeating mathematics. Recognizing patterns is the beginning of using them to understand mathematics and solve problems. You should be able to describe, extend, analyze, and create a wide variety of patterns.
7. Reasoning and proof are not just things to be learned by older students; they are also an essential part of elementary school mathematics.
8. There are three kinds of reasoning used in understanding mathematical ideas and solving mathematical problems: inductive, deductive, and intuitive reasoning.
9. Communication is an essential aspect of mathematics—being able to communicate with yourself as you learn mathematical ideas and solve problems, and being able to communicate with others. Tools to improve communication include diagrams, vocabulary, and symbols.
10. There are many aspects to being able to develop mathematical connections: making connections between the problem and what is inside your head, connecting new concepts to old concepts, making connections among different concepts,

connecting different models for the same concept, connecting conceptual and procedural knowledge, and making connections between school mathematics and “real life” and between mathematics and other disciplines.

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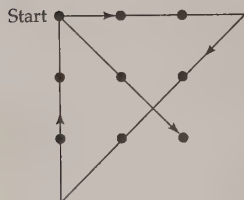
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Fundamental Concepts

- ◆ 2.1 Sets
- ◆ 2.2 Algebraic Thinking
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In this chapter, we will explore three topics that occur throughout elementary school mathematics: sets, functions, and numeration. However, before we explore sets, we will touch on each of these three topics.

When we talk about a herd of deer, a bunch of carrots, an army of ants, a band of jackals, or a pack of lies, we are talking about a *set* of objects that have something in common. In mathematics, we use set language in many situations. For example, we talk about the set of even numbers or the set of common denominators of 20 and 30. Many mathematical concepts and operations are defined in set language, and set language is often useful when we are talking about mathematical situations.

In many situations, both in daily life and in people's work lives, one set is *functionally* related to another set. The relationship may be simple—for example, the relationship between weight and postage—or it may be complex—for example, the relationship between income and tax.

The base 10 *numeration system*, which most people use without thinking about it, is one of the greatest inventions in human history. The analogy to a guitar is useful. Just as one can learn a few chords and memorize a few songs, one can compute with virtually no understanding of the structure of our numeration system (base, place value). However, a deeper understanding of how the guitar works enables a person to play more complex songs and even to adapt songs. A deeper understanding of our numeration system increases a person's ability to make estimates and to solve more complex problems, to understand connections between whole numbers and decimals, to understand *why* the computation procedures work, and much more.

One of the foundations of the NCTM standards is the notion that mathematics ought to make sense to more people. (You may be surprised to hear that

the idea that mathematics is an esoteric subject that only “smart” people can really understand is held almost solely in the United States.) One part of making sense of mathematics is understanding its vocabulary and symbols. Throughout this book, you will encounter many mathematical words and symbols, some of which will be familiar and some of which will be new to you. It is important that you realize that these words (for example, *set*, *intersection*, *proportion*, and *parallelogram*) and symbols (e.g., $\cup$, $+$, $\%$, $\neq$, $\frac{a}{b}$) were created to make it easier for people to discuss mathematical ideas. As you encounter new vocabulary and symbols, you might find it helpful to keep the following quote in mind:

Mathematics is often considered a difficult and mysterious science, because of the numerous symbols which it employs. . . . [T]he technical terms of any profession or trade are incomprehensible to those who have never been trained to use them. But this is not because they are difficult in themselves. On the contrary they have invariably been introduced to make things easy. So in mathematics, granted that we are giving any serious attention to mathematical ideas, the symbolism is invariably an immense simplification.¹

When you have finished this chapter, it might be useful to look back at the symbols introduced in the chapter. To what extent do the symbols make things easier? Is it important for the average person to know the concepts, or are they needed only by some people in some occupations? What do you think?

SECTION

2.1

SETS

WHAT DO YOU THINK?

- How do we use and apply set concepts in everyday life?
- Do we have to use circles to make Venn diagrams?

When mathematics educators discuss how much mathematics a teacher needs to know in order to teach well, there are two extreme positions. Those at one extreme argue that the content of this course should consist only of those concepts that the teachers will actually use with their future students. Those at the other extreme argue that the content of the course should be rigorous so that the teachers will know the mathematics at a much higher level.

My own position is somewhat in the middle. I believe that future teachers will generally be more motivated to learn concepts if they can see that their future students will work with these concepts too. I also feel that the teachers' knowledge needs to be at a higher level than they will teach at, for three reasons. First, if you have a strong understanding of the mathematical ideas, you can be more flexible in your lesson planning and in classroom discussions. Second, you will see more of the connections between concepts. Third, if you have a sense of where your students are headed, you can serve them better. That is, if an elementary teacher knows the concepts that middle school and high school teachers will build on, then the transition from elementary school mathematics to middle and high school mathematics will not be as abrupt as it has been for all too many students.

Sets as a Classification Tool

With this context, let us now look at why you might need to have some knowledge of various set concepts in order to teach more effectively at the elementary level. Children use set ideas in everyday life as they look for similarities and

¹ *Introduction to Mathematics* (New York, 1911), pp. 59–69, cited in Robert Moritz, *On Mathematics* (New York: Dover Publications, 1914), p. 199.



CHILDREN

Learning theorists tell us that a large part of the cognitive development of young children is driven by classification, and classification involves the idea of sets. For example, certain people belong in the set called "family." Most young children divide the human race into two sets: good people and bad people. Children grapple with the defining characteristics of sets; for example, there are many different-looking objects that are called chairs.

differences between sets; for example, they want to know why lions are in the cat family and wolves are in the dog family. Children also look for similarities and differences within sets; for example, they look within a set of blocks for the blocks that they can stack and the blocks that they can't.

Whether we realize it or not, we are classifying many times each day, and our lives are shaped by classifications we and others have made.



INVESTIGATION 2.1 Classifying Quadrilaterals

Without classifying various objects and ideas, it would not be possible to have mathematics. This introductory investigation will help you to connect set language and concepts to other, more concrete mathematical ideas. Look at the eight shapes below. How can you classify these shapes into two groups so that each group has a common characteristic? Give a name to each group if you can. Work and then read on. . .



DISCUSSION

There are many possible answers to this question. Let us examine some common ones.

One answer:

Those shapes with four sides (quadrilaterals)



Those shapes with five sides (pentagons)



Another answer:

Those shapes in which all sides are equal



Those shapes in which not all sides are equal



Another answer:

Those shapes with at least one right angle



Those shapes with no right angles

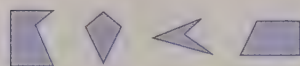


Another answer:

Those shapes in which opposite sides are equal (parallelograms)



Those shapes in which opposite sides are not equal



Defining Sets

In Investigation 2.1, we began with a **set**, which is a collection of objects, and classified that set into smaller groups (*subsets*) having certain common features. In general, a **subset** is a set that is part of some other set. (A more precise definition will be given later.) Some subsets have names—for example, the subset of the set of shapes in Investigation 2.1 consisting of parallelograms. Other subsets are well defined but have no names; for example, there is no name for the set of geometric figures with at least one right angle.

We speak of individual objects in a given set as **members** or **elements** of the set. The symbol $\in$ means “is a member of.” The symbol $\notin$ means “is not a member of.” For example, if E is the set of even numbers, then $4 \in E$ but $3 \notin E$.

Describing Sets

There are three different ways to describe sets:

1. We can use words.
2. We can make a list.
3. We can use *set-builder notation*.

Words and lists In many cases, one of these representations is simpler or easier than the others. Let us examine some important mathematical sets and how we can describe them.

The first set of numbers that young children learn is called the set of **natural numbers**. We can use words to describe this set, or we can describe this set with a list.

N is the set of natural numbers or counting numbers.

$$N = \{1, 2, 3, \dots\}$$

We use braces to indicate a set. The three dots are referred to as an *ellipsis* and are used to indicate that the established pattern continues indefinitely.

At some point, children realize that zero is also a number, and this leads to the next set: the set of **whole numbers** (W), which we can describe with words or with a list:

W is the set of positive whole numbers and zero.

$$W = \{0, 1, 2, 3, \dots\}$$

Later, children become aware of negative numbers, so we have the set of **integers** (I):

$$I = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Set-builder notation Another important set is the set of **rational numbers** (Q), which we can describe in words:

Q is the set of all numbers that can be represented as the ratio of two integers as long as the denominator is not zero.

We cannot represent this set by making a list. Why is this? . . .  In this case, and in many other cases, it is actually easier (as long as you understand the notation) to describe the set using what we call set-builder notation:

$$Q = \left\{ \frac{a}{b} \mid a \in I \text{ and } b \in I, b \neq 0 \right\}$$

This statement is read in English as “ Q is the set of all numbers of the form $\frac{a}{b}$ such that a and b are both integers, but b is not equal to zero.”

Set-builder notation always takes the form $\{x \mid x \text{ has a certain property}\}$. Although elementary school students do not use this notation to describe sets, it is helpful occasionally for you to see where your students will go. In high school, when students are working with more complex ideas, this notation makes communication easier.

Let us now apply these different ways of describing sets.

INVESTIGATION 2.2 Describing Sets

Consider the following set:

$$T = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120\}$$

Try to describe this set with words and with set-builder notation. What do you see as advantages and disadvantages of each of the three ways to describe this set?

If you have difficulty with this question, stop and think what might help. . . . 

You might read the previous section again. You might write down the definitions and examples (or say the important ideas out loud to yourself). Or you might find that discussing this problem with a friend helps. The point here is to develop active learning habits rather than simply reading on to find the answer.

DISCUSSION

Verbal description:

T is the set of all multiples of 10 that are less than 130.

Set-builder notation:

$$T = \{x \mid x = 10n, 1 \leq n \leq 12, n \in N\}.$$

Many students initially have difficulty with set-builder notation. Therefore, let us examine the heart of the description: $x = 10n, 1 \leq n \leq 12, n \in N$.

This notation tells us that we are looking at all numbers that have the form “ $10n$,” i.e. multiples of 10. Because we are not talking about *all* multiples of 10, we have to let the reader know which multiples of 10 are in T . The mathematical phrase $1 \leq n \leq 12$ simply tells us that we are looking for multiples of 10 beginning with $10 \cdot 1$ and ending with $10 \cdot 12$. The last part of the description, “ $n \in N$,” simply lets us know that n must be a natural number.

The table below summarizes some of the advantages and disadvantages of each way of describing the set.

	Advantage	Disadvantage
Verbal description	Relatively simple	Subject to misinterpretation; in this case, some readers may not know whether 10 is included
List	No confusion	A bit impractical in some cases
Set-builder notation	Concise, clear	Can be imposing to a novice

Kinds of Sets

Finite and infinite sets In the course of this book, we will sometimes refer to *finite* and *infinite* sets. If the number of elements in a set is a whole number, that set is said to be **finite**. Some finite sets are small—for example, the set of Nobel Prize winners. Some finite sets are very large—for example, the grains of sand on all the beaches in the world. An **infinite set** has an unlimited number of members.

Consider the following infinite set. Does each way of describing the set make sense?

Verbal description	E is the set of positive even numbers.
List	$E = \{2, 4, 6, 8, \dots\}$
Set-builder notation	$E = \{x \mid x = 2n, n \in N\}$

Well-defined sets Thus far, each of the sets we have studied has been a **well-defined set**. That is, there is no ambiguity concerning whether a particular element is a member of the set. For example, 1 is not an element of the set of even numbers, and Virginia is not an element of the set called “states in New England.” However, in everyday life, many sets are not well defined. For example, “the set of large numbers” is clearly not a well-defined set, whereas “the set of whole numbers greater than one billion” is well defined. Similarly, “the set of wealthy people” is not well defined. We can sharpen the description of this set by saying, for example, “the set of people whose net income is over \$100,000.” However, tax experts would still not consider this set to be well defined: If a law were enacted that imposed a tax on people whose net income was over \$100,000, “net income” would need to be more precisely defined. Over the course of this book, we will frequently encounter problem situations in which part of the problem is that the problem is not

MATHEMATICS

Many students still falsely believe that one of the ways in which mathematics is different from other disciplines is that mathematics is “black and white”; for example, an answer is right or wrong, and there is one “best” way to solve a problem. With respect to sets, historically, a set is a set only if it is well defined. In some of the newer fields of mathematics, this black and whiteness is breaking down. In some fields, mathematicians speak of *fuzzy sets* and of the probability that a certain element is or is not in that set.

well defined; in some cases, the set (of numbers or objects) in the problem is not well defined.

Defining Subsets

The notion of subset is critical because many practical applications of sets involve subsets. What do you think a subset is? Write your current definition of a subset and then read on. . . . 

One informal definition of a subset is that a set is a subset of another set if it has fewer elements. Although this informal definition of subset makes sense, it presents problems in some situations. Thus, we give a more formal definition of subset:

A set X is a **subset** of a set Y if and only if every member of X is also a member of Y .

The symbol $\subseteq$ means “is a subset of.” Thus, we say $X \subseteq Y$. On the other hand, if a set X is not a subset of a set Y , we say $X \not\subseteq Y$.

There is actually another symbol that we can use when talking about subsets. This symbol ($\subset$) is used when we want to emphasize that the subset is a **proper subset**. A subset X is a proper subset of set Y if and only if the two sets are not equal *and* every member of X is also a member of Y . In the case of finite sets, this means that the proper subset has fewer elements than the given set.

If and only if We use the abbreviation **iff** to stand for “if and only if”; iff essentially means that the statement is true “backwards and forwards.” For example, the definition of subset is equivalent to *both* statements below.

- If X is a subset of Y , then every member of X is also a member of Y .
- If every member of X is also a member of Y , then X is a subset of Y .

Does this definition make sense to you? If it does, that’s great. If it doesn’t, you need to find out why and take steps to understand the definition. One way to do this is to connect it to your original sense of subset, which might be “a smaller piece of the larger set.” How would you explain the need for the formal definition of *subset*?



BEYOND THE CLASSROOM

Sets and subsets occur often in the business world. In fact, databases are closely connected to the idea of sets. In my office, I have a collection of articles on teaching mathematics. Let’s say one article is entitled “Using Cooperative Learning to Teach Fractions.” Physically, I have to store that article in one file folder—the folder called Cooperative Learning or the folder called Fractions or the folder called Elementary Math Methods. This presents a problem, for I wind up losing

articles. I could make copies and have a copy in each folder, but that is costly and wasteful. With a database program, I can enter the article in a file and use the following descriptors: cooperative learning, fractions, and elementary math methods. When I want to review my resources on cooperative learning, I give a command to the computer saying, in the language of this section, “Look in the set of teaching resources and find the subset ‘cooperative learning.’”

INVESTIGATION 2.3 How Many Subsets?

This investigation serves several purposes. It opens the idea of families of subsets, it shows how subsets might be useful, and it provides an opportunity to develop problem-solving tools.

Let's say that you and your friends decide to go out and get a large pizza. Let T represent the set of toppings that this restaurant offers:

$$T = \{\text{onions, sausage, mushrooms, peppers}\}$$

List all the possible different combinations of pizza that you could order, such as a mushroom and onion pizza. Then read on. . . 

DISCUSSION

The text will contain occasional reminders about the process of problem-solving. In this investigation, we focus on understanding the problem and looking back. Before you made your list, did you make sure you understood the question? After you make your list and before reading on, take a few moments to look back: How can you check your solution to make sure you didn't miss any combinations? . . . 

STRATEGY 1: List all the combinations systematically

Begin with all combinations that involve onion, from the simplest to the most complex, and then do the same for each of the other toppings (see Table 2.1).

TABLE 2.1

Onion	Sausage	Mushroom	Pepper
o	s	m	p
o, s	s, m	m, p	
o, m	s, p		
o, p	s, m, p		
o, s, m			
o, s, p			
o, m, p			
o, s, m, p			

If you didn't use this strategy, do you understand it? If you cover up the table, can you reconstruct it? If you can't, then chances are you don't really understand this strategy.

There are many patterns within Table 2.1. How many can you see? How would you describe them so that someone who didn't see any patterns could see them from your description?

STRATEGY 2: List all the combinations using another system

Begin with all the ways to have one topping, then all the ways to have two toppings, etc. (see Table 2.2).

If you didn't use this strategy, do you understand it? If you cover up the table, can you reconstruct it? If you can't, then chances are you don't really understand this strategy.

There are many patterns within Table 2.2. How many can you see? How would you describe them so that someone who didn't see any patterns could see them from your description?

TABLE 2.2

One of the four	Two of the four	Three of the four	All of the choices
o	o, s	o, s, m	o, s, m, p
s	o, m	o, s, p	
m	o, p	o, m, p	
p	s, m	s, m, p	
	s, p		
	m, p		



MATHEMATICS

Do you remember Pascal's triangle from Chapter 1? It occurs in Tables 2.1 and 2.2. Can you find it?

One of the common mistakes that students make in such problems is to overlook a combination. There are patterns in both of the tables above that could help you to find all the combinations. Write down all the patterns you see and then read on. . . . 

One pattern (in the “Three of the four” column) is that each of the four toppings should be equally represented. Let's say you had somehow omitted the last combination: s, m, p. Looking over the three combinations you had found, you would notice that onion occurred *three* times but that sausage, mushroom, and pepper each occurred only *two* times. This observation itself names the combination you had missed: sausage, mushroom, and pepper.

Mathematically, there are 16 possible subsets of a set containing four elements. However, if you look at either of the tables above, there are only 15 subsets. What is the missing subset? If you haven't noticed it yet, stop and think before reading on. . . . 

The Empty Set

The missing subset is plain pizza. How would we represent this subset using set notation? Recall that the subset called “mushroom pizza” can be represented as {m}, and the pizza with everything on it can be represented as {o, m, p, s}. Write your thoughts before reading on. . . . 

One way is to put nothing inside the brackets { }. We also use the following symbol to represent a set that is empty: $\emptyset$. We use the terms **empty set** and **null set** interchangeably to mean the set with no elements.

This pizza problem also illustrates two mathematical statements that often baffle students and that will be left as exercises:

- Every set is a subset of itself.
- The empty set is a subset of every set.

Equal and Equivalent Sets

In many situations, the relationship between two sets is important. Let us examine two relationships that will come up occasionally.

Two sets are said to be **equal** iff they contain the same elements. For example, $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$.

Two sets are **equivalent** iff they have the same number of elements. More precisely, two sets are equivalent if their elements can be placed in a **one-to-one correspondence**. In such a correspondence, an element of either set is paired with exactly one element in the other set. We use the symbol $\sim$ to designate set equivalence. For example, $\{\text{United States, Canada, Mexico}\} \sim \{1, 2, 3\}$.



HISTORY

Earlier in this century, anthropologists discovered a tribe in New Guinea that did not have numbers for large amounts. For example, tribe members had no way of counting how many warriors went off to war. Without numbers, how do you think the tribe was able to determine how many warriors, if any, had been killed? The solution involved one-to-one correspondence. Before leaving, each warrior would place a large stone in the center of the village. When he returned, he would take one stone out of the pile. The number of stones remaining told the village how many warriors had not returned.

This definition appears rather formal and intimidating to many students, who exclaim, “Why can’t you just say that two sets are equivalent if they have the same number of elements?” The answer is that in most cases, this less formal description of equivalence works.

Equivalence and counting One reason for defining sets in terms of a one-to-one correspondence has to do with the difficulty young children have counting objects accurately. Watch young children trying to count the number of objects in a collection. Initially, they do not realize that each number has to match one and only one object, and so they can count a set several times and arrive at a different number each time! At some point, they realize (and you can see it in their pointing) that there must be a one-to-one correspondence between their words and the physical objects. Thus, although our definition of equivalence appears unnecessarily formal to many students, it reflects what children actually go through in learning to count objects accurately.

Venn Diagrams

Let us now explore applications of set theory. One way to represent sets is to use **Venn diagrams**, which are named after John Venn, the Englishman who invented these diagrams to illustrate ideas in logic. You have probably seen Venn diagrams used in other contexts, and we used them in Chapter 1.

Several years ago, I was teaching a graduate course in cooperative learning. One teacher explained how she had used the Venn diagram in Figure 2.1 to help her students understand the similarities and differences between butterflies and moths. In this Venn diagram, one region represents the set of moths’ characteristics, another region represents the set of butterflies’ characteristics, and the overlapping region represents the set of characteristics common to both.

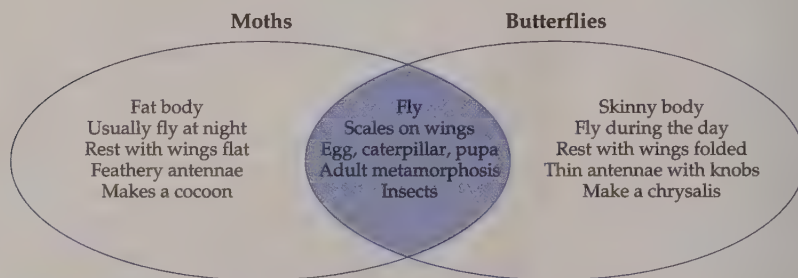


FIGURE 2.1

Operations on Sets

We will use Venn diagrams as we examine three operations on sets: intersection, union, and complement. Just as a doctor’s operation consists of something the doctor does to a patient, a mathematical operation consists of something we do to a set of objects. In Chapter 3, we will examine the mathematical operations of addition, subtraction, multiplication, and division.

When we perform operations on sets of objects, it is often useful to refer to the set that consists of all the elements being considered as the **universal set**, or the **universe**, and to refer to it as U . We represent U in the Venn diagram with a rectangle.

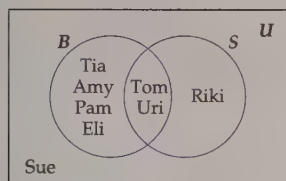


FIGURE 2.2

In the following discussions, we will let U be the set of students in a small class:

$$U = \{\text{Amy, Uri, Tia, Eli, Pam, Sue, Tom, Riki}\}$$

We begin with two subsets of U :

$$B = \{\text{students who have at least one brother}\}$$

$$S = \{\text{students who have at least one sister}\}$$

Figure 2.2 represents the students in this hypothetical class.

Before we discuss the various operations, does the diagram make sense to you? If you were in the class, where would your name appear? . . .

Intersection We can group this class of eight students into various subsets. How would you describe the subset consisting of Tom and Uri? Think and then read on. . . .

One way to describe this subset is “Those students who have at least one brother and at least one sister.”

Mathematically, we call this subset the *intersection* of sets B and S . In mathematical language, we say that the **intersection** of two sets B and S consists of the set of all elements common to both B and S .

We represent the intersection of B and S by shading it (see Figure 2.3).

Using set-builder notation, we write

$$B \cap S = \{x \mid x \in B \text{ and } x \in S\}$$

The symbol $\cap$ is used to denote “intersection.”

In Figure 2.1, the characteristics common to both moths and butterflies are listed in the intersection area. Connecting the concept of intersection to previous notation, we can say $\text{Tom} \in B \cap S$ (that is, Tom is a member of the set of students who have a brother and a sister), and we can say $(B \cap S) \subset U$ (that is, the set of students who have a brother and a sister is a subset of the entire class).

Union Let us examine another subset of the class: {Tia, Amy, Pam, Eli, Tom, Uri, Riki}.

How would you describe this subset in everyday English? Think and then read on. . . .

There are actually several ways to describe this subset:

- Those students who have at least one brother or sister.
- Those students who have at least one sibling.
- Those students who are not an only child.

Mathematically, we describe this subset as the *union* of sets B and S . In mathematical language, we say that the **union** of two sets B and S consists of the set of all elements that are in set B or in set S or in both sets B and S .

We represent the union of B and S by shading it (see Figure 2.4).

Symbolically, we write

$$B \cup S = \{x \mid x \in B \text{ and/or } x \in S\}$$

The symbol $\cup$ is used to denote “union.”

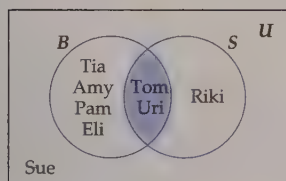


FIGURE 2.3

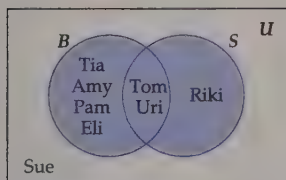
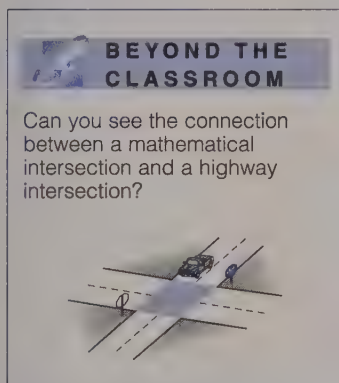


FIGURE 2.4



MATHEMATICS

If we denote the universal set as the set of integers, then the complement of the set of even numbers is the set of odd numbers. Do you see why?

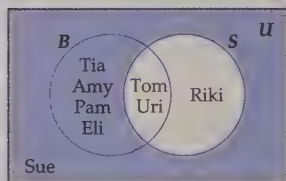


FIGURE 2.5

Connecting the concept of union to previous notation, we can say $\text{Tom} \in B \cup S$, and we can also say $B \subset (B \cup S)$. Do you see why?

Complement Let us examine another subset of the class. Look back to Figure 2.2 and consider this subset of the class: {Tia, Amy, Pam, Eli, Sue}. How would you describe this subset in everyday English? Think and then read on. . .

One way to describe this subset is, "Those students who have no sisters." We describe this subset as the *complement* of set S . In mathematical language, the **complement** of set S consists of the set of all elements in U that are *not* in S .

We represent the complement of S by shading it (see Figure 2.5).

Symbolically, we write $\bar{S} = \{x \mid x \notin S\}$. We represent the complement of a set by placing a line over the set's letter.

Some people understand complement better if they think of the complement of S as "not S "—that is, all elements that are not in set S .

INVESTIGATION 2.4 Finding Information from Venn Diagrams

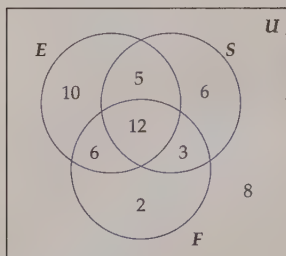


FIGURE 2.6

The owners of Marksalot (a company that makes all sorts of markers) believe that healthy, happy employees are productive employees, so they had a consultant conduct an anonymous survey about employees' exercise habits, their smoking habits, and one particular aspect of their eating habits. Unfortunately, the consultant quit before the job was done (she got a teaching position). Before quitting, she gave Marksalot a Venn diagram (Figure 2.6) that represents the survey results.

U = the employees of Marksalot

E = the set of people who exercise regularly

S = the set of people who do not smoke at all

F = the set of employees who average at least five servings of fruits and vegetables each day, the amount recommended by the Food and Drug Administration

Marksalot has asked you to help the company interpret the diagram. Before reading on, take a few minutes to write what conclusions you can make from the diagram. Try to write your conclusions in everyday English. If possible, compare your conclusions with those of a classmate. . .

Now try to answer the four questions below on your own and then check your answers. . .

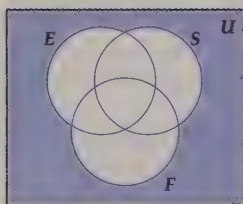


FIGURE 2.7

- How many employees exercise regularly but don't average five daily servings of fruits and vegetables? Represent that subset visually and symbolically.
- Describe the 8 employees who are outside all three circles (see Figure 2.7), first in everyday English and then with symbols.
- Describe the following subset both in everyday English and visually:

$$(E \cap F) \cup (E \cap S)$$
- Represent the results of the survey other than with a Venn diagram.

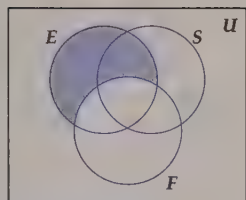


FIGURE 2.8

DISCUSSION

A If we represent the question symbolically, we find that we want to look at the regions that are in E but not in F . There are 15 persons who match the subset “ E but not F ”—that is, $E \cap \bar{F}$. Visually, we have Figure 2.8.

B In everyday English, these people do not exercise regularly, they do smoke, and they average less than five servings of fruits and vegetables a day.

Symbolically, this set is $\bar{E} \cup S \cup \bar{F}$. Because $E \cup S \cup F$ describes the subset of employees who fit into one or more of the healthy categories, the complement of this set describes the subset of employees who fit into none of the healthy categories. This is the subset that will probably require a disproportionate amount of medical services over their lifetimes. We will explore the concepts of probability and proportions in Chapters 6 and 7.

C Many students find themselves intimidated by the apparent complexity of this problem. Looking at an analogous arithmetic problem may be useful. What if you were asked to find $(17 \times 34) + (30 \times 76) + (12 \times 56)$? Take a moment to think about how you might solve this problem and how this problem might be connected to finding $(E \cap F) \cup (E \cap S)$. . .

Most students have little difficulty with the arithmetic problem. You know that you have to multiply the numbers inside the parentheses before you can add. Many mathematical operations, such as multiplication, are **binary operations**—operations that can be performed on only two elements at a time. If you had trouble with Question C, can you solve it now? Try to do so before reading on. . .

Applying the idea of binary operations, we first find $(E \cap F)$ and $(E \cap S)$, then we find the union of those two sets. See Figure 2.9.

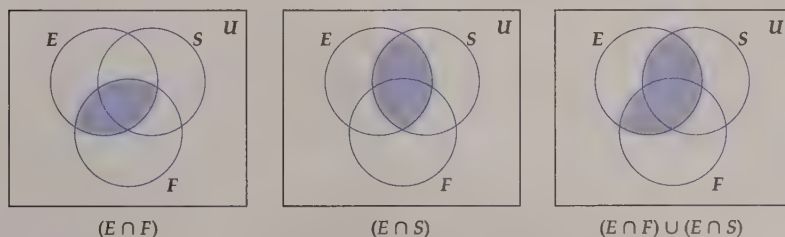


FIGURE 2.9

One description of this set in everyday English is “Those employees who exercise regularly and who also don’t smoke and/or who average at least five servings of fruit and vegetables each day.” Some readers find this description to be awkward. Using the word *subset*, we can describe this group more clearly: we are talking about that subset of regular exercisers who also don’t smoke and/or who eat at least 5 servings of fruits and vegetables each day.

If this explanation helped and you want to do more problems to see whether it does make sense, try some problems in the Exercises for which answers are at the back of the book. One note of caution: Being able to “get the answer” is not the same as “understanding the process.” The larger goal of this book is for the mathematical concepts and operations to make sense.

D Another way to represent the results of the survey is to list the numbers of employees in each of the eight different regions created by the Venn diagram:

- 33 people checked E

- 23 people checked F
- 26 people checked S
- 18 people checked E and F
- 17 people checked E and S
- 15 people checked F and S
- 12 people checked all 3
- 8 people checked none

We could also represent the results with a bar graph. If you don't see this possibility, look at Exercise 23.

Venn Diagrams as a Communication Tool

People other than mathematicians often use Venn diagrams in the same sense that “a picture is worth a thousand words.” *Mathematics in Science and Society*, a book for elementary and middle school mathematics teachers, uses Venn diagrams to discuss teaching methodology. In the introduction to one of the chapters in the book, the authors make a teaching point: “This section is devoted to three valuable teacher tools: laboratory approaches, concrete materials and small groups of students. All three of these may—and quite often do—appear in the same lesson.” The authors then use Venn diagrams to illustrate the kind of lesson they would be discussing. Try to describe, in everyday English, the kind of lesson represented in Figure 2.10. Then read on. . . .

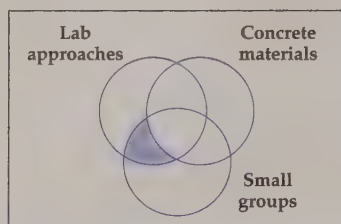


FIGURE 2.10

Source: Mathematics Resource Project, *Mathematics in Science and Society* (Palo Alto, Calif.: Creative Publications, 1977)

They would be describing a lesson in which the students would be doing a math lab in small groups but would not be using concrete materials.

Summary

In this section, we have investigated those concepts from set theory that are most likely to occur in elementary mathematics. Although you are not likely to use these ideas in their formal sense, you will find that you encounter sets and subsets both in elementary school and in everyday life. Most sets in everyday life are finite, but many mathematical sets are infinite. Both in everyday life and in mathematics, there is confusion if a set is not well defined or clearly described. The notion of equivalent sets is one that early childhood educators see young children grapple with as they seek to make sense of relationships. Venn diagrams are used both in elementary school and in other places to describe relationships between various sets. These relationships involve the intersection, union, and complements of sets.

EXERCISES 2.1

- Rewrite the following statements using mathematical symbols:
 - 0 is not an element of the null set.
 - 3 is not an element of set B .
- Rewrite the following statements using mathematical symbols:
 - The set D is not a subset of the set E .
 - The set A is a subset of the set U .
- Sets can be described in three ways: verbally, by listing the elements, or with set-builder notation. In each of the statements below, a set has been described verbally. Either describe the set in the other two ways or explain why it would be impossible to do so. Then explain which of the three descriptions you think would be most useful and why.
 - The set of letters in the word *elementary*.
 - The set of countries in Europe.

- c. The set of prime numbers less than 100.
 d. The set of fractions between 0 and 1.
 e. The set of students in your class.
4. Which of the sets in Exercise 3 are not well defined? Re-define those sets so that they are well defined.

5. Let U be the set of all colors and S be the subset {red, orange, yellow, green, blue, violet}.

For the following questions, place the appropriate symbol in the blank: $\subset$, $\not\subset$, $\in$, $\notin$

- a. $S \underline{\hspace{1cm}} U$ b. red $\underline{\hspace{1cm}} U$
 c. {magenta} $\underline{\hspace{1cm}} U$ d. {green, blue} $\underline{\hspace{1cm}} S$

Which of the following are true? Briefly explain your answer.

- e. $S \subseteq U$ f. red $\subseteq U$
 g. gray $\in S$ h. {green, blue} $\subseteq S$

6. Fill in the most appropriate symbol for each of the following: $\subset$, $\not\subset$, $\in$, $\notin$. Briefly justify your choice.

- a. $3 \underline{\hspace{1cm}} \{1, 2, 3\}$ b. $\{3\} \underline{\hspace{1cm}} \{1, 2, 3\}$
 c. $\{1\} \underline{\hspace{1cm}} \{\{1\}, \{2\}, \{3\}\}$ d. $\{a\} \underline{\hspace{1cm}} \{a, b, c\}$
 e. $\{ab\} \underline{\hspace{1cm}} \{a, b, c, d\}$ f. $\{\} \underline{\hspace{1cm}} \{1, 2, 3\}$

7. a. How many subsets does $A = \{p, i, c, k, l, e\}$ have?
 b. Can you make a generalization about the relationship between the number of elements in a set and the number of subsets? *Hint:* Look at Investigation 2.3.

8. Can you apply what you learned in Investigation 2.3 to answer the following question? There are 6 members on the Student Council. A committee consisting of 2 members is to be made. How many different committees are possible?

9. Justify the following statements, as though you were talking to a student who had not read this section.

- a. Every set is a subset of itself.
 b. The empty set is a subset of every set.

10. Make a Venn diagram with four circles. Use a compass or another device to make good circles. How many distinct regions are there in this diagram? What patterns do you notice in this diagram?

11. Find a Venn diagram from a newspaper, magazine, or elementary school mathematics book. Describe the Venn diagram in words.

12. Let $U = \{x \mid x \text{ is an American}\}$

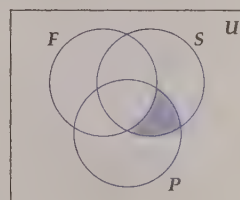
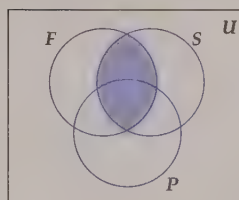
$$F = \{x \mid x \text{ is a female}\}$$

$$S = \{x \mid x \text{ is a smoker}\}$$

$$P = \{x \mid x \text{ has a health problem}\}$$

- a. Represent the following description with a Venn diagram and with symbols: An American nonsmoking healthy female.
 b. Represent $F \cap (\bar{S} \cup P)$ with a Venn diagram and in everyday English.

- c. Represent $(F \cup P) \cap \bar{S}$ with a Venn diagram and in everyday English.
 d. Convert the information from the left-hand diagram to everyday English and to symbols.
 e. Convert the information from the right-hand diagram to everyday English and to symbols.



13. In the following Venn diagram,

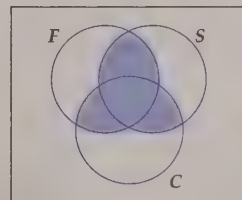
F = the set of students in the film club

S = the set of students in the science club

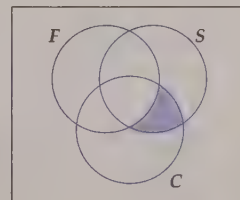
C = the set of students in the computer club

Describe the following sets in English and in symbols.

a.



b.



- c. Describe the following subsets in symbolic language and with a Venn diagram.
 d. Those people who are in the science and film clubs but not the computer club
 e. Those people who are in none of the three clubs

14. Use the sets below to answer parts (a) through (f):

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$$

A : the numbers in U that divide 12 with no remainder

B : the numbers in U that divide 15 with no remainder

C : the numbers in U that divide 20 with no remainder

- a. Make a *clear* Venn diagram showing the sets U , A , B , and C .
 b. Represent the following subset with symbols: $\{7, 8, 9, 11, 13, 14, 16, 17, 18, 19\}$.
 c. Represent the following subset in everyday English and with symbols: $\{2, 4\}$.
 d. Represent the following subset in everyday English and with a diagram: $\overline{A \cap B}$.
 e. Represent the following subset in everyday English and with a diagram: $A \cup B$.

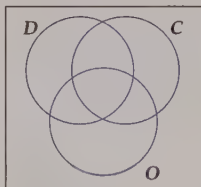
- f. Represent the following subset with a diagram and with symbols: Those numbers that divide 12 or 15 with no remainder.

15. An elementary teacher has asked her students to place their names in the region that represents their answers to the question "Which pets live in your home?"

D = the set of students who have at least one dog

C = the set of students who have at least one cat

O = the set of students who have a pet that is not a dog or a cat



Describe the following sets in everyday English and then with a diagram:

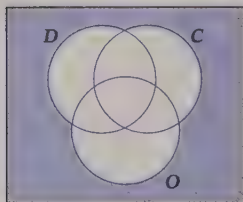
- a. $C \cap D$ b. $\overline{D \cup C}$ c. $C \cap O \cap D$

Describe the following sets in symbols and then with a diagram:

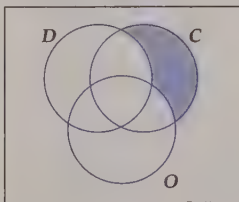
- d. Students who have dogs but not cats
e. Students who have at least one pet
f. Students who have other pets but neither cats nor dogs

Describe the following sets in everyday English and then with symbols:

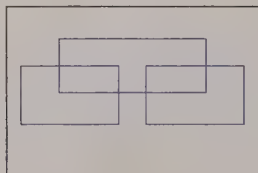
g.



h.



16. Why do you think we use circles instead of squares or triangles or other shapes when we make Venn diagrams? For example, is the representation below equivalent to the standard Venn diagram with three overlapping circles? Explain your answer as though you were talking to a fellow student who does not understand.



17. Recall the different kinds of sets of numbers described earlier: natural numbers, whole numbers, integers, and rational numbers. Make a Venn diagram to represent how these sets are related.

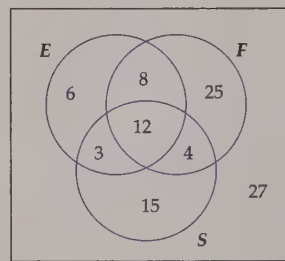
18. Pollsters often ask people's opinions. Politicians want to know how their position on an issue is viewed by particular constituencies—for example, by young voters, by African Americans, by women, by those who belong to the Sierra Club. Decisions about policies are often made on the basis of this polling information. Let's say that a public opinion survey was conducted to determine how much support there was for the president's policies. People were asked three questions:

Do you support the president's economic policy?

Do you support the president's foreign policy?

Do you support the president's social policy?

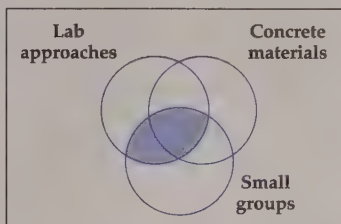
Let E , F , and S denote the sets of persons responding yes to the first, second, and third questions, respectively. The results of the survey are shown in the Venn diagram below. The numbers represent the percent of respondents.



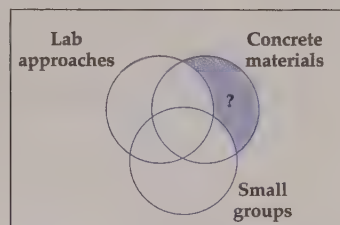
- What percent agree with his economic policy?
 - What percent agree with just one of his policies?
 - Describe the subset that would be represented by $F \cup (E \cap S)$, either in everyday English or by shading the appropriate portion of the diagram.
 - If the president could make one single region of the Venn diagram larger (that is, make it have more members), which would it be? Why?
19. In a group of 120 students, 75 know how to use a Macintosh, 65 know how to use an IBM-compatible computer, and 20 do not know how to use either kind.
- How many students know how to use both kinds of computers?
 - Are the sets in this problem well defined or not? Justify your response.
20. An advertising firm found that a certain ad that ran on both radio and TV was only heard on the radio by 21 percent of the people and was only seen on TV by 33 percent of the people. Just 10 percent of the population both heard the ad on the radio and saw it on TV.
- What percent of the people in the area has neither seen nor heard the ad?
 - What percent of the people in the area only heard the ad on radio or only saw the ad on TV?

21. Refer to the discussion about the book *Mathematics in Science and Society* in the last subsection.

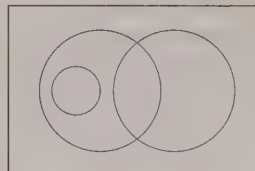
a. What would be the focus of the authors' discussion following the Venn diagram below? Describe your answer in everyday English.



b. The Venn diagram below appeared in the same book.

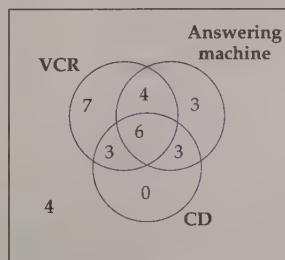
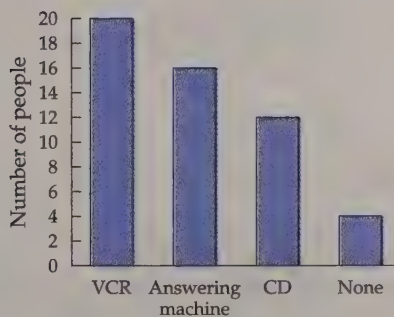
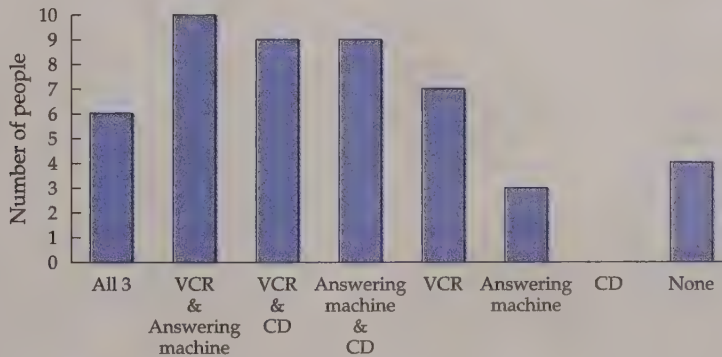


22. Make up a situation for which the following Venn diagram is appropriate.



23. I was curious about the number of high-tech machines that my friends have, so I asked them if they had a VCR, an answering machine, and/or a CD player. I have represented the results of my survey with a Venn diagram and with two different bar graphs.

Number of People Who Own Various Machines



- a. Which of the three graphs would you choose as the most useful? Why?
 - b. Jack thinks there is a mistake in the bar graph at the top. Assuming that the Venn diagram is accurate, do you agree with Jack? If you do, explain the mistake. If you don't, explain why, as though you were trying to convince Jack.
 - c. How many friends did I question?
 - d. From the Venn diagram, what conclusions could you draw about the relationships that characterize ownership of the three items?
24. Consider the following two subsets of U , the set of all people:

I = the set of intelligent people

S = the set of successful people

 - a. By yourself, answer the following question: How would you represent these two subsets with a Venn diagram?
 - b. Discuss your responses with other classmates. What did you learn from this activity? How did the Venn diagram contribute to this learning?

SECTION

2.2

ALGEBRAIC THINKING

WHAT DO YOU THINK?

- What does it mean to say that two sets are functionally related?
- What are some examples of functions in everyday life?
- Can we have functions without numbers?
- What is the reason for developing algebraic thinking in elementary school?

The NCTM repeatedly stresses that from the earliest grades, the curriculum should give students opportunities to focus on recognizing, describing, and extending patterns. Work with patterns helps them to see that “regularity is the essence of mathematics.”² From these explorations, “algebra evolves as a general language to describe patterns in the world around us.”³ The NCTM states that elementary teachers should have sufficient familiarity with the “ideas of algebra”⁴ so that their students will explore and develop, before they enter high school, the concepts and skills that are essential prerequisites to formal algebraic thinking. Students at the elementary school level can explore many kinds of relationships between two variables—for example, the relationship between the slope of a ramp and the distance traveled by a marble. Understanding when relationships are and when they are not functional is one of the cornerstone ideas of algebra. However, algebraic thinking is more than just patterns and functional relationships. When I inform my students that they need to understand algebraic thinking, “hooray” is not the first reaction I hear. Thus, in order for you to want to learn the ideas in this section, I must answer, to your satisfaction, two related questions: (1) What do we mean by algebraic thinking? (2) What does algebraic thinking have to do with elementary school mathematics?

² NCTM, *Curriculum and Evaluation Standards for School Mathematics* (Reston, VA: NCTM, 1989), p. 60.

³ Elizabeth Phillips et. al., *Patterns and Functions, Addenda Series. Grades 5–8* (Reston, VA: NCTM, 1991).

⁴ Arthur F. Coxford, ed., *The Ideas of Algebra, K–12: 1988 Yearbook* (Reston, VA: NCTM, 1998).



HISTORY

The word *algebra* first appears in “Hisab al-jabr walmuqabala” (The Science of Reduction and Cancellations) by Al-Khwarizmi around A.D. 825. It is regarded as the first book to have been written on algebra. Before the development of base 10 and modern notation, solving even very simple equations was difficult. His book was intended to be very practical; he introduced algebra as a means of finding quicker ways to solve problems that were part of everyday life at that time. Over time, *algebra* came to mean the study of equations.

First, we need to address what algebra is and what it isn’t. Algebra is not *just* a set of rules and procedures, though these rules and procedures are helpful when we are solving certain kinds of problems. Algebra is also a set of structures that make it easier for us to understand connections between certain kinds of problems and make it easier to solve problems. Algebra is also the study of relationships among variables—when variables are related in “nice” ways, we call that relationship a function. Finally, algebra can be seen as generalized arithmetic. We will explore each of these conceptions in turn, and I promise to present, in a basic way, the fundamental role or value of each conception and to show how this conception is manifested in the elementary classroom.⁵

Now, let me say a little bit more about what algebra isn’t. Algebra is not a discrete subject that should be taught in ninth grade but rather a strand that should be integrated with other branches of mathematics, including geometry, statistics, and discrete mathematics. In fact, there is a trend in U.S. high schools to reconnect algebra and geometry rather than to teach them as separate courses.

Let us now explore and examine each of these conceptions of algebra in turn.

Algebra as a Set of Rules and Procedures

Think of something you do well now, such as skiing, playing softball, cooking, or knitting. Each of these areas has a set of rules and procedures that you need to learn in order to perform well. Now think of when you first learned to ski, to play softball, to cook, or to knit. You probably remember making mistakes and feeling overwhelmed at times, and you probably remember having fun too.

All this can be true in algebra as well. Now some of you might be thinking “Yeah, except for having fun.” But skiing, playing softball, cooking, and knitting, are not fun for everyone either. The point here is that learning the rules and procedures takes some time, and when most people understand the rules and procedures in a given situation or setting, they generally feel confidence and enjoyment. An important realization is that many of the rules and procedures in algebra are actually extensions of rules and procedures in arithmetic. As I stated in the Preface, seeing connections is a key theme of this book, and algebra has rich connections both to arithmetic and to geometry. We will explore some of these connections in more detail soon, but right now we are going to focus on one aspect of symbol manipulation in algebra that begins in elementary school.

⁵ Several of the ideas in this section are adapted from “Conceptions of School Algebra and Uses of Variables” by Zalman Usiskin, published in *The Ideas of Algebra, K–12*, which was the 1988 Yearbook of the NCTM.

INVESTIGATION 2.5 A Variable by Any Other Name Is Still a Variable

**CHILDREN**

More than one high school teacher has reported a variation of the following response by a student: "If $x = 6$, what did you call it x for?"

**HISTORY**

The notation that we use to express algebraic ideas developed over a period of hundreds of years, and the driving force for inventing this notation and language was to make it easier to solve problems and communicate solutions and ideas. For example, in the 1500s, the expression $x^2 + 2x - 8$ would have been written as $Zp\ 2Rm\ 8$; R for an unknown, Z for zensus (an unknown squared), p for plus, and m for minus.

A crucial idea in algebraic thinking is that of the variable. In each of the following examples, the variables have a slightly different meaning. On a separate piece of paper, write down what each of the variables means, and then read on. . . .



$$C = \pi d$$

$$5x = 30$$

$$\sin x = \cos x \cdot \tan x$$

$$1 = n \cdot (1/n)$$

$$y = kx$$

DISCUSSION

Most people will call the first example a formula. C and d stand for circumference and diameter, whose values vary according to the circle; however, the value of π (3.14) does not vary. The second example is generally called an equation. Although x is the variable, in this case its value is 6. The third example is an example of an identity; it is true no matter what the value of x .

We generally refer to the fourth example as a property; in this case, n is used as a symbol to represent a property that is true for all numbers (except 0); the formal name of this property is the multiplicative inverse property. In the fifth example, the k is used to enable us to represent a family of functions in which the independent variable (x) and the dependent variable (y) are related in a certain way—in this case, a linear relationship.

Table 2.3 illustrates some of the important differences in these examples.

TABLE 2.3

Example	What it is usually called	What the variables stand for
$C = \pi d$	Formula	C and d are concrete quantities. π is a constant.
$5x = 30$ $\sin x = \cos x \cdot \tan x$	Equation Identity	x is the unknown. x is an argument of a function.
$1 = n \cdot (1/n)$	Property	n is a symbol to represent a generalization.
$y = kx$	An equation of a function of direct variation	x is the independent variable, y is the dependent variable, and k is a constant.

My sense is that some of the readers are now asking, "So what does this have to do with elementary school?" Lots! If children simply learn their addition and multiplication facts by rote, such as $6 + 5 = 11$ and $9 \cdot 7 = 63$, their ability to apply this knowledge is limited. The people who write curriculum

CHILDREN

This is a true story. A child comes home from the first day in first grade. His mother asks him what he learned, and he says, "I learned that five plus three equals eight." "That's great," says his mother, at which point the child asks, "Mommy, what does equals mean?"

materials know that, and thus you will also find problems like the following throughout elementary school textbooks:

Fill in the blank: $\square + 6 = 14$ and $\square \times 12 = 96$.

They do this because the goal is for students not just to know facts but also to see relationships between operations. Recall the conception of understanding, presented in Chapter 1, as the quantity and quality of connections that the learner has made. For example, in the first case, "something plus 6 equals 14" is translated as "that something is equal to 14 minus 6." Students need this flexibility, which comes from consistent application in classrooms of the process standards—problem-solving, reasoning, communication, connections, and representation.

There is another aspect of symbol manipulation that has to do with elementary school mathematics. In all of the five examples above, the equations can be manipulated. For example, we can change $C = \pi d$ into $\pi = \frac{C}{d}$. This is not just a manipulation but also a different representation of this idea. This representation highlights the fact that π is the ratio of the circumference of any circle to the diameter of any circle. During elementary school, children will learn lots of relationships that are expressed with algebraic language; for example, $A = LW$, $a + b = b + a$, an average (mean) is obtained by adding all the data and dividing by the number of data, etc.

Since the development of the ability to manipulate algebraic equations and expressions is one goal of high school mathematics, we will not spend much time on manipulation of algebraic expressions in this course, except to remind you that it is important for you to realize that algebra does have its own notation and language. Some of you are more comfortable and familiar with this language than others, (think about the different approaches people took to Investigation 1.1 with the pigs and chickens). What is crucial in elementary school is that children develop the habit of making sense of manipulation of numbers. From the very beginning of school mathematics, children learn to manipulate mathematical symbols. During grade school, they learn to manipulate number sentences and to solve computation problems by breaking them down; for example, 8×6 is double 4×6 —that is, $8 \times 6 = 2(4 \times 6)$. They also learn to use a variety of symbols: $=$, $+$, $-$, $\times$, $\div$, $>$, and $<$, to name but a few.

Algebra as the Study of Structures

All mathematical structures can be described using algebraic notation. Many of these structures are at the heart of the elementary school curriculum:

- Properties of operations—closure, commutative, associative, identity and inverse, distributive, etc.
- Properties of numbers—for example, an "even" number can be divided into two equal halves, whereas an odd number cannot. The algebraic representation of an even number is $2n$, and the algebraic representation for an odd number is $2n + 1$ or $2n - 1$.
- Connections between operations, such as $a - b = a + ^{-}b$

We will explore these structures in more detail in Chapters 3 through 6. Let me offer one example here. I have had the privilege of being in a classroom at the moment when a child has discovered that $2 + 9$ (which is pretty hard for many first graders) is "the same as" $9 + 2$. This discovery is quickly

generalized as “You can switch the numbers when adding.” In this case, the student has discovered a very important structure in the system of whole numbers that we call the commutative property of addition and is stated in its most concise form as $a + b = b + a$ for any whole numbers a and b . In elementary school, coming to understand structures often comes from “playing” with patterns and asking, “What do you see?” As the quote from Mary-Baratta Lorton in Chapter 1 indicates, if children come to expect mathematics to make sense and look for patterns, the transition from arithmetic to algebra will be so much smoother than it was for most readers of this book.

Is it crucial that second graders can say and spell “commutative”? I don’t think so. Is it crucial that they can apply this concept when adding and that they know it is not true when subtracting? Absolutely. Researchers have found that the mistake shown below is common in second grade. In the following example, the student not only had trouble with place value and “borrowing” but also mistakenly assumed commutativity in subtraction. That is, the child assumed that $2 - 6 = 6 - 2$.

$$\begin{array}{r} 52 \\ - 16 \\ \hline 44 \end{array}$$

In high school, the students are expected to learn to do formal proofs and to know the properties in formal language. In elementary school, we want the students to continue to ask “why” and “what if” questions (Why doesn’t the order matter when you add but it does when you subtract? What if you multiply? What if you are dealing with big numbers? Will it work with fractions too?) and we want them to have an understanding of basic structures and be able to apply this knowledge when solving problems. We will revisit this conception of algebra throughout the book, especially in Chapter 3 as we examine the operations of addition, subtraction, multiplication, and division with whole numbers.

Algebra as the Study of Relationships Among Quantities



HISTORY

The word *function* was originally introduced by Leibniz, one of the inventors of calculus, to denote any quantity connected with a curve. Many famous mathematicians had a conception of functions that would be considered erroneous today. For example, Bernoulli regarded a function as any expression made up of a variable and some constants, and Euler regarded a function as any equation or formula involving variables and constants. The familiar notation $f(x)$ that we use today was introduced by Clairaut and Euler in the 1700s.

“The concept of function is an important, unifying idea in mathematics. Functions which are special correspondences between the elements of two sets are common throughout the curriculum. In arithmetic, functions appear as the usual operations on numbers, where a pair of numbers corresponds to a single number, such as the sum of the pair; in algebra, functions are relationships between variables that represent numbers; in geometry, functions relate sets of points to their images under motions such as flips, slides, and turns; and in probability, they relate events to their likelihoods. The function concept also is important because it is a mathematical representation of many input-output situations found in the real world, including those that recently have arisen as a result of technological advances”

(Curriculum Standards, 1989, p. 154).

Many students come to college confusing functions with equations and then find college mathematics very difficult. Success at higher levels of mathematics rests on more than just being able to manipulate equations. In elementary school, children will not investigate functions and other algebraic ideas at a formal level. However, through their investigations, they do need to realize

that relationships often exist between different variables (for example, the area of a rectangle varies according to the length and the width of the rectangle); that patterns and relationships can be represented in various ways (with words, symbols, graphs, and pictures); and that being able to communicate clearly the patterns they see is important.

If students engage in these kinds of investigations during elementary and middle school, then they will see many of the concepts that are developed at the formal level in high school as extensions of what they encountered earlier, not as something that is entirely new and foreign. We will now focus on the concept of a function, which is one of the concepts that underlies virtually all algebraic thinking.

A simple example of a function is the hourly wage. If you know that you are being paid at the rate of \$8.00 per hour, you expect to receive \$320 for working 40 hours, or \$160 for working 20 hours, or \$40 for working 5 hours. In other words, there is a clear, consistent relationship between the variable “hours worked” and the variable “dollars earned.” We can create a table of values for this function (see Table 2.4).

Formally, we define a **function** as a relationship between two sets in which each element of the first set, called the **domain** of the function, is matched with *exactly* one element of the second set, called the **range** of the function. In Exploration 2.4, you found functional relationships between other pairs of variables.

In the wage example, the first column (hours worked) represents the domain, and the second column (dollars earned) represents the range. Technically, what is in these columns is a subset of the domain and a subset of the range, respectively.

It is important to note that not all relationships between two sets are functions. For example, consider the relationship between the set of positive whole numbers and their square roots. In this case, if we ask, “What is the square root of 9?” the answer is $+3$ and -3 . That is, each member of the set of positive whole numbers is matched with two square roots.

When we go outside the classroom to look for instances of functions in situations, we find that their value is often the predictability of the relationship. Consider the following questions:

- If I sell 50 computers, what will be my commission?
- If 75 students sign up for a course, how many sections will be offered?
- What are the consequences when you chew gum in Appleby High School?

If Jack sells 50 computers and receives \$1000, whereas Jill sells 50 computers and receives only \$700, then there is not a functional relationship between the number of computers sold and the commission.

If one department creates three sections when 75 students preregister and another department creates only two sections, then we say that at that college there is not a functional relationship between the number of students and the number of sections of a course.

If Cindy chews gum in one class and gets detention and chews gum in another class and receives no detention, then there is not a functional relationship between chewing gum and consequences.

Inputs, outputs, and function notation Many elementary teachers use a game called “What’s my rule?” to introduce children to the idea of functions. The students give a number (the *input*); then the teacher performs an operation and

TABLE 2.4

Hours worked	Dollars earned
1	8
1.5	12
2	16
3	24
4	32
5	40
40	320

gives them the number associated with that input (the *output*). The students then try to guess what the teacher is doing to the inputs. A game of “What’s my rule?” is given below.

Teacher: What’s my rule?

Student: 3

Teacher: 6

Student: 4

Teacher: 7

Student: I think I know the rule.

Teacher: What is your rule?

Student: You are adding 3. If I give you 10, you will say 13.

Teacher: That’s right!

The teacher is adding 3. Algebraically, we would say $y = x + 3$. We can also use **function notation** to write the rule for adding 3 to any number: $f(x) = x + 3$. This mathematical sentence is read “ f of x equals x plus 3.” That is, $f(x)$ means “function of x .”

Function notation is more precise because it emphasizes the relationship between the **input**, which is another name for a member of a function’s domain, and the **output**, which is the corresponding member of its range. That is, the output is a function of the input. In this particular case of “What’s my rule?” the output $f(x)$ is determined by adding 3 to the input x , and so we have

$$f(3) = 3 + 3 = 6$$

$$f(4) = 4 + 3 = 7$$

$$f(10) = 10 + 3 = 13$$

Let’s play another round. What is the rule? Think and then read on. . . . 

Input	Output
1	4
2	7
3	10

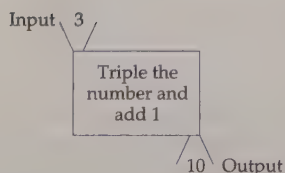
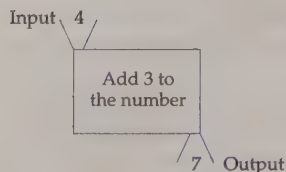


FIGURE 2.11

Using everyday English, we could say that the rule (or function) is to triple the input and add 1. Using function notation, we could say that $f(x) = 3x + 1$.

Elementary teachers often make a “function machine” to illustrate these situations. Function machines for the two situations just presented can be seen in Figure 2.11.

In this game, the domain and range do not have to be sets of numbers. Can you think of an example in which the domain and range are not numbers? Try to do so before reading on. . . . 

Here is one example from Margie Hoey, a second-grade teacher with whom I worked. One student placed some students in one group and some students in another group. The determining factor was whether the student was wearing blue or not. That is, the domain set consisted of the set of students in the

class. The range set had two elements: “is wearing blue” and “is not wearing blue.” For example,

Input	Output
Anastasia	→ Is wearing blue
Benito	→
Cecilia	↗
Damon	↗
Emilia	→ Is not wearing blue

Do you see why this student’s “rule” represents a function? Try to make up more rules that do not involve numbers.

INVESTIGATION 2.6 Baby-sitting

Let us extend our understanding of functions by examining another relationship between the two variables “hours worked” and “dollars earned.” Let’s say Ellen baby-sits and charges \$6 per hour. Describe how you would determine how much to pay Ellen. Determine whether the relationship between the two variables—hours and dollars—is a function. Whether you say it is or is not a function, can you justify your response? Then read on. . . .



TABLE 2.5

h	d
1	6
2	12
3	18
4	24
5	30
⋮	⋮

DISCUSSION

A common response to describing how to pay Ellen is to multiply the number of hours sat by 6. Let us use this situation to introduce several different ways to represent functions, and then we will examine more closely the relationship between time sat and dollars paid.

For purposes of simplicity, let us first look at different representations of this function using only whole-hour amounts.

Tables We can represent this function with a table in which h represents hours and d represents dollars (see Table 2.5). We can then use the table to determine how much money Ellen will make from baby-sitting.

Equations We can represent this function with an equation in which h represents hours and d represents dollars:

$$d = 6h$$

Graphs We can represent this function with a graph (see Figure 2.12).

In some cases, like this one, many people find it more convenient to refer to the input as the **independent variable** and to the output as the **dependent variable**. In this case, we say that the independent variable is the hours baby-sat and the dependent variable is the dollars earned. That is, the number of dollars you earn is dependent on the number of hours you baby-sit.

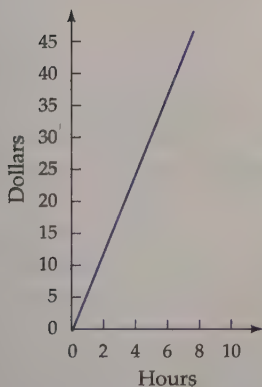


FIGURE 2.12

Ordered pairs We can represent this function as a set of **ordered pairs** in which the first element of each ordered pair represents hours and the second element represents dollars:

$$B = \{(1, 6), (2, 12), (3, 18), (4, 24), \dots\}$$

Mappings Finally, we can represent functions with arrow diagrams. Mathematicians often refer to this representation as a **mapping** of one set onto another set. We shall soon use this representation when we look at different kinds of functions.

h	d
1	$\rightarrow$ 6
2	$\rightarrow$ 12
3	$\rightarrow$ 18
4	$\rightarrow$ 24

A closer look at paying the baby-sitter At first glance, the question of how much to pay the baby-sitter is simple: Multiply by 6. However, let us use the problem-solving strategy “act it out” to examine this problem more closely. For example, what if Ellen baby-sat from 7 to 11:10? How much would you pay her? Think before reading on. . . . 

Some people say \$24. Some people say \$27—they round up to the nearest half-hour. In actuality, different people have different ways of determining how much to pay a baby-sitter. Let us examine the case of a couple, whom we will call the Alomars, who determine how much to pay their sitter in the following way: They round up the time to the nearest half-hour. We could now represent the Alomars’ process for paying the baby-sitter in each of the ways we have just examined. Is the relationship between time sat and dollars earned a functional relationship in the case of the Alomars? How can you justify your answer? Think and then read on. . . . 

Many students find that they need to represent the relationship to help them answer this question. Let us examine one representation more closely. If you were to graph this relationship (using the Alomars’ process), what would the graph look like? Try to make your own graph before reading on. . . . 

The graph is shown on page 110. Take a look at the graph and see whether you can make sense of it before reading on.

If you are having trouble making sense of the graph, consider a few examples. Let’s say the sitter sits for 2 hours and 50 minutes. Using the Alomars’ process, we round this time to 3 hours and pay Ellen \$18. As you can see, Ellen will receive \$18 for all times between 2 hours and 46 minutes and 3 hours. So is this a function?

It is, because for any specific time (input), there is exactly one dollar amount (output) associated with that time.

The graph that represents this baby-sitting function is often confusing to students who see it for the first time. However, it is relatively common in real-world mathematics and is a member of the subset of functions called step functions.

Mathematical modeling As you have discovered in this section and in other mathematics courses, upon closer examination, a problem is often more complicated than it first appears. This is what makes “real-life” problems so challenging. The NCTM has stated that mathematical modeling should be an important part of mathematics education in school. Let us take some time to understand this concept and its importance. We will begin by examining the word *model*. A model of an object is not the object and is often a scaled-down version of the object. We speak of model airplanes, and architects and engineers



HISTORY

René Descartes (1596–1650) invented the Cartesian coordinate system (named after him) to illustrate such relationships on graphs. Legend has it that he got the idea from watching a fly crawl across a ceiling. He realized that a coordinate system could represent the fly’s location with one symbol—that is, the ordered pair.

build a model of a building or a bridge before constructing the actual object. In one sense, a model is a representation of the original object. There are two important features of many models. First, the model contains many of the properties of the original object. Second, the model can be manipulated and studied to help us better understand the (usually larger and more complex) object. A **mathematical model** is a mathematical structure that approximates the features of a situation. A mathematical model can be an equation (or a set of equations), a graph, or some other kind of diagram.

In this case, we have seen that the original straight-line graph turns out not to be a useful or accurate model of the baby-sitting situation. The step graph is a more accurate representation. This process of examining a situation and then developing a model that accurately represents that situation is called **mathematical modeling**. Constructing and interpreting mathematical models is one of the more important uses of mathematics in the real world. Over the course of this book, you will have many opportunities to construct and interpret mathematical models.

INVESTIGATION 2.7 Choosing Between Functions

Jackie has been promoted to chief salesperson of Southside Computers. She has two choices as to how she will be paid. She can receive \$50 for every computer system she sells, or she can receive \$250 a week plus \$25 for every computer system she sells. Looking at the past year, she finds that she has averaged 9.3 sales per week, and she figures that she can do a little better this year. What would you recommend? Work on this problem for a bit and then read on. . . . 

DISCUSSION

You may want to start by plugging in a few numbers—that is, using guess-check-revise. Do you see that if Jackie sells only a *few* computers, she will be better off with the second plan but that the first plan is better if she sells a *lot* of computers? This realization leads to a specific mathematical question: At what point will the two plans give her the same money? Once she can answer that question, she can base her decision on whether she believes she will sell more or less than that amount. Below we see three very different strategies for solving this subproblem: guess-check-revise, use algebra, and make a graph. If you already solved the problem using one strategy, try to use the other strategies now and then read on. . . . 

STRATEGY 1: Guess-check-and revise

This strategy is illustrated in Table 2.6.

TABLE 2.6

Sales	Plan A	Plan B	Reflection/Analysis
0	0	250	Plan A goes up 100; plan B goes up 50. Plan A is still \$200 below plan B. Each time you increase sales by 2, plan A increases \$ <i>more</i> than plan B does.
2	100	300	
10	500	500	

What does the last row mean? Think and read on. . . 

It means that Jackie makes the same amount under both plans when she sells 10 computers a month. If she thinks she will sell more than 10, then plan A is better for her.

STRATEGY 2: Use algebra

Let y = Jackie's weekly salary.

Let x = the number of computer systems she sells.

The equation that represents plan A is

$$y = 50x$$

The equation that represents plan B is

$$y = 250 + 25x$$

Do you understand how both of these equations were constructed?

We have two equations in two unknowns that we can solve:

$$50x = 250 + 25x$$

$$25x = 250$$

$$x = 10$$

STRATEGY 3: Make a graph

Where the two lines cross will show the point at which her earnings under both plans will be equal.

The two lines cross at the point (10, 500) (see Figure 2.13). In other words, the ordered pair (10, 500) is an element of both graphs.

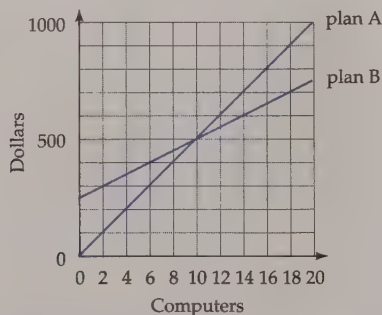


FIGURE 2.13

INVESTIGATION 2.8 Interpreting Graphs of Functions

We see graphs of functional relationships every day in newspapers and magazines. The purpose of this investigation is to examine some of the nuances involved in interpreting graphs.

The graph in Figure 2.14 shows the value of the stock of a company over the course of a year.

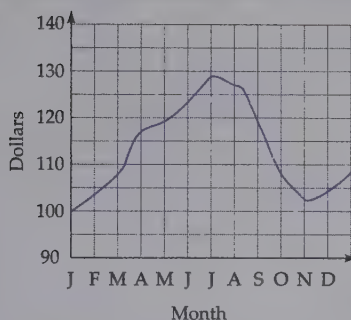


FIGURE 2.14

Write your responses to the following questions. . . . 

- Approximately when was the value of the stock \$110?
- In which month did the stock's value show the biggest decline?
- How much did the value of the stock decrease during the third quarter?

DISCUSSION

Frequently, interpreting graphs requires us to examine the question and the graph carefully. What problem-solving tools did you use in this problem?

A The value of the stock first reached \$110 just to the right of the M standing for March. Many students are not sure whether the line above the M represents the beginning of March or the end of March. If this applies to you, what could you do to make sense of the graph? Think and then read on. . . . 

If we look at the left side of the graph, we can see that the beginning of the year (January 1) is at the beginning of the graph, and so the first vertical line represents the beginning of January; the second vertical line therefore represents the beginning of February; and so on. Therefore, the value of the stock first reached \$110 just after the beginning of March. The graph is not large enough or marked precisely enough for us to determine the exact date on which the stock's value was \$110. However, we can be more precise than simply saying "sometime in March." What do you think? 

A rough estimate is that the value reached \$110 about $\frac{1}{4}$ to $\frac{1}{3}$ of the way from March 1 to April 1. Thus we could say "early March" or "somewhere in the second week of March."

Did you see that the value of the stock actually hit \$110 twice during the year? The second time occurred in the last week of September.

B We can determine the month with the greatest decline by finding all months that show a decline, estimating the value of the stock at the beginning and ending of these months, and determining which month has the biggest difference. Another way to determine the biggest decline is to recall the algebraic concept of slope. From this perspective, we can focus on those months during which the stock's value declined and find the month with the steepest slope. September shows the biggest decline, although August is a close second.

C. Students often encounter two problems with this question. The first is not knowing what is meant by the third quarter. If you didn't know this term, what did you do in order to find out (rather than just reading ahead)? This is important because resourcefulness is an important quality of good problem-solvers and good teachers.

The other problem is that once we find out that the third quarter means from the end of June to the end of September (or, equivalently, from the first of July to the first of October), we must make sure that we locate those points correctly on the graph. The value of the stock at the end of June is about \$127 or \$128, and the value of the stock at the end of September is about \$107 or \$108. Thus, the value of the stock declined about \$20. Do you see why we say about \$20?

If we take the biggest possible difference ($128 - 107$), we get 21. If we take the smallest possible difference, ($127 - 108$), we get 19. Alternatively, we could say that the value of the stock declined between \$19 and \$21 during this period; this response tells the reader that the graph (the data available to us) is not precise enough for us to give an exact answer.

Algebra as Generalized Arithmetic

This conception of algebra is related to algebraic structures and to functions, but having you see the connectedness between algebra and arithmetic is so important that we will explore it in another conception. In Investigation 2.9, we open the discussion of this conception of algebra.

INVESTIGATION 2.9 Triangular Numbers

In Exploration 2.5, you worked with **figurate numbers**, a type of number whose name comes from the shape it represents. Figurate numbers present a rich area for investigation: They exhibit many interesting patterns, they have nice visual representations, so that patterns are readily observable; they are connected to other mathematical ideas; and they can be investigated throughout the elementary curriculum.

Figurate numbers seem to have been invented by the Pythagoreans around the sixth century B.C. The first set of figurate numbers we will explore here is the set of triangular numbers.

Figure 2.15 shows the first four triangular numbers.

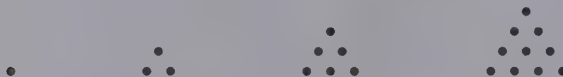


FIGURE 2.15

Write your responses to the following questions. If possible, compare your responses with those of someone else. . . . 

- A.** Assume that a friend who missed the class asks you to describe triangular numbers over the phone. How would you describe them?
- B.** How many dots are in the fifth triangular number? in the 10th triangular number? in the n th triangular number?

DISCUSSION

A. By having you express the concept of triangular numbers in words, we are emphasizing the development of communication in mathematics. There is no

one “correct” description. Chances are that if you gave your description to several people and each of them successfully produced the set of triangular numbers, your description was fine. Many students find that what seems very clear to one person doesn’t make sense to another. That is, a description that seems clear to you may make no sense to someone who has no prior experience with the concept. Many teachers, including me, have experienced this phenomenon with our students.

Consider the following description, which makes perfect sense to me, the teacher. Does it make sense to you? What parts of this description might prove difficult for someone who has never heard of triangular numbers? “Let me explain how to draw the fifth triangular number: Begin by drawing one dot; then underneath it draw two dots so that, if the single dot dropped down, it would be right in the middle of the two dots. The third row consists of three dots, the fourth row of four dots, and the fifth row of five dots. Each row is staggered so that each dot in a row is directly above a dot two rows below.”

B. Determining how many dots are in the fifth, tenth, and n th triangular numbers is an interesting challenge. Were you able to discover a relationship between each triangular number and how many dots constitute that number? If so, what tools helped? If not, follow the discussion below actively.

There are often many patterns that will lead us to understand and state a relationship in its most general form. It turns out that the difference between successive triangular numbers makes an interesting pattern. Look at Table 2.7. Do you understand what each column represents? Think and then read on. . . . 

TABLE 2.7

n	T_n	$T_n - T_{n-1}$	Determining T_n
1	1	—	1
2	3	2	$1 + 2$
3	6	3	$1 + 2 + 3$
4	10	4	$1 + 2 + 3 + 4$
5	15	5	$1 + 2 + 3 + 4 + 5$

HISTORY

Few readers realize that representing the value of the n th triangular number in the form $n(n + 1)/2$ is a very recent development. Modern notation is generally assumed to have originated in the 1500s and much of it did not crystallize in its current form until the nineteenth century. Thus, before 1500, this relationship could only have been stated in words: To find the value of any triangular number, multiply the number of the term that you are dealing with by one more than that number and then divide that result by 2. I don’t know about you, but $n(n + 1)/2$ seems a lot easier to me!

The column at the far left tells us which triangular number we are looking at. The second column tells us how many dots it takes to make that triangular number. The third column simply notes the difference between the number of dots in the triangular number and the number of dots in the preceding triangular number. The last column shows one way of counting the total number of dots in that triangular number. Extend the table for a few more numbers if you wish. How would you describe what happens to the difference as we go along? Think and then read on. . . . 

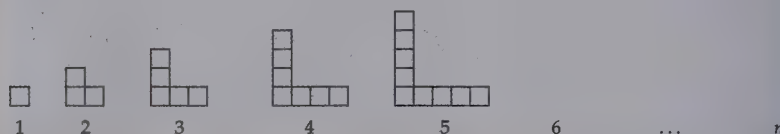
One way to describe what happens is that the difference increases by 1 each time. This is interesting, but it does not lead most students to be able to predict the 10th or the n th triangular number. However, if you noticed the similarity between the sums in the right column and finding the sum of the first 100 numbers (Investigation 1.5), you can find the 10th triangular number and write a rule for determining the n th triangular number. Do you see why?

Using the notation introduced in Investigation 1.5, we can say that the n th triangular number, T_n , is equal to $\frac{1}{2}n(n + 1)$. For example, to determine how many dots make up the 10th triangular number, we simply take one-half of 10 times 11 and get 55.

Let us consider another example so that you can get a deeper understanding of this goal of moving from reasoning with numbers to using letters to symbolize the essential relationship between the two variables.

INVESTIGATION 2.10 Looking for Generalizations

The figure below shows another growth pattern. In this case, I have chosen to represent this pattern with squares rather than dots (the reason will become apparent later). This set of figurate numbers is often called the L numbers, because the shapes resemble the letter L. How many little squares will it take to make the n th figure in this pattern? Work on this and then read on. . .



DISCUSSION

As we discussed in the first chapter, there are often different strategies that can be used to solve a problem and that embody different representations of the problem. I will show two different solution paths that come from representing the problem numerically (with a table) and geometrically (by taking the figure apart). In both cases, the solution comes from seeing a pattern (in the numbers and in the shapes).

STRATEGY 1: Make a table and look for patterns

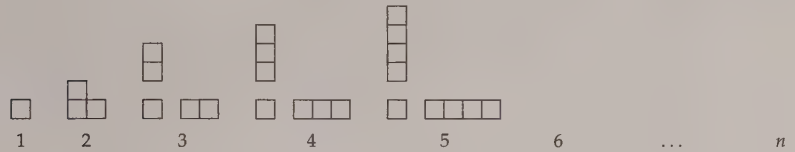
Figure number	1	2	3	4	5	6	7
Number of squares	1	3	5	7	9	11	13

Looking at the relationship between the corresponding numbers in the first and second sets (input, output), many people will quickly see that the output number is always 1 less than double the input number. Thus, the number of squares in the n th L number is simply $2n - 1$. The third row in the table below illustrates this relationship very clearly.

Figure number	1	2	3	4	5	6	n
Number of squares	1	3	5	7	9	11	
Number of squares	1	$2 \cdot 2 - 1$	$2 \cdot 3 - 1$	$2 \cdot 4 - 1$	$2 \cdot 5 - 1$	$2 \cdot 6 - 1$	$2 \cdot n - 1$

STRATEGY 2: Look at the structure

Each of the L shapes can be broken down (decomposed) into two “arms” and a base (or “arm connector”).

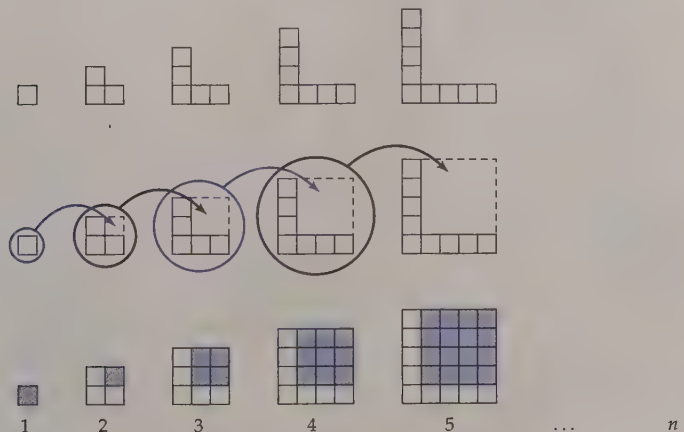


The base is always a single cube. In each case, the arms are the same length, and the length is 1 less than the figure number. That is, the length of each of the arms of the 5th figure is 4 (1 less than 5). Thus, the n th figure will have two arms, each of whose length is $(n - 1)$, plus one square that represents the base. Thus, the number of squares of the n th figure will be equal to $(n - 1) + (n - 1) + 1$, which is mathematically equivalent to $2n - 1$.

One of the characteristics of a worthwhile mathematical task is that it can be extended; in other words, “there is more than meets the eye!” Let’s say we kept a cumulative total of the number of squares as we went along. What would be the total number of squares needed to make n of these L numbers? Think and then read on. . . .

Figure number	1	2	3	4	5	...	n
Number of squares	1	3	5	7	9		
Sum	1	4	9	16	25		

In this case, it is not too difficult to deduce that the relationship between each figure number and the sum of that many L numbers is n^2 ; that is, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, etc. An interesting question here is “Why do you think this happens?” Please cover up the figure below before thinking about this question. Then read on. . . .



I have added dotted lines to the figures in the second row so that you can see how each L number needs a square in order to “fill out” to become a square itself, and that square that is needed just happens to be equal to the size of the square made by the previous L numbers put together. The arrows help you to see the progression.

Now, it is beyond the reach of elementary students to prove formally that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. However, it is not beyond the reach of elementary students to see squares in these shapes. In fact, one of the signs that your elementary classroom is functioning well would be that some of the students are able to make this geometric connection. This goes back to the Baratta-Lorton quote from Chapter 1: “Looking for patterns trains the mind to search out and discover the similarities that bind seemingly unrelated information together in a whole. . . . A child who *expects* things to ‘make sense’ *looks* for the *sense* in things and from this sense develops understanding. A child who does not see patterns often does not expect things to make sense and sees all events as discrete, separate, and unrelated” (*italics added*).

Summary

In this section, we have explored the notion of algebraic thinking. I hope that when someone mentions the importance of algebraic thinking in elementary school, you will have a sense of what this means and will believe that it is important for this kind of thinking to begin in elementary school, not in high school. What all four conceptions have in common is sense making, looking for patterns, and the five process standards. Just as we take off the training wheels when children have become confident at riding a bicycle, a crucial role of elementary teachers is to help students move beyond reasoning with actual numbers to more general reasoning—about relationships, about quantity, and about what symbols mean. In elementary school, it is crucial that students become skilled at seeing and communicating patterns; this in turn helps them to see relationships, which then lead to generalizations, which point to important mathematical structures and ideas. That is why the role of elementary teachers in working with patterns is so important for the children’s future success in mathematics.

There are two other points that bear emphasizing. The notion of multiple representations was introduced in Chapter 1, and you saw its importance in this section in the many different ways to represent functions. As you saw in Investigations 2.7 and 2.10, different representations can lead to the same solution. It is just as important for children to become fluent at translating between representations. The notion of representations is related to the idea of modeling; in many problem situations, we make a model of the situation in order to solve the problem.

The other point is connections. In order to solve many problems, the solver needs to connect the problem to a diagram or table used to represent the problem. It is also important that you, and your future students, realize that arithmetic and algebra are connected. This is so that algebra will be seen not as a totally new subject but as one in which many of the rules are simply rules from arithmetic stated in a more general form. Realization of the connections between algebra and geometry is also crucial. Virtually all algebraic ideas can be represented geometrically, and virtually all geometric ideas can be represented algebraically.

EXERCISES 2.2

- Go to the Web or to journals to find an example in pre K-2, grades 3-5, or grades 4-6 of an exploration that develops the notion of algebraic thinking.
 - Briefly describe the activity/exploration.
 - Briefly summarize the important mathematical knowledge that can come out of this activity.
 - Name one of the process standards that you feel will be developed during this activity, and briefly justify your choice.
- The table below shows the process by which a college department determines how many sections of a course to offer.

Number of students	Number of sections
Less than 6	Course not offered this semester
6-29	1
30-59	2
60-89	3
90-119	4

- Make a graph to represent this process.
 - Is the relationship between the number of students and the number of sections a function? Justify your answer.
- Let's say you have children, you hire a baby-sitter frequently, and you pay the baby-sitter \$6 per hour. Describe how *you* would pay the baby-sitter, first in words, then with a graph. In order to receive full credit, the reader must be able to pay the baby-sitter the same amount that *you* would pay, based on your descriptions.
 - Let's say the entire freshmen class of Wannago High School is going on a field trip. The number of buses needed is a function of the number of people going. To keep matters simple, let's say that the school bus company has buses that can hold 36 passengers and that there will be two adults per bus.
 - How many buses will be needed for 232 students?
 - Represent this function graphically.
 - At the time this book was written, the postage on a letter was determined by the following set of rules:
The first ounce costs 33¢.
Each additional ounce, or fraction thereof, costs 22¢.
 - How much would it cost to mail a letter weighing 6.3 ounces?
 - Draw a graph of this function.
 - Archaeologists can estimate a person's height from the length of the femur (thigh bone). The formula for doing so is

$$H = 2.3L + 61.4$$

where all measurements are in centimeters. Let's say an archaeologist has found a femur of a human, and the bone is 45 centimeters long. How tall do we think the person was at the time of death?

- In Investigation 2.3 in Section 2.1, we found that a set containing 4 elements has a total of 16 subsets.
 - Fill in the blanks in the table that follows.
 - Describe in words the relationship between the number of elements in a set and the number of subsets. Refer to the table.
 - Describe the relationship with an equation.

Number of elements in the set	Number of subsets
1	?
2	?
3	?
4	16
5	?

- Have you ever ridden on a bicycle where the saddle was too high or too low? The height of a saddle can be determined by adjusting it until the rider says, "It feels right at this height," but not all people guess right. Bicycle manufacturers have found that for the average person, the saddle height can be determined by the following formula: $h = 1.08i$, where h represents the saddle height and i represents the inseam.
 - How tall should the saddle height be when you ride a bicycle?
 - If you have a bicycle, measure the saddle height. If it is different from the number derived by applying the formula above, change it and see whether the height is more comfortable now.
- Let's say a beautician charges \$15 per haircut. She also determines that overhead not associated with number of haircuts (rent, utilities, equipment) is \$150 per week. Consider the relationship between number of haircuts and profit. Is this a proportional function? That is, does double number of haircuts mean that profit is doubled?
- Describe a relationship from Chapter 1 that involves a functional relationship between two sets.
- John says that Investigation 1.1 (Pigs and Chickens) does not represent a function because two different inputs can have the same output. For example, (7, 17) and (0, 31) both map to the same number of feet. Do you agree with John? Why or why not?
- Is it mathematically correct to say that one's grade is a function of the amount of time one spends on the course? Justify your response.

13. There is a relationship between the successive squares of triangular numbers. This relationship can be expressed in everyday English as follows: "Select any triangular number and square it. Then take the previous triangular number and square it. Now take the difference between the two numbers."

- Express this relationship in notation.
- Is this relationship a function? Justify your response.

14. Jack and Jill run a catering business. Their fee for catering banquets is \$150 plus \$4 per person.

- Express this relationship with a table.
- Express this relationship with an equation.
- Express this relationship with a graph. What would be a realistic domain for the graph?
- Jack and Jill catered a banquet for which they charged \$462. How many people attended?
- Their goal is to gross \$2000 a month. How many jobs do they need to average in order to make their goal? (There is no single right answer.)

15. My parents used to live in Park Rapids, Minnesota. When I visited them, I flew to Minneapolis and rented a car. It is about 160 miles to Park Rapids. The last time I visited them, I spent three days with them, counting my arrival and departure days. The car rental agency offered two options: \$45 a day plus 30¢ per mile or \$185 for the week. Which do you recommend?

16. Let's say you have a choice between two checking plans. With the first plan, you will be charged \$2 per month plus 15¢ per check. With the second plan, you will be charged \$5 per month regardless of how many checks you write. If you write only a few checks, the first plan is clearly cheaper. Similarly, if you write 200 checks a month, the second plan is clearly cheaper.

- Determine the number x that will enable you to make the following statement: If you write 1 to x checks per month, choose the first plan. If you write more than x checks per month, choose the second plan.
- Determine the number x using a method different from the one you used in part (a).
- Which of the plans are functions: both, only one, or neither? Justify your response.

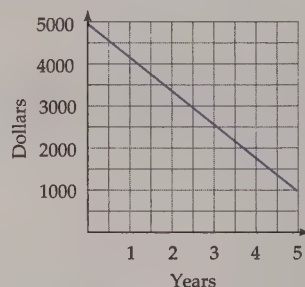
17. There is a functional relationship between the frequency of cricket chirps and the temperature. The rule is to count the number of chirps in one minute, divide by 4, and add 40.

- What temperature does 50 chirps per minute correspond to?
- Without calculating, will 100 chirps per minute correspond to twice the temperature of 50 chirps per minute? That is, does chirping at twice the rate mean that the temperature is twice as high? Explain your reasoning.

- Translate the rule into an equation.
- Draw a graph to represent this relationship.

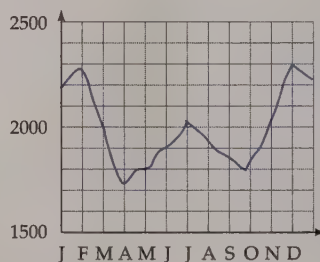
18. Consider the graph below, which shows the depreciation of a copy machine over 5 years.

- What is the value of the machine when it is 2 years old?
- By how much does the machine decrease in value in 1 year?
- Predict when the value of the copy machine will be zero.



19. The graph below shows the value of the Dow Jones Industrial Average over the course of a year.

- Approximately when was the Dow Jones 2100?
- Which month showed the biggest decline?
- How much did the Dow Jones gain during the second quarter (from the end of March to the end of June)?

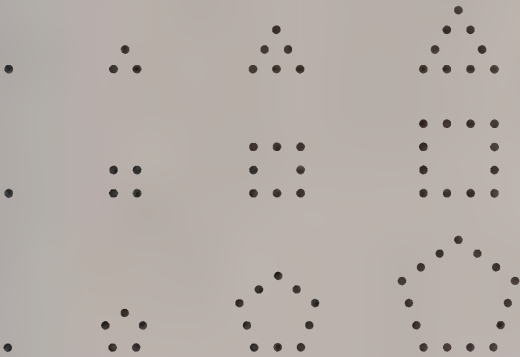


20. I can recall trying to explain to my children that thunder is the sound that lightning makes. It took some time for them to understand this because there is often a delay of many seconds between seeing the lightning and hearing the thunder. Lightning travels at 186,000 miles per second, whereas thunder travels at approximately 750 miles per hour.

Let's say that you see a flash of lightning and 5 seconds later you hear the thunder. How far away did the lightning strike?

21. The set of figurate numbers can be extended to different kinds of numbers. The diagrams following show the

first four trinumbers, quadrinumbers, and pentinumbers, respectively.



- What is the value of the fifth trinumber? the 10th trinumber? the n th trinumber?
- What is the value of the fifth quadrinumber? the 10th quadrinumber? the n th quadrinumber?
- What is the value of the fifth pentinumber? the 10th pentinumber? the n th pentinumber?
- With respect to part (a), describe how you could use what you learned about triangular numbers to help you answer that question.

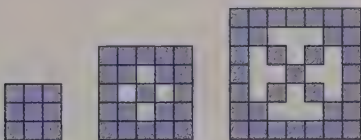
22. Look at the growth pattern below, in which each figure resembles the letter C. How many squares will it take to make the n th figure in this pattern?



23. Look at the growth pattern below. How many squares will it take to make the n th figure in this pattern?



24. Look at the growth pattern below. How many squares will it take to make the n th figure in this pattern?



25. What if we made a pattern by joining pentagons? If the length of each side of the pentagon is 1 unit, then what would be the perimeter of n pentagons joined together?

Note: The perimeter is the distance around the outside of the figure.

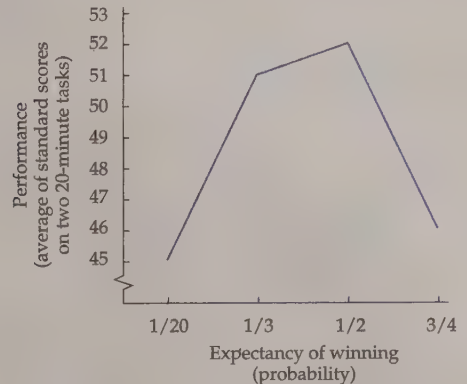


- What is the surface area of a tower of cubes like the one shown at the left below?
- What is the surface area of a tower of cubes like the one shown at the right below?



27. There is a mathematical relationship between motivation and perceived chance of success. Carnival operators use this relationship masterfully. Think of the ring toss, in which you toss rings at bottles. If your ring lands on one of the bottles, you win a prize. If the bottles are too close, a high percentage of people win and the carnival loses money. However, if the bottles are too far away, virtually no one will participate. Therefore, the ideal distance (from the carnival owner's perspective) is determined by a maximization function—that is, the distance for which the value of total ticket sales minus cost of prizes is the greatest.

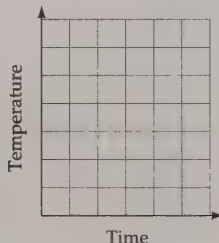
The graph below accompanied a research paper that reported the authors' investigation of motivation. Translate this graph into words. That is, briefly describe what the graph is saying.



28. I usually have a cup of tea by my side in my office at school and in my office at home. For years, I kept my tea in plastic cups with lids so that the tea would cool down more slowly than it would in ceramic mugs. However, it still cooled down, and the plastic mugs are not as pretty as the ceramic mugs. Several years ago, my wife gave me a cup warmer so that my tea would not cool down. I was very grateful. This gift, however, led me to wonder about how well different containers insulate liquids.

Let's say that you have two different cups and you want to know which one will keep your tea or coffee hot longer.

- Design an experiment to answer this question.
- Predict what the graph might look like.



- Carry out the experiment.
 - Report your results. How might you compare the efficiency of the two containers?
- Find a graph from a newspaper or magazine that represents a functional relationship between two variables.
 - Explain why the relationship is functional.
 - Describe the relationship in words, as though you were talking on the phone to someone who cannot see the graph.
 - Make up and answer one question based on data in the graph.
 - Describe a real-life scenario involving a functional relationship between two variables.

SECTION

2.3

NUMERATION

WHAT DO YOU THINK?

- Why and when did humans invent numbers?
- Why do many mathematicians regard the invention of zero as one of the most important developments in the entire history of mathematics?

"Through and through the world is infested with quantity: to talk sense is to talk quantities. It is no use saying the nation is large—how large? It is no use saying that radium is scarce—how scarce? You cannot evade quantity. You may fly to poetry and music, and quantity and number will face you in your rhythms and your octaves."⁶

Numbers permeate our everyday lives. In Chapter 3, we will examine how we operate with numbers (addition, subtraction, multiplication, and division) and how we use exact numbers and estimates. In this section, we will lay the foundation for that work by examining the numeration system we use. Most adults don't think twice about our numeration system. Yet it represents one of the greatest advances in human history, and its widespread use is surprisingly recent—less than 500 years. If you look at Standard 1, you can see that the NCTM stresses the importance of students not just understanding how to count but also understanding how our numeration system works. Rather than read about why our numeration system is so powerful and so much "better" than other systems, you will appreciate our system and understand its properties better if you participate in the reenactment of the invention of numeration systems, which is the aim of Exploration 2.7: Alphabitia.

Origins of Numbers and Counting

History Did you know that the base 10 numeration system you use every day is called the Hindu-Arabic numeration system because it was invented by the Hindus and transmitted to the West by the Arabs? Did you know that this system has been in widespread use in the West for only 500 years? Did you know that people had to invent counting?

The earliest systems must have been quite simple, probably tallies. The oldest archaeological evidence of such thinking is a wolf bone over 30,000 years old, discovered in the former Czechoslovakia (see Figure 2.16). On the bone are

⁶Alfred North Whitehead, cited in Julian Weissglass, *Exploring Elementary Mathematics* (Dubuque, IA: Kendall Hunt Publishing, 1990), p. 100.

55 notches in two rows, divided into groups of five. We can only guess what the notches represent—how many animals the hunter had killed or how many people there were in the tribe.

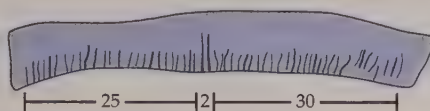


FIGURE 2.16

Other anthropologists have discovered how shepherds, throughout the ages, have been able to keep track of their sheep without using numbers to count them. Each morning as the sheep left the pen, the shepherds made a notch on a piece of wood or on some other object. In the evening, when the sheep returned, they would again make a notch for each sheep. Looking at the two tallies, they could quickly see whether any sheep were missing. Anthropologists also have discovered several tribes in the twentieth century that did not have any counting systems!

The beginnings of what we call civilization were laid when humans made the transition from being hunter-gatherers to being farmers—that is, from a nomadic life to a settled life. Archaeologists generally agree that this transition took place almost simultaneously in many parts of the world some ten to twelve thousand years ago. It was probably during this transition that the need for more sophisticated numeration systems developed. For example, a tribe need kill only a few animals, but one crop of corn will yield many hundreds of ears of corn.

The invention of numeration systems was not as simple as you might think. The ancient Sumerian words for one, two, and three were the words for man, woman, and many. The Aranda tribe in Australia used the word *ninta* for one and *tara* for two. Their words for three and four were *tara-ma-ninta* and *tara-ma-tara*.

Requirements for counting In order to have a counting system, people first needed to realize that the number of objects is independent of the objects themselves. Look at Figure 2.17. What do you see?



FIGURE 2.17

There are three objects in each of the sets. However, the number three is an abstraction that represents an amount. Archaeologists have found that people didn't always understand this. For example, the Thimshians, a tribe in British Columbia, had seven sets of words in their language for each number they knew, depending on whether the word referred to (1) animals and flat objects, (2) time and round objects, (3) humans, (4) trees and long objects, (5) canoes, (6) measures, and (7) miscellaneous objects. Whereas we would say three people, three beavers, three days, and so on, they would use a different word for "three" in each case. There is evidence in some of

our own words of such origins. For example, we have many different nouns denoting two: a pair, a couple, a brace (of pheasants), a span (of horses), a duet.

Having a counting system also requires that we recognize a one-to-one correspondence between two equivalent sets: the set of objects we are counting and the set of numbers we are using to count them. (Notice the use of set language as we develop the concepts related to counting.) I watched both of my children miscount objects for some time, either counting too many or too few, because they had not yet realized that they needed to say the next number each time they touched the next object. It took some time for them to realize that each object represented the next number, as shown in Figure 2.18—that is, that there is a one-to-one correspondence between the set of objects and the set of numbers.

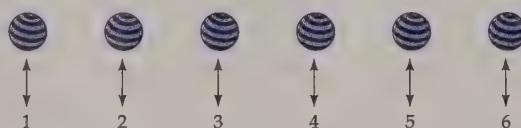


FIGURE 2.18

There is another aspect of counting that needs to be noted also. Most people think of numbers in terms of counting discrete objects. However, this is only one of the two major contexts in which numbers occur. For example, in Figure 2.19, there are 3 balls, there are 3 ounces of water in the jar, and the length of the line is 3 centimeters. In the first case, the 3 tells us how many objects we have. However, in the two latter cases, the number tells how many of the units we have. In this example, the units are ounces and centimeters.

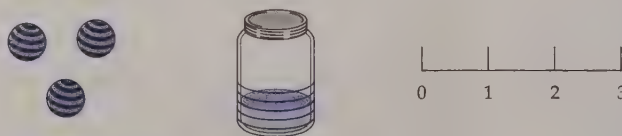


FIGURE 2.19

Working with numbers that represent discrete amounts is more concrete than working with numbers that represent measures. Recall the Thimshian language. Virtually all of the seven categories involved discrete sets. Ascribing numbers to continuous objects or events, such as length, time, or area, is more abstract.

Note that we distinguish between **number**, which refers to the amount (being counted or measured), and **numeral**, which refers to the symbol(s) used to designate the amount.

Patterns in Counting

As humans developed names for numbers larger than the number of fingers on one or two hands, the names for the larger numbers were often combinations of names for smaller numbers. Can you see the patterns and fill in the blanks

for the three systems in Table 2.8 below? What patterns did you see? Did you find any surprises after you checked your answers on page 110?

TABLE 2.8

Number	Greenland Eskimos	Aztecs	Luo of Kenya
1	atauseq	ce	achiel
2	machdlug	ome	ariyo
3	pinasut	yey	adek
4	sisasmat	naui	angwen
5	tadlimat	maculli	abich
6	achfineq-atauseq (other hand one)	chica-ce	ab-achiel
7	achfineq-machdlug	chic-ome	ab-ariyo
8	-----	chicu-ey	-----
9	achfineq-sisasmat	chic-naui	-----
10	qulit (first foot)	matlacti	apar
11	achqaneq-atauseq (first foot one)	matlacti-on-ce	apar-achiel
12	-----	-----	apar-ariyo
13	-----	-----	-----
15	achfechsaneq (other foot)	caxtulli	-----
16	-----	-----	-----
20	inuk navdlucho (a man ended)	cem-poualli	piero-ariyo

People who have investigated the development of numeration systems, from prehistoric tallies to the Hindu-Arabic system, have discovered that many of the numeration systems had patterns, both in the symbols and in the words, around the amounts we call 5 and 10. However, a surprising number of systems also show patterns around 2, 20, and 60. For example, the French word for eighty, *quatre-vingts*, literally means “four twenties”.

As time went on, people developed increasingly elaborate numeration systems so that they could have words and symbols for larger and larger amounts. We will examine three different numeration systems—Egyptian, Roman, and Babylonian—before we examine our own base 10 system. Essentially, though, all numeration systems are simply a set of rules used with a set of symbols so that each amount can be represented by a unique element or combination of elements from that set of symbols! Do you see the connection to functions?

The Egyptian Numeration System

The earliest known written numbers are from the Egypt of about 5000 years ago. The Egyptians made their “paper” from a water plant called papyrus that grew in the marshes. They found that if they cut this plant into thin strips, placed the strips very close together, then placed another layer crosswise, and finally let it dry, they could write on the substance that resulted. Our word *paper* derives from their word for papyrus.

Symbols in the Egyptian system The Egyptians developed a numeration system that combined picture symbols (hieroglyphics) with tally marks to represent numbers. Table 2.9 gives the primary symbols in the Egyptian system.

TABLE 2.9

1,000,000	100,000	10,000	1,000	100	10	1
						
Astonished person	Polliwog or burbot fish	Pointing finger	Lotus flower	Scroll	Heelbone	Staff, stroke

The Egyptians could represent amounts using combinations of these basic numerals. To deepen your understanding of how the Egyptian system worked, use it to answer the following questions. Check your answers on page 111.

What is the value of each of the following numbers?

1.       
 2.      

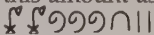
How would the Egyptians represent the following amounts?

3. 1,202 4. 304

Working with the Egyptian system Take a few minutes to think about the following questions. Write your thoughts before reading on. If possible, compare your responses with those of a classmate. . . . 

1. What do you notice about the Egyptian system? Do you see any patterns?
2. What similarities do you see between this and the more primitive systems we have discussed?
3. What limitations or disadvantages do you find in this system?
4. Try to describe the rules for this system, as though you were talking to someone on the phone.

The Egyptian numeration system resembles many earlier counting systems in that it uses tallies and pictures. In this sense, it is called an *additive system*. Do you see why?

Look at the way this system represents the amount 2312. In one sense, the Egyptians saw this amount as $1000 + 1000 + 100 + 100 + 100 + 10 + 1 + 1$ and wrote it as . In an **additive system**, the value of a number is literally the sum of the numerals.

However, this system represents a powerful advance: The Egyptians created a new numeral for every *power of 10*. Do you understand this?

They had a numeral for the amount 1. To represent amounts between 1 and 10, they simply repeated the numeral. For the amount 10, they created a new numeral. All amounts between 10 and 100 can now be expressed using combinations of these two numerals. For the amount 100, they created a new numeral, and so on.

These amounts for which they created numerals are called **powers of ten**.

1	10	$10 \cdot 10$	$10 \cdot 10 \cdot 10$	$10 \cdot 10 \cdot 10 \cdot 10$	$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$	$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$
1	10	100	1000	10,000	100,000	1,000,000

We can express $10 \cdot 10$ as 10^2 . If you recall your work with exponents from algebra, you can express 10 as 10^1 and 1 as 10^0 . Thus, we can express the value of each of the Egyptian numerals as a power of 10:

$$10^0 \quad 10^1 \quad 10^2 \quad 10^3 \quad 10^4 \quad 10^5 \quad 10^6$$

The Egyptian system was not a full-blown base 10 system, as we shall find out, but it was a remarkable achievement for its time. Egyptian rulers could represent very large numbers. One of the primary limitations of this system, as we shall see in Chapter 3, was that computation was extremely cumbersome. It was so difficult, in fact, that the few who could compute enjoyed very high status in the society.

The Roman Numeration System

The Roman system is of historical importance because it was the numeration system used in Europe from the time of the Roman Empire until after the Renaissance. In fact, several remote areas of Europe continued to use it well into the twentieth century. Some film makers still list the copyright year of their films in Roman numerals.

Symbols in the Roman system Table 2.10 below gives the primary symbols used by early Romans and later Romans.

TABLE 2.10		
Amount	Early Roman	Later Roman
1	I	I
5	V or Δ	V
10	X	X
50	$\downarrow$	L
100	$\odot$	C
500	$\mathcal{D}$ and $\mathcal{D}$	D
1000	$\textcircled{1}$ or ∞	M

Again, to deepen your understanding of how the Roman system worked, use it to answer the following questions. Check your answers on page 111.

What is the value of each of the following numbers (using the numerals of the Later Roman system)?

1. MXI
2. MDXCVII

The X precedes the C in MDXCVII above. Notice how this placement affects the value. We will discuss this aspect of the Roman system later.

How do you think the Romans would represent the following amounts?

3. 1102
4. 319

Working with the Roman system Take a few minutes to think about the following questions. Write your thoughts before reading on. If possible, compare your responses with those of a classmate. . . . 

1. What do you notice about the Roman system? Do you see any patterns?
2. What similarities do you see between this system and the Egyptian system?
3. What limitations or disadvantages do you find in this system?
4. Try to describe the rules for the Roman system, as though you were talking to someone on the phone.

Like the Egyptians, the Romans created new numerals with each power of 10, that is, 1, 10, 100, 1000, etc. However, the Romans also created new

numerals at “halfway” amounts—that is, 5, 50, 500, etc. Why do you think they did this? Think and then read on. . . . 

This invention reduced some of the repetitiveness that encumbered the Egyptian system. For example, 55 is not XXXXXIII but LV.

Basically, the Roman system, like the Egyptian system, was an additive system. However, the Later Roman system introduced a *subtractive* aspect. For example, IV can be seen as “one before five.” This invention further reduced the length of many large numbers. For example, when writing the amount 444, users of the Later Roman system no longer had to write CCCCXXXIII, but rather could write . . . well, what do you think? 

Check your answer on page 111.

As in the Egyptian system, computation in the Roman system was complicated and cumbersome, and neither system had anything resembling our zero.

The Babylonian Numeration System

The Babylonian numeration system is a refinement of a system developed by the Sumerians several thousand years ago. Both the Sumerian and Babylonian empires were located in the region occupied by modern Iraq. The Sumerians did not have papyrus, but clay was abundant. Thus, they kept records by writing on clay tablets with a pointed stick called a stylus. Thousands of clay tablets with their writing and numbers have survived to the present time; the earliest of these tablets were written almost five thousand years ago.

Symbols in the Babylonian system Because they had to make their numbers by pressing into clay instead of writing on papyrus, their symbols could not be as “fancy” as the Egyptian symbols. In fact, this physical constraint was probably the impetus for the advances that their system represents. They had only two symbols, an upright wedge that symbolized “one” and a sideways wedge that symbolized “ten.” In fact, the Babylonian writing system is called *cuneiform*, which means “wedge-shaped.”



HISTORY

We are not sure why the Babylonians chose 60, although many hypotheses have been set forth. One is that 60 divides 360 evenly, and the Sumerians may have originally thought that there were 360 days in a year. Another possibility is that because many numbers divide 60 evenly (1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60), this choice made calculations with fractions much easier. Their choice of 60 affects our lives today, for it was they who divided a circle into 360 parts and divided an hour into 60 minutes.

Amount	Symbol
1	▼
10	◄

Amounts could be expressed using combinations of these numerals; for example, 23 was written as ◄▼▼▼

However, being restricted to two numerals creates a problem with large amounts. The Babylonians’ solution to this problem was to choose the amount 60 as an important number. Unlike the Egyptians and the Romans, they did not create a new numeral for this amount. Rather, they agreed (among themselves) that they would have a new *place*. For example, the amount 73 was represented as ▼ ◄▼▼. That is, the ▼ at the left represented 60 and the ◄▼▼ to the right represented 13. In other words, they “saw” 73 as 60 + 13.

Similarly, ▼▼ ◄▼ was seen as six 60s plus 12, or 372.

▼▼

Thus, we consider the Babylonian system to be a **positional system** because the value of a numeral depends on its position (place) in the number.

As with the other two systems, use the following questions to reinforce your understanding of their system. Check your answers on page 111.

What is the value of each of the following numbers?

1. $\text{II} \lll$ 2. $\text{IIII} \lll \lll$ 3. $\text{I} \lll \lll$
 $\text{II} \ll$

How do you think the Babylonians would represent the following amounts?

4. 1202 5. 304

Working with the Babylonian system Take a few minutes to think about the following questions. Write your thoughts before reading on. If possible, compare your responses with those of a classmate. . . . 

1. What do you notice about the Babylonian system? Do you see any patterns?
2. What similarities do you see between this system and the Egyptian and Roman systems?
3. What limitations or disadvantages do you find in the Babylonian system?
4. Try to describe the rules for this system, as though you were talking to someone on the phone.

Place value All three systems have vestiges of tallies for the first four numbers: IIII, IIII, IIIII. The Romans invented a new numeral for 5. The Egyptians and Babylonians waited until 10 to create a new numeral. To represent larger amounts, the Babylonians invented the idea of the value of a numeral being a function of its place in the number. This is the earliest occurrence of the concept of **place value** in recorded history. With this idea of place value, they could represent any amount using only two numerals.

We can understand the value of their system by examining their numbers with expanded notation. Look at the following Babylonian number:

$\text{II} \lll \lll \lll$

Because the $\lll$ occurs in the first (or rightmost) place, its value is simply the sum of the values of the numerals—that is, $10 + 10 + 1 = 21$. However, the value of the $\lll \lll \lll$ in the second place is determined by multiplying the face value of the numerals by 60—that is, $60 \cdot 23$. The value of the II in the third place is determined by multiplying the face value of the numerals by 60^2 —that is, $60^2 \cdot 2$.

The value of this amount is

$$60^2 \cdot 2 + 60 \cdot 23 + 21 = 7200 + 1380 + 21 = 8601$$

Thus, in order to understand the Babylonian system, you have to look at the face value of the numerals *and* the place of the numerals in the number. Therefore, although the Babylonian system still has additive aspects (for example, $5 = \text{IIIIII}$), the value of a number is no longer determined simply by adding the values of the numerals. One must take into account the place of each numeral in the number.

This later Babylonian system is thus considered by many scholars to be the first place value system⁷ because the value of every symbol depends on its place in the number and there is a symbol to designate when a place is empty.

⁷ Georges Ifrah, *From One to Zero: A Universal History of Numbers*, p. 373. Translated from the French 1985.

Exercise 10 looks at another very different place value numeration system that was developed by the Mayas in Central America between A.D. 300 and A.D. 900.

The Babylonian system is more sophisticated than the Egyptian and Roman systems. However, there were some “glitches” associated with this invention. Before you read on, imagine that you were a Babylonian. What problems could you see with this new idea of place value?

What if there were nothing in a place? For example, how could the Babylonians represent the amount 3624? Try this and then read on. . . . 

The need for a zero If we represent this amount from the Babylonian perspective, we note that $60^2 = 3600$. Thus, the Babylonians saw 3624 as $3600 + 24$. They would use ∇ in the third place to represent 3600, and they would use $\llcorner \llcorner \llcorner \llcorner$ in the first place to represent 24, but the second place is empty. Thus, if they wrote $\nabla \llcorner \llcorner \llcorner \llcorner$, how was the reader to know that this was not $60 + 24 = 84$? Again, try to imagine yourself as a Babylonian. How might you solve this problem? Think and read on. . . . If possible, compare your ideas with those of a classmate. 

Archaeologists have found evidence that later Babylonian writers experimented with several different notations to represent an empty place. A Babylonian mathematical table from about 300 B.C. contains a new symbol $\nabla \nabla$ that acts like a zero. Using this convention, they could thus represent 3624 as

$\nabla \nabla \llcorner \llcorner$
 $\nabla \nabla$

The slightly sideways wedges indicate that the second place is empty, and thus we can unambiguously interpret this numeral as representing

$$60^2 + 0 + 24 = 3624$$

In the exercises, you will find that the Mayans also created a numeration system with a zero. As you may have discovered in your work as an Alphabitian in Exploration 2.7, creating a system with a **base** and with a symbol to act as a place holder is not a simple task. This is why it took humans so long to create such systems.

The Development of Base 10: The Hindu-Arabic System

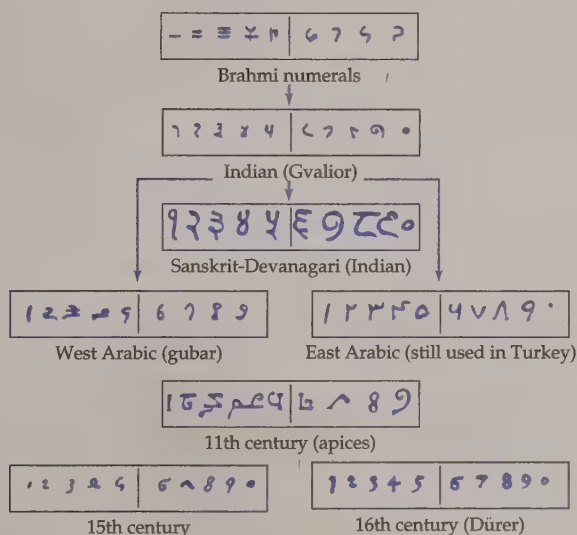
We have now seen how the Egyptian and Roman systems represented important advances over tallying, allowing people to count and represent large amounts with numerals. We have seen how the Babylonians struggled with the concept of the value of a numeral depending on its place in the number. You are now in a better position to understand and appreciate the structure of the base 10 numeration system we use today—the Hindu-Arabic system.

We can still only guess at early stages of this system because we have no early archaeological evidence, as we have for the Egyptian and Babylonian systems. However, at some point, someone or some group discovered that all numbers could be represented using combinations of just ten symbols. One of these symbols, zero, constituted a tremendous leap in abstraction, because it both represented nothing and also represented an empty place. You have seen how the Babylonians struggled with this dilemma for many hundreds of years.

Archaeologists tell us that the Hindu numeration system emerged around A.D. 600 and that the earliest known zero is found inscribed on the wall of a temple in central India dated A.D. 870. By A.D. 800, news of this system came to Baghdad, which had been founded in A.D. 762 by followers of Mohammed.

Leonardo of Pisa, whom you encountered in Chapter 1, traveled throughout the Mediterranean and the Middle East, where he first heard of the new system. In his book *Liber Abaci* (translated as *Book of Computations*), published in 1202, he argued the merits of this new system, but it took some time for the Hindu-Arabic system to replace the Roman system that almost everyone in Europe used. In fact, a thirteenth-century law forbade bankers of Florence to use these Hindu-Arabic numerals, because it was felt that forgeries were easier in the new system than in the Roman system. However (as you will discover in Chapter 3), one of the biggest advantages of this new system was ease of computation, and it was not long before the merchants realized how much easier computing was in this new system.

Figure 2.20 below traces the development of the ten numerals that make up our numeration system.⁸ Just as the ancient Egyptians used papyrus and the Sumerians used clay tablets, the people in ancient India used dried palm leaves to write with. Some historians believe that the Indians could have joined up these strokes, so that $\equiv$ became $\approx$ and $\equiv$ became $\approx$.



Source: From Karl Menninger, *Number Words and Number Symbols: A Cultural History of Numbers*, 1969, p. 418. Reprinted by permission of M.I.T. Press.

FIGURE 2.20

The development of numeration systems from the most primitive (tally) to the most efficient (base 10) has taken tens of thousands of years. Although the base 10 system is the one you grew up with, it is also the most abstract of the systems and possibly the most difficult initially for children. The next section describes the essential mathematical structure of the Hindu-Arabic numeration system. However, rather than just reading this section, stop first and reflect on what you have learned thus far in your own investigations. Imagine describing this system to a Babylonian, Egyptian, or Roman who has suddenly been transported into our time. Imagine that this person can understand English and has a counting table like Table 2.11 but is still struggling to make sense of this

⁸ Karl Menninger, *Number Words and Number Symbols: A Cultural History of Number*, trans. Paul Bonebrake (Cambridge, MA: M.I.T. Press, 1969), p. 418.

system. How would you describe the essential features of this system to that person? Do so before reading on. . . 

TABLE 2.11

Egyptian	Roman	Babylonian	Hindu-Arabic
I	I	▼	1
II	II	▼▼	2
III	III	▼▼▼	3
IIII	IV	▼▼▼▼	4
IIII I	V	▼▼▼▼▼	5
IIII II	VI	▼▼▼▼▼▼	6
IIII III	VII	▼▼▼▼▼▼▼	7
IIII IIII	VIII	▼▼▼▼▼▼▼▼	8
IIII IIII I	IX	▼▼▼▼▼▼▼▼▼	9
↪	X	↪	10
↪↪	XX	↪↪	20
↪↪↪↪↪	L	↪↪↪↪↪	50
↪↪↪↪↪↪	LX	↪	60
↻	C	↪↪↪↪↪	100

Advantages of Base 10

Our base 10 numeration system has several characteristics that make it so powerful.

No tallies The base 10 system has no vestiges of tallies. Any amount can be expressed using only 10 **digits**: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. In fact, the word *digit* literally means “finger.”

Decimal system The base 10 system is a **decimal** system, because it is based on groupings (powers) of 10. The value of each successive place to the left is 10 times the value of the previous place:

100,000 10,000 1000 100 10 1

Ten ones make one ten.

Ten tens make one hundred.

Ten hundreds make one thousand.

Ten thousands make ten thousand.

Expanded form When we represent a number by decomposing it into the sum of the values from each place, we are using **expanded form**. There are different variations of expanded form. For example, all of the expressions below emphasize the structure of the number, 234—some more simply and some using exponents.

$$\begin{aligned}
 &200 + 30 + 4 \\
 &= 2 \cdot 100 + 3 \cdot 10 + 4 \cdot 1 \\
 &= 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0
 \end{aligned}$$

Note: $10^1 = 10$ and $10^0 = 1$.

Expanded form is an important tool for two reasons. First, it is one of the many tools that mathematicians use to represent the process of decomposing numbers into smaller amounts to help us better understand the concepts and

procedures that we use. Second, in Chapter 3, we will use expanded form to understand why the procedures we use to add, subtract, multiply, and divide actually work.

The concept of zero The fourth reason why the base 10 system is powerful is the invention of the concept of zero, represented by the symbol 0. “The invention of zero marks one of the most important developments in the whole history of mathematics.”⁹ This is the feature that moves us beyond the Babylonian system. Recall the Babylonians’ attempts to deal with the confusion when a place was empty. It was the genius of some person or persons in ancient India to develop this idea, which made for the most efficient system of representing amounts and also made computation much, much easier. One of the most difficult aspects of this system is that the symbol 0 has two related meanings: In one sense, it works just like any other digit (it can be seen as the number 0), and at the same time, it also acts as a place holder. It takes young schoolchildren several years to understand this fully and accept it. I recall my five-year-old daughter saying that zero meant “nothing” and insisting that it was not a number!

Do you understand these four aspects of our numeration system? When you were exploring the Alphabitian system in Exploration 2.7, did you struggle with one or more of these concepts? Do you recall having difficulties with any of them yourself in elementary school? Exploration 2.9 is a game that helps people to understand more deeply this idea of place value.

Connecting Geometric and Numerical Representations

Let me now ask a question that will help you to assess your understanding of these concepts and to extend your understanding: What do you think the fifth place in our **base 10 manipulatives** looks like? I ask this question of my students, and many are baffled by the question and need a hint. If you find yourself baffled, read the following.

If you mentally step back and look at this system, you will notice that the first place is represented by a unit, which we will call a “small cube”; the second place is represented by a long; the third place is represented by a flat; and the fourth place is represented by a block (a “big cube”). If you were to draw a picture of the fifth place, what would it look like? Think about this and read on. . . . 

Rather than give a direct answer to the question, we will lead up to it. As we look at each place, we see some amazing patterns. Let us start at the beginning, with units.

- Ten ones make one ten. In a physical sense, ten units become one long.
- Then ten tens become one hundred. In a physical sense, ten longs become one flat.
- Then ten hundreds become one thousand. In a physical sense, ten flats become a “big cube.”

Continuing this pattern, we see that ten thousands become one ten-thousand. In a physical sense, we can represent ten thousands as a “big long.” There are other representations; for example, we could have represented 10,000 as a pile of ten “big cubes,” as shown in Figure 2.21.



FIGURE 2.21

⁹ H.A. Freebury, *A History of Mathematics* (New York: Macmillan, 1958), p. 170.

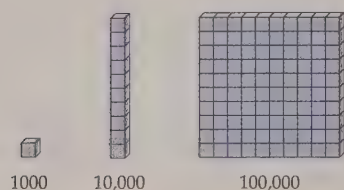


FIGURE 2.22

Both representations show a value of 10,000. Mathematically, we say that the former representation is more powerful because it makes a better connection. One of the reasons it makes a better connection is that it fits a larger pattern that you will soon see.

If we continue the pattern in this way, we can represent ten 10,000s (big longs) as one 100,000, and this amount can be represented visually as a “big flat,” as shown in Figure 2.22.

Table 2.12 below shows the name of each place, the amount it represents, and its shape.

TABLE 2.12

Word	Million	Hundred thousand	Ten thousand	Thousand	Hundred	Ten	One
Symbol	1,000,000	100,000	10,000	1000	100	10	1
Shape	huge cube	big flat	big long	big cube	flat	long	small cube

Do you see now why the representation of 10,000 as a “big long” is more powerful than the other representations?

Think about the terms, look at the physical shape of each place, and look at the way we write numbers. All three representations are closely connected.

- The name changes every three terms—thousand, million, billion, etc.
- The shape changes every three terms—cube, long, flat.
- We place commas after every three terms, e.g., 345,234,186.

Is this a coincidence, or is this the reason why we separate large numbers by groups of three?

Summary

Almost certainly, base 10 has become the universal counting system today because we have 10 fingers on both hands. We also could have selected base 5—fingers on one hand—or base 20—digits on fingers and toes, not an uncommon choice when we look at historical records. (As you will find out in Chapter 3, computation would be much easier for most people if our species had one more finger on each hand, in which case we would have used base 12.)

We have explored different counting systems to give you an appreciation of the significance of base 10 and its abstractness—it took humans many thousands of years finally to invent such a powerful numeration system. In the course of the *Explorations*, you have come to appreciate the importance of mathematical vocabulary, including the terms *digit* and *place* (ones place, tens place, etc.). You have realized that with the concepts of base and place value, and a symbol to represent “nothing,” we can represent any amount using only a few digits (in base 10, we use 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9). You have been introduced, through expanded form, to the tool of decomposing a number into its constituent parts. This notion of breaking an object or idea into its component parts is an essential tool in all scientific disciplines.

In this section, you have probably gained a deeper appreciation of both the power and the abstractness of our base 10 numeration system. To use an old phrase, I wish I had a dollar for every student who has said something like “Wow, no wonder it’s hard for little kids to learn how to count; I never thought

of it [our system] that way before." In Chapter 3, we will carefully examine the meanings of the four basic operations of addition, subtraction, multiplication, and division. In that chapter, you will come to understand better the value of expanded form. Then we will learn how the algorithms, which most adults take for granted, really work. By the end of Chapter 3, your appreciation of the Hindu-Arabic system should be even greater!

EXERCISES 2.3

- Fill in the blanks in these counting systems.
- What patterns do you see in these systems?
- Describe the "rules" of this system as though you were talking on the phone to a friend who missed the class when this system was discussed.

Base 10	Maya	Luli of Paraguay	South American
1	hun	alapea	tey
2	ca	tamop	cayupa
3	ox	tamlip	toazumba
4	can	lokep	cajesa
5	ho	lokep moile alapea (four and one) or is alapea (one hand)	teente
6	uac	lokep moile tamep	teyente-tey
7	uuc	?	teyente-cayapa
8	uaxac	lokep moile lokep	?
9	bolon	lokep moile alapea	teyente-cajesa
10	lahun	is yaoum (both hands)	caya-ente
11	buluc	is yaoum moile alapea (hands and one)	caya-ente-tey
12	lah-ca	is yaoum moile tamep	?
13	ox-lahun	?	?
15	ho-lahun	?	toazumba-ente
16	?	?	?
20	hunkal	is eln yaoum (hands, feet)	?
21	?	?	?
22	?	?	?

- Find the base 10 equivalent of each of these numerals.

-
-
- MDCLXVI
- MDXIX
- CIX
-
-
-

- Represent these base 10 amounts in Egyptian, Roman, and Babylonian symbols.

- 312
- 1206
- 6000
- 10,000
- 123,456

- An amusing exercise is to convert English words to Roman values. For example, the English word LID would be worth $50 + 1 + 500 = 551$.

- What is the value of MIX?
- What is the most valuable English word made up only of letters in the Roman numeration system?

- What is the most valuable English word that you can find, if we allow any English word but determine its value by adding only those letters that have values in the Roman system?

- Represent the following base 10 numbers by sketching the base 10 manipulatives.

- 345
- 2001

- Represent the following base 10 numbers in expanded form.

- 345
- 2001
- 10,101

- Rewrite each of the following in standard form.

- $4 \cdot 10^3 + 8 \cdot 10^2 + 5 \cdot 10 + 9$
- $3 \cdot 10^4 + 2 \cdot 10^2 + 4 \cdot 10$
- $7 \cdot 10^5 + 5 \cdot 10^4 + 3$

- The symbols developed by the ancient Greeks are visually quite fascinating and remind me of the children's game called hangman. The basic symbols for amounts up to 10,000 are given below.

1	5	10	50	100	500	1,000	5,000	10,000
	⌒	Δ	⌒	⌒	⌒	X	⌒	⌒

Translate the following Greek numbers into base 10:

-
-

Translate the following base 10 numbers into Greek:

- 347
- 5555

- An ancient Chinese mathematician named Sun-Tsu who lived in the first century A.D. described the use of calculating rods (made of bamboo) for representing numbers. The digits for 1 to 9 are represented as follows:

1	2	3	4	5	6	7	8	9
					⊥	⊥	⊥	⊥

However, when the digits from 1 to 9 appear in the tens column, they are represented as follows:

1	2	3	4	5	6	7	8	9
—	=	≡	≡	≡	⊥	⊥	⊥	⊥

Thereafter, every time you move over one place, you change from one form to the other.

Thus, the number 4763 would be represented as and the amount 8888 would be represented as

Translate the following numbers into base 10:

-
-

Represent the following numbers in the Chinese system:

c. 346

d. 12,345

10. One of the most impressive of the ancient numeration systems comes from the Mayans of Central America. Their numeration system was developed around 400 or 300 B.C. They used only three symbols: dot, line, and oval. The oval functions in a manner very similar to our zero. They also wrote their numbers vertically.

•	••	•••	••••	—	•	••	•••	••••	—
1	2	3	4	5	6	7	8	9	10
•	••	•••	••••	—	•	••	•••	••••	—
									
20	40	60	80	100	120	140	160	180	200

Many mathematics historians credit the Mayans with being the first civilization to develop a numeration system with a fully functioning zero. At first, the Mayan system seems to be a base 20 system. If this were the case, however, the values of their places would be 1, 20, 20^2 , 20^3 , 20^4 , etc. Instead, however, the values of their places are 1, 20, 360 , 360×20 , 360×20^2 , 360×20^3 , etc. We are fairly certain that their choice of this system has to do with the fact that at the time of its creation, they thought that the length of a year was 360 days. Below are three base 10 numbers and how they would be written in the Mayan system.

46 =	300 =	407 =
(2×20)	(15×20)	(1×360)
+ 6	+ 0	+ (2×20)
		+ 7
••	≡	•
•		••
—		••

Translate the following Mayan numbers into base 10:

a. 	b. 	c. 
--	--	--

Write the following base 10 numbers in Mayan:

d. 245	e. 500	f. 3600
--------	--------	---------

In Exercises 11 through 22, recall your examination of different bases in the *Explorations*.

11. Tell what comes after:

a. 34_5	b. 1011_2	c. 99_{12}	d. 7099_{12}
e. 101_2	f. 111_{12}	g. 124_5	h. 405_6

12. Tell what comes before:

a. 1010_5	b. 340_5	c. 100_{12}	d. 1110_2
e. 1010_2	f. 110_{12}	g. 120_4	h. 60_7

13. Convert the following numbers into base 10.

a. 41_5	b. 55_6	c. 210_5	d. 2104_5
e. 101_5	f. 1111_6	g. 303_4	h. 606_7
i. 1101_2	j. 10001_2	k. 99_{12}	l. 909_{12}

14. Convert the following numbers from base 10 into the designated base.

a. $44_{10} = ?_5$	b. $152_{10} = ?_5$	c. $92_{10} = ?_2$
d. $206_{10} = ?_2$	e. $72_{10} = ?_{12}$	f. $402_{10} = ?_{12}$
g. $44_{10} = ?_6$	h. $1252_{10} = ?_6$	i. $144_{10} = ?_{12}$
j. $100_{10} = ?_5$	k. $99_{10} = ?_{12}$	l. $1052_{10} = ?_5$
m. $2,500,000_{10} = ?_5$	n. $2,500,000_{10} = ?_2$	

15. In what base does $25 + 25 = 51$?
16. Tell what base makes the following statement true: $23_{10} = 25_x$
17. For what base x is this statement true? $598_{10} = 734_x$
18. If $44_x = 28_{10}$, how many base x candy bars will fit into a box holding 110_{10} candy bars?
19. Convert 222_3 into base 6 directly, as though you had never heard of base 10. A diagram is helpful but not required. In order to receive credit, you must include a written explanation of how you converted from the given base 3 number to the base 6 number.
20. Convert $204_6 = ?_3$ directly, without going through base 10.
21. Explain the following conversion problems as though you were talking to a struggling student.
- a. $234_{10} = ?_5$ b. $405_8 = ?_{10}$
22. One common mistake that young children make when counting is illustrated below: twenty-six, twenty-seven, twenty-eight, twenty-nine, twenty-ten, twenty-eleven, etc.

Can you describe the nature of the child's mistake? That is, what is the child not understanding or what misconception of counting causes the mistake? Do you see this mistake differently now than you would have before this course? If so, please explain.

23. Another common mistake that young children make when counting is illustrated below: twenty-six, twenty-seven, twenty-eight, twenty-nine, thirty-one, thirty-two, etc.

Can you describe the nature of the child's mistake? That is, what is the child not understanding or what misconception of counting causes the mistake? Do you see this mistake differently now than you would have before this course? If so, please explain.

24. We speak of so many children having little or no understanding of place value. What does "place value" mean to you? Define the concept in your own words. This involves giving some meaningful explanation of what "place value" means mathematically.
25. We have discussed the development of more sophisticated numeration systems in class, culminating in the Hindu-Arabic base 10 system that we currently use. Describe the advantages of the Hindu-Arabic system over the Roman system.
26. In addition to a better understanding of place value, what else have you learned about base 10 by working in other bases in the *Explorations*?

27. The first edition of this book was published in 1997, which is in the twentieth century, not the nineteenth century. Can you explain why?
28. When we write large numbers, why do we insert a comma every three numbers instead of every two or four numbers?
29. You are Zirkle, from Zordon, and you have just finished your trip to the solar system. Your planet has managed to develop interplanetary travel without ever developing a sophisticated number system. You have examined three different bases (base 2, base 5, and base 6), and you are preparing your recommendations for your home planet. Summarize the pros and cons of each base, and then tell which base you recommend that your planet adopt. *Note:* Your species has no fingers, only paws. Therefore, none of the bases has an advantage with respect to your anatomy.

- a. Summarize the pros and cons of each base in a table like the one below. *Note:* Entries in the cells need to demonstrate that you understand the advantage or disadvantage/limitation. That is, an entry should state *what* the advantage or disadvantage is and also give a brief *explanation* of the advantage or disadvantage.

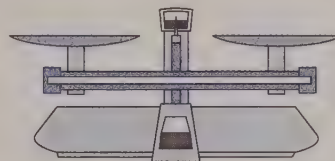
Base	+	-
2		
5		
6		

- b. Write your final recommendation: an essay of at least one paragraph (more than two sentences). You may refer to your chart, so that you don't have to be redundant.

30. Balance scales usually come with a variety of weights. The balance scale below works in the following manner: You put the object you want to weigh in one pan, and you place weights in the other pan until the arrow points straight down.

Let's say that the scale below comes with the following weights: 1 gram, 2 grams, 2 grams, 5 grams, 10 grams, 10 grams, 20 grams, 50 grams.

Thus, using only these eight weights, the heaviest object we can weigh is 100 grams. Suppose you could choose whatever weights you wanted. If you were restricted to eight weights, is there a different combination that would enable you to weigh objects heavier than 100 grams but still weigh any object to the nearest gram?



CHAPTER SUMMARY

- Many mathematical concepts and problems involve various kinds of sets and subsets of those sets.
- When two sets or variables are said to be functionally related, this tells us that the relationship between those two sets or variables is such that knowing an element of one set (or the value of the independent variable) enables us to make predictions.
- Our current (Hindu-Arabic) numeration system evolved over thousands of years, and people from many cultures developed ideas that contributed to our system.
- Our numeration system has four characteristics:
 - Any amount can be expressed using only the basic 10 digits.
 - It is based on groupings (powers) of 10.
 - The value of any digit depends on its place.

- d. Zero can represent both a value and an empty place.

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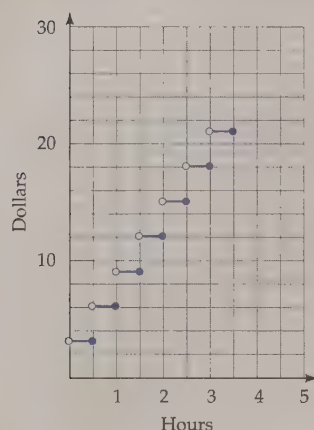
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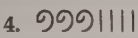
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Answers to questions in text:**Paying the baby-sitter: graph, p. 82****Patterns in counting: missing words in Table 2.6, p. 97**

Number	Greenland Eskimos	Aztec	Luo of Kenya
8	achfineq-pinasut		ab-adek
9			ab-angwen
12	achqaneq-machdlug	matlacti-on-ome	
13	achqaneq-pinasut	matlacti-on-yey	apar-adek
15			apar-abich
16	achfechsaneq-atauseq	caxtulli-on-ce	apar-ab-chiel

The Egyptian numeration system: symbols, p. 98

1. 1323 2. 143,022 3.  4. 

The Roman numeration system: symbols, p. 99

1. 1011 2. 1597 3. MCII 4. CCCXIX

Working with the system

The Later Roman symbol for 444 was CDXLIV.

The Babylonian numeration system: places, p. 101

1. $120 + 12 = 132$
 2. $5 \times 60 + 42 = 342$
 3. $1 \times 60^2 + 11 \times 60 + 21 = 3600 + 660 + 21 = 4281$
 4. 1202 would be seen as $20 \times 60 + 2 =$ 
 5. 304 would be seen as five 60s plus 4, or 

The Four Fundamental Operations of Arithmetic

- ◆ 3.1 Understanding Addition and Subtraction
- ◆ 3.2 Understanding Multiplication and Division
- ◆ 3.3 Mental Arithmetic and Estimation

In this chapter, we will examine what are called the four fundamental operations of arithmetic: addition, subtraction, multiplication, and division. Understanding these four operations “is central to knowing mathematics” (*Curriculum Standards*, p. 41). This is a critical chapter because so many seemingly different problem situations can be represented by a single operation. Furthermore, in many real-life situations, a problem may require multiple operations, and it is not always obvious which operation is needed. For example, if a hospital has 35 ounces of medicine and each dose requires $\frac{2}{3}$ of an ounce, how many doses does the hospital have? In this case, do we multiply or divide? Do you know why? In the course of the text and the explorations, we will “develop meaning for the operations by modeling and discussing a rich variety of problem situations” (*Curriculum Standards*, p. 41) so that you can understand at a deep level what each of the operations means and how these four operations are related to one another.

In the first two sections, we will explore addition, subtraction, multiplication, and division. You will come to understand that each operation has multiple meanings, that there are several different ways to represent these operations, and that this has implications for problem-solving. Our exploration of patterns and properties not only will give you important tools for solving problems but also will enable you to understand better the procedures, called algorithms, that we use in computation. The third section will focus on developing a repertoire of strategies for estimating, because much (if not most) of the computation done in daily life involves estimation rather than determining exact answers. In the course of investigating these concepts in the text and the *Explorations*

volume, your “operation sense” and “number sense” will become stronger. These are two crucial components to being able to use these four operations skillfully.

A Nonroutine, Multistep Problem

Before examining each of the operations separately, let us explore a real-life problem that requires several of the operations—that is, a nonroutine, multistep problem. As mentioned earlier, much research indicates that although many of our students can do routine problems (that is, problems “just like” the ones the teacher did or the ones shown in the book) and single-step problems, most of our students perform poorly when the problems are not routine and when they require several steps and/or computations. However, *this* is what most problems outside the classroom are like! In order to become successful at these kinds of problems, one must have a deeper understanding of fundamental mathematical concepts and procedures. As you do the problem, do you feel confident about which operation to use? Do you see more than one way to arrive at the answer?

It is the beginning of the month, and Annie wants to buy a bicycle for \$300. She has \$173 in the bank, and she just got a birthday check for \$40 from her grandparents. Her source of income is baby-sitting, for which she charges \$4 an hour. How many hours will she need to baby-sit if she wants to be able to buy the bicycle at the end of the month? She also figures that she will spend about \$25 this month on recreation.

In order to do this problem, most people perform several mathematical operations, and not everyone does it the same way. Some students add Annie’s resources ($173 + 40 = 213$), subtract her expenses ($213 - 25 = 188$), and then see how much she needs (that is, how far 188 is from 300). Other students note that $40 - 25$ is a net difference of 15, add this to 173 to get 188, and now find the difference between 188 and 300. Still other students determine how much money she needs by adding or subtracting the numbers in the order in which they occur: $300 - 173 - 40 + 25 = 112$.

Now we need to divide \$112 by \$4 per hour, and we find that she will achieve her goal if she baby-sits 28 hours this month.

What does this problem have to do with operations on whole numbers? In one sense, it serves as what is called an advance organizer for this chapter. Most real-life problems involve more than one operation (they are not what the NCTM calls routine one-step problems). In this problem, there are many different ways to arrive at the solution, not just one “right” way. In order to solve the problem, you first have to make sense of it in order to determine what operations to do and in what order.

Seeing These Operations Through the Eyes of a Child

Before we begin to explore the four operations, it is important to step back for a moment. Many of my students tell me that the Alphabitia explorations forced them to think carefully about the important mathematical ideas of counting, place value, base, and zero. In one sense, they say that because of Alphabitia they can better understand both the excitement and the confusion that young children encounter when learning these ideas for the first time. It is essential that you keep this freshness of perspective as you work through this book. Let



HISTORY

I cannot overemphasize how recent most of our everyday mathematics actually is. During the Middle Ages, many mathematicians considered an operation to be any mathematical technique or procedure that was considered important. In fact, especially when the Hindu-Arabic numeration system was still new in Europe, many mathematicians considered numeration to be an operation. It has only been in the last century that mathematicians have linked the concept of operation to the concept of function.¹

¹ Frank J. Swetz, *Capitalism and Arithmetic: The New Math of the 15th Century* (La Salle, IL: Open Court, 1987), p. 181.

me set the stage with an example. Most adults can quickly determine the appropriate operation needed to solve problems. For example, if Mannie has \$24 and he wants to buy a diskman that costs \$40, how much money does he need? Do this problem and then read on. . . . 

Most adults quickly see that this is a subtraction problem and subtract 24 from 40 to determine that he needs \$16. Some adults see this as “24 plus what is 40?” and there are various ways in which they can determine that $24 + 16 = 40$. However, a problem like this is very challenging for most first and second graders and even for some third and fourth graders. If you cannot put yourself in their shoes, then you will not be as helpful to your students.

There are four tools that will help you be more helpful to your future students. The first has already been mentioned: Your instructor and I have to search for investigations and explorations that require you to *think* as opposed to just mechanically getting the answer. The second is to spend time with children, asking them questions and listening to their responses. Many colleges don’t have field placements early in the program, so you might have to do this on your own. Another possibility is to look at videos. There are many web sites that have video clips of children learning. If you go to the web page for this book, you can get started. The third tool to help you understand the child’s view is some research on children’s learning. Researchers have found that when children are presented with arithmetic story problems, they tend to focus first on the *action* in the problem and then on the *unknown*. For example, in the problem above, the action is comparing—How much bigger is 40 than 24? The unknown is the second addend (if seen as an addition problem) or the difference (if seen as a subtraction problem). The fourth tool is to keep the NCTM process standards alive constantly as you read about the definitions, the contexts, the models, and the properties of each operation. Consider what *problem-solving* tools children might use, pay attention to the *whys* as well as the *hows* so that your *reasoning* develops, make sure that the new vocabulary makes sense as your *communication* improves, look for *connections* between operations, and be aware of different *representations* for the operations.

SECTION

3.1

UNDERSTANDING ADDITION AND SUBTRACTION

WHAT DO YOU THINK?

- What other meanings are there for subtraction beyond “take away”?
- How are addition and subtraction related in terms of parts and wholes?
- Why do math educators discourage the use of “carry” and “borrow”?

If you did Explorations 3.1 through 3.3, you probably found that trying to add and subtract in Alphabitia threw you back into the kind of confusion that young children often encounter when they try to make sense of adding and subtracting in base 10. Now that you have a deeper understanding of numeration, we will develop a deeper understanding of addition and subtraction. The understanding of numeration and these operations will enable you to understand why the various computation procedures actually work. Without this understanding, your “mathematical power” is significantly lessened.

Addition

Children encounter addition even before they have mastered counting. They naturally combine objects and want to know how many. For example, there are 4 people in our family, and 2 guests have come for dinner. How many people will be eating?

Examine the following addition problems and then write your responses to the following questions in your own words. Pretend that you are a young child who has not yet learned addition. Obviously, finding the answers to the

problems will be easy for you. Thus, you should focus on the following questions before reading on. . .

1. What action words describe what is happening in these and other addition problems? One action word is *combining*. What others can you think of?
2. Other than “they all involve addition,” can you think of other ways in which all four problems are alike? In what ways are some of the problems different from each other?

Four addition problems

1. Andy has 3 marbles, and his older sister Bella gives him 5 more. How many does he have now?
2. Betty and Joe each drank 6 ounces of orange juice. How much juice did they drink in all?
3. Linnea has 4 feet of yellow ribbon and 3 feet of red ribbon. How many feet of ribbon does she have?
4. Josh has 4 red trucks and 2 blue trucks. How many trucks does he have altogether?

There are several classical stages in children’s understanding of addition. At the most basic level, the child counts to determine the sum. For example, in the first problem, many young children would answer the question by putting the marbles on the floor and then counting the two groups. Interestingly, the traditional definition of addition is in set language, and children intuitively begin with this model (see Figure 3.1).

Let A = the set of Andy’s marbles, and let B = the set of Bella’s marbles.

Then $n(A) = a$, the number of marbles Andy has, and $n(B) = b$, the number of marbles Bella has.

We then define **addition** in the following way:

If A and B are disjoint sets, containing a and b elements, respectively, then $a + b = n(A \cup B)$.

In other words, the **sum** of a and b is equal to the number of elements in the union of the two sets. Do you see why the sets have to be disjoint?

The numbers a and b are called **addends**.

In the next stage of development, the child “counts on.” That is, the child begins with the first number and counts however many more the second number represents. A child solving Problem 1 in this manner would say, “4, 5, 6, 7, 8,” probably keeping track of the second addend (5) with fingers. At the next level, the child realizes that it is possible to begin with the larger number (i.e., $3 + 5 = 5 + 3$) and counts, “6, 7, 8.” Finally, the child simply knows that $3 + 5 = 8$.

Contexts for Addition

Problems 1 and 4 are easier for most children, because the child can see the actual marbles and trucks, and count 1, 2, 3, 4, 5, 6, 7, 8 marbles and 1, 2, 3, 4, 5, 6 trucks. Problems 2 and 3 are more abstract in that the child cannot actually see 6 ounces or 4 feet. The child might represent the problem with concrete objects, such as 4 buttons for the yellow ribbon and 3 buttons for the red ribbon. These problems represent two basic contexts in which we operate on numbers. Some numbers represent **discrete** amounts, and some numbers represent measured (**continuous**) amounts.

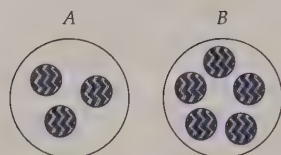


FIGURE 3.1

A Pictorial Model for Addition

One of the themes of this book is the power of “multiple representations”—that there is generally more than one way in which we can represent most mathematical concepts and problems. If we draw diagrams to represent the four addition problems, they seem on the surface to be quite different, except for the marbles and trucks problems (see Figure 3.2).

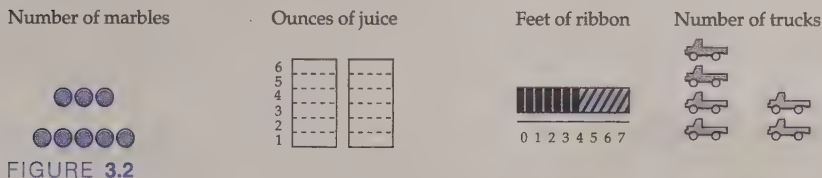


FIGURE 3.2

In each of these cases, however, we are joining two sets or we are increasing a set. We can highlight the similarities among the problems with the representations in Figure 3.3.

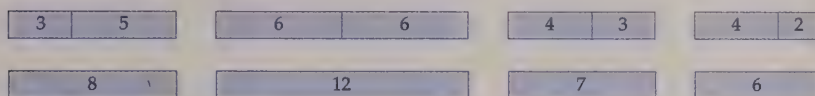


FIGURE 3.3

In general, we can represent any addition problem $a + b = c$ as shown in Figure 3.4.

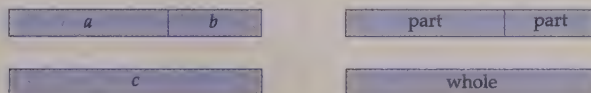


FIGURE 3.4

Do you see any advantages of this general model over having no model at all? If you do, what advantages do you see? Think and then read on. . . 

1. This model captures the way in which *all* addition problems are similar—that is, joining and combining two amounts to make a larger amount.
2. This model is also related to the notion of parts and wholes, an abstraction that is important in the development of whole-number ideas and in understanding other mathematical ideas, like fractions.
3. This model also works well whether the elements to be combined are discrete objects or measurements.
4. Perhaps the most important advantage of the pictorial model is that we can define all four operations in this context, as we shall soon see. This reveals the essential connectedness of the four operations. When students see this connectedness and the connection between numbers and place value, they are likely to be more successful with nonroutine and multistep problems.

Did your list include these advantages? The same ideas in different words? Other advantages?

One researcher who has worked extensively with young children and looked at how children come to make sense of numbers and operations on

numbers tells us that “(p)robably the major conceptual achievement of the early school years is the interpretation of numbers in terms of part and whole relationships.”² It is important to note that understanding part-whole relationships with whole numbers allows numbers to be interpreted simultaneously as positions on the mental number line and as compositions of other numbers. For example, 18 is the number after 17 and before 19, but 18 can also be seen as $10 + 8$, $9 + 9$, $20 - 2$, and so on. Understanding that a number can be composed (put together) and decomposed (broken into parts) is essential for being able to work confidently with the four operations. This notion of composing and decomposing is one of the big ideas of elementary mathematics, and we will come back to it repeatedly throughout the text.

Number Lines: An Addition Model

The number line plays an important role throughout arithmetic, but research shows that American schoolchildren tend to have trouble understanding and using this model. Therefore, we will take a little time to investigate the number line further as a tool for representing (and solving) addition problems. We will come back to the number line when we examine the other operations. Before you read on, try to think of physical examples of number lines in everyday life. . . . 

Number lines are found on rulers, clocks, graphs, and thermometers. A number line can be constructed by taking a line (not necessarily a straight line) and marking off two points: zero (the origin) and one. The distance from 0 to the point 1 is called the unit segment, and the distance between all consecutive whole numbers is the same. Although number lines are most commonly used to represent length, they may be used to model all kinds of problems. For example, we could use a number line to indicate time, with each unit representing one unit of time—day, minute, year, etc.

How could we have represented the ribbon problem with a number line? Write down your thoughts before reading on. . . . 

In the diagram at the left in Figure 3.5, we first draw an arrow (in this case representing the length of the ribbon) 4 units long. We draw another arrow 3 units long and connect the two arrows. The arrow at the top represents the combined length of the two shorter arrows.

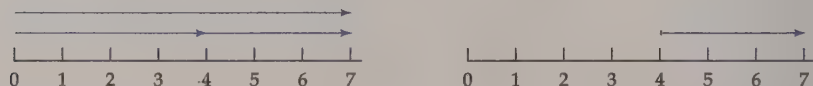


FIGURE 3.5

In the diagram at the right, we start at the point on the line representing the length of the first ribbon and then draw an arrow 3 units long (representing the second ribbon). The location where the arrow ends tells us the combined length of the two ribbons.

Both diagrams represent $4 + 3$ on the number line, although the one on the left more closely resembles the actual laying of the two ribbons end to end.

²Lauren Resnick, “A Developmental Theory of Number Understanding,” in *The Development of Mathematical Thinking*, ed. Herbert Ginsburg (New York: Academic Press, 1983), p. 114.

Properties of Addition

Another aspect of mathematical power comes from understanding the properties of the various operations. As usual, this is best grasped via exploration rather than presentation. Look at Table 3.1, the addition table for base 10.

TABLE 3.1

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Sometimes students in this course forget how new addition is to young children. Recall how strange the Alphabitan system was to you. Think back to your work with Alphabitan: What observations made adding more comfortable for you? Can you think of analogous observations that might make the learning of base 10 addition facts easier for young children?

When young children start to learn about adding, Table 3.1 has been the traditional method of representing the 100 “addition facts” that they have to learn. It can look imposing to some children! However, understanding some properties of addition and base 10 can unlock its potential as a learning tool. Take a few moments to think about what addition means and to look at the addition table. What do you observe (insights or patterns) that might make learning the addition facts easier for children? Think before reading on. . . . 

One observation (in children’s language) is that “adding zero doesn’t change your answer.” Mathematicians call this the **identity property of addition**. It is represented symbolically as

$$a + 0 = 0 + a = a$$

Children discover that when you add 1 to any number, you get the next number from counting. In other words, they connect addition and counting. Did you make this connection when doing the Alphabitan explorations?

We also find that when we add any two numbers, we get the same sum regardless of the order in which we added; that is,

$$a + b = b + a$$

This property is known as the **commutative property of addition** and is one that all children come to recognize. I have been fortunate enough to be with a few children when they made this discovery. They were as excited as

Edison probably was when he finally invented the light bulb. Their faces just shone when they shared with me the discovery that you get the same number either way!

This discovery occurs in different ways. Some students, more naturally curious than others, simply wonder whether the order matters; they discover that it doesn't. For other students, the discovery is of a visual nature. That is, if we draw a diagonal line from the top left corner to the bottom right corner of Table 3.1 and then fold the table in half along that line, we find that the two halves of the table are identical. Mathematically, we speak of the symmetric nature of the addition table.

There is another discovery, called "bridging with 10," that makes learning addition easier. For example, if you ask a child, "What is $7 + 5$?" and then ask how the child got the answer, many children will say something like, " $7 + 3$ is 10 and 2 more is 12." What happens is that students quickly learn the combinations that make a long (that is, 10) when using base 10 manipulatives. Many will intuitively decompose $7 + 5$ into $7 + 3 + 2$.

This leads to a third property, called the **associative property of addition**. For example, $7 + (3 + 2) = (7 + 3) + 2$. Formally, we say

$$(a + b) + c = a + (b + c)$$

One last property of addition often seems almost trivially obvious; it is called the **closure property of addition**. Stated in everyday language, the closure property means that the sum of any two whole numbers is a unique whole number. There are two parts to this property: (1) uniqueness (the sum will always be the same number) and (2) existence (the sum will always be a whole number).

For most college students, this property evokes one of two reactions: either "so what" or "here we go again, making something obvious look complicated." Let me turn the tables on you. How might this property relate to a question a child might actually ask? Think and read on. . . . 

Children will literally ask closure questions: "Will *any* two numbers make a whole number?"

From a more formal perspective, not all sets are closed under addition. For example, consider the set of odd whole numbers $\{1, 3, 5, \dots\}$. The sum of two odd numbers is not in the set of odd numbers; thus, we say that the set of odd numbers is not closed under addition. What about the set of even numbers? In higher mathematics, there are many different kinds of sets, and it is often important to know whether the set is closed under the various operations.

INVESTIGATION 3.1 A Pattern in the Addition Table

6	7
7	8

As you will come to appreciate by the end of this course, patterns can help make the learning of mathematics easier and more interesting. Exploration 3.2 revealed patterns in the Alphabetian addition table. Let us investigate one of many patterns in adding in base 10. In the addition table, if you look at any 2×2 **matrix** (that is, a rectangular array of numbers or other symbols), the sums of the numbers in each of the two diagonals are equal. For example, in the matrix to the left, $6 + 8 = 7 + 7$. Can you justify this pattern mathematically? Work on it and then read on. . . . 

DISCUSSION

Description 1

At a verbal level, one could justify this pattern by saying that in any 2×2 matrix in the table, the two numbers in one diagonal are always identical and the other two numbers are always 1 less and 1 more than this number. Therefore, the sum of the two other numbers will “cancel out” so that you get the “same” sum in either case. If you had a hard time following the previous two sentences, you are not alone, although I have heard second graders explain this pattern in language as elaborate as that above.

Description 2

We can use some notation to make the description less ambiguous. Noting that the value of each number increases by 1 each time that we move across (or down) the table, we can let x represent the number in the top left corner of the diagonal. Thus, in relation to x , the values of the other three numbers are

x	$x + 1$
$x + 1$	$x + 2$

It is now a simple algebraic exercise to demonstrate that the sum of each diagonal is $2x + 2$.

Description 3

Yet other students will say that the sums of the diagonals are equal because “it’s the same numbers in both cases.” What do you think such a student might be seeing? Think before reading on. . . . 

Let’s say that we are looking at the matrix formed by the intersection of the 2 row and the 3 row and the 4 column and the 5 column. The numbers are 6, 7 and 7, 8. However, if we represent the numbers by their origin, we have

	4	5
	$\vdots$	$\vdots$
2...	$2 + 4$	$3 + 4$
3...	$2 + 5$	$3 + 5$

The sum of the top-left-to-bottom-right diagonal is $(2 + 4) + (3 + 5)$. However, because of the commutative and associative properties, this sum is equal to $(2 + 5) + (3 + 4)$. In other words, we are indeed using the same numbers!

We can now generalize this cell by saying that the matrix formed by the intersection of the a row and the b row and the c column and the d column is

	c	d
	$\vdots$	$\vdots$
$a \dots$	$a + c$	$b + c$
$b \dots$	$a + d$	$b + d$

and

$$a + c + b + d = a + d + b + c$$



MATHEMATICS

We have defined addition in the context of whole numbers. In later chapters, we will examine how to extend/modify this definition so that it will fit other numbers, for example,

$$\frac{3}{4} + \frac{2}{5}$$

$$^{-}4 + ^{-}5$$

$$4.342 + 5.6$$

$$x + 22 = 37$$

Can you see that the simplistic notion of addition as joining and combining doesn't fit all of these situations? As we examine extensions of the number line, we will consider how our definition of addition (and other operations) must be refined.

Let us now examine how addition with larger numbers is performed.

Algorithms

If our only goal were to master computation, we could simply do it by rote with lots of practice. However, if our goal is to be able to solve problems, then we need to understand how addition procedures work. Although it is true that calculators (and computers) can do the tedious computations for us, the calculator cannot do the thinking. Remember Investigation 1.15 about dividing the wire into segments and having to find how much wire was wasted? The computation was relatively easy, but interpreting the computation was difficult for many students. In this section, we will examine the meaning of the procedures for adding and subtracting that you have learned. It is critical that as future teachers, you know not only how to compute but also why these established procedures work. The word **algorithm** simply means a standardized procedure, one that can be described in step-by-step terms.

The NCTM curriculum standards emphasize the importance of connecting what we call “procedural understanding” to “conceptual understanding.” In less formal language, they are saying that teachers and students must not only know how to do the procedures but also understand their structure—that is, why they work. Specifically, “It is important to tie these conceptual ideas to more abstract procedures . . . [so that] children will not perceive of mathematics as an arbitrary set of rules; [they] will not need to learn or memorize as many procedures; and [they] will have the foundation to apply, re-create, and invent new ones when needed” (*Curriculum Standards*, p. 32).

With these thoughts in mind, let us investigate the problem of adding larger numbers efficiently. At the end of the discussion, you will understand why different addition algorithms work.

Adding Without Base 10

Imagine that you were a Roman. How might you add these two numbers?

$$\begin{array}{r} \text{XXXVII} \\ + \text{XLVI} \\ \hline \end{array}$$

As you found in Section 2.3, Romans couldn't just add the VII to the VI and "carry" the III.

We are not certain how the Romans actually added. There is evidence that the Babylonians did calculations in the sand with pebbles and inscribed the results on clay tablets. In fact, our word *calculate* comes from *calculus*, which means "small stone." This pebble method is very possibly a precursor of the abacus, which has been used extensively in Asia. The term *carrying* may have also come from carrying the pebbles from one column to the next. We do know that computation was rather cumbersome and that few people in those days could do what we call basic arithmetic. They were accorded high status in the society because of their knowledge!

Investigating Addition Algorithms in Base 10



LEARNING

The National Assessment of Education Progress (NAEP) has found that third graders' success in computation decreases considerably when they move from two-digit problems to three-

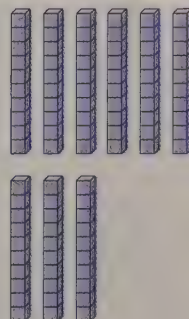
digit problems. Further research indicates that many students are memorizing rather than understanding the process.

Let us look at one of the more difficult of the three-digit addition problems, one in which we encounter zeros. Many of my students find that, even though procedurally we treat zero "just like any other number," they feel just a bit uncomfortable when they run into zeros. Therefore, an analysis of such a problem is often helpful.

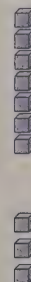
Add $267 + 133$ as you normally would. Can you explain the "whys" of each step? Try to do so before reading on. You may or may not wish to use base 10 blocks. 

In the context of our base 10 numeration system, we are combining 2 hundreds, 6 tens, and 7 ones with 1 hundred, 3 tens, and 3 ones. If the student understands the process of combining and regrouping, then this problem is *not* substantially more difficult than one with smaller numbers; it is only longer.

Using manipulatives



Using words

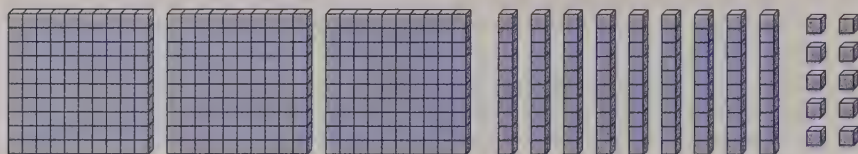


2 hundreds, 6 tens, 7 ones

+

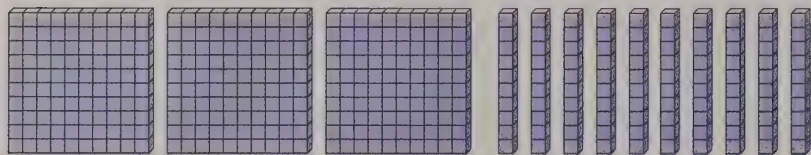
1 hundred, 3 tens, 3 ones

Recalling what addition means, when we literally combine the two sets, we have 3 hundreds, 9 tens, and 10 ones.



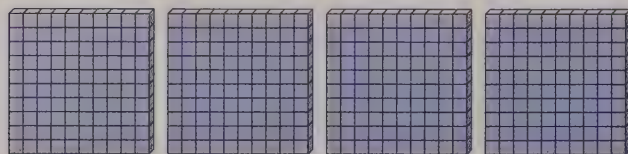
3 hundreds, 9 tens, 10 ones

However, this is not a valid answer. Thus, we have to regroup. After we “trade” the 10 ones for 1 ten, we now have 3 hundreds, 10 tens, and 0 ones.



3 hundreds, 10 tens, 0 ones

Because we now have 10 tens (once again not a valid representation in base 10), we trade them for another hundred, giving us 4 hundreds, no tens, and no ones—that is, 400.



4 hundreds, 0 tens, 0 ones

A commonly used algorithm Now look at one **common algorithm for addition** (the one you probably learned in grade school). Using this algorithm, we go from right to left (that is, we begin with the ones place).

$7 + 3 = 10$; place the 0 in the ones place and put the 1 above the tens place.

$$\begin{array}{r} 1 \\ 267 \\ 133 \\ \hline 0 \end{array}$$

$1 + 6 + 3 = 10$; place the 0 in the tens place and put the 1 above the hundreds place.

$$\begin{array}{r} 11 \\ 267 \\ 133 \\ \hline 00 \end{array}$$

$1 + 2 + 1 = 4$; place the 4 in the hundreds place. The sum is 400.

$$\begin{array}{r} 11 \\ 267 \\ 133 \\ \hline 400 \end{array}$$

Can you explain the *whys* behind the *hows* of this algorithm? That is, in the language of the NCTM standards, how can you justify this procedure? Think about this before reading on. . . . 

There are two levels of justification of this algorithm. First, let us focus on justifying the actual procedure. Then, at a more formal level, we will examine the properties that underlie the procedure.

Focusing on the procedure, when we add 7 ones and 3 ones, we have 10 ones, but in our numeration system, we can have at most 9 (of any unit) in a place. Therefore, we must regroup; in this case, 10 ones become 1 ten and 0 ones. Therefore, we place the numeral 0 in the ones place and the numeral 1 in the tens place. Looking at the tens place, we find that we now have 1 ten plus 6 tens plus 3 tens, making 10 tens. As before, this is not allowed in our numeration system, and so we regroup the 10 tens as 1 hundred and 0 tens. As before, we place the numeral 0 in the tens place and the numeral 1 in the hundreds place. Looking at the hundreds place, we find that we have 1 hundred plus 2 hundreds plus 1 hundred, giving us 4 hundreds. Reading our sum, we have 4 hundreds, 0 tens, and 0 ones—that is, 400. Do you see the relationship between this justification and what we did physically with the base 10 blocks?

We can also examine this procedure from a more formal perspective. Representing the problem in expanded form enables us to understand why it works. Knowing that addition can be seen as combining, we can represent our problem in the following way.

Statement	Justification
$267 + 133$	
$= (2 \cdot 100 + 6 \cdot 10 + 7) + (1 \cdot 100 + 3 \cdot 10 + 3)$	Expanded form
$= (2 \cdot 100 + 1 \cdot 100) + (6 \cdot 10 + 3 \cdot 10) + (7 + 3)$	Commutative and associative properties
$= (2 + 1)100 + (6 + 3)10 + (7 + 3)$	Distributive property
$= 3(100) + 9(10) + 10$	Addition
$= 3(100) + (9 + 1)10$	Distributive property
$= 3(100) + 10(10)$	Addition
$= 3(100) + 100$	Multiplication
$= (3 + 1)100$	Distributive property
$= 4(100)$	Addition
$= 400$	Multiplication



LEARNING

A friend of mine, David Sobel, was talking about mathematics with his six-year-old daughter, Tara. They were doing double-digit addition. David had just shown Tara that $20 + 20 = 40$. Tara thought for a moment and then proudly announced that $50 + 50$ must be 70. When David asked how she had got that answer, she said, "When you add the same numbers that have a zero at the end, you just skip ten!" In so many mathematical situations, there are many ways to interpret the data. It takes time and practice to improve your ability to analyze and interpret patterns. It is important to note that both David and I were encouraged, not discouraged, by Tara's thinking. Why was this?

Seeing how numbers can be *composed* and *decomposed* makes it possible to understand the algorithm deeply. To understand the addition algorithm, we **decompose** the number using expanded form; we can then see how those different parts can be reconfigured—**composed**—to make our new whole, that is, the sum. When you were challenged to add amounts in Alphabitia in Exploration 3.1, at first you may have found that the manipulatives were essential, both because their concreteness made you more comfortable (for example, BAC was B flats, A longs, and C units) and because you could "see" the addition process actually happening. In one sense, the manipulatives let you see the parts (expanded form) and the whole at the same time.

Subtraction

Now let us investigate the operation of subtraction, which presents some interesting issues for young children. We will connect the pictorial model developed for addition to subtraction, but first let's look at some questions to make sure you are actively involved.

Assume that you are a young child who has not yet learned subtraction. How might you solve these problems—using manipulatives or diagrams, or in your head using common sense and other mathematical knowledge? Purposely “forgetting” what you already know about subtraction will probably clear the way for you to see beyond the mechanics of the operation to its underlying meaning—and that's what this investigation is about. Note your thoughts before reading on. . . . 

Five subtraction problems

1. Joe had 7 marbles. He lost 2 in a game. How many does he have left?
2. Billy has 23 marbles and Yaka has 32. How many more does Yaka have?
3. At the beginning of the week, the hospital had 32 ounces of insulin. During the week, 14 ounces of insulin were used. How much insulin did the hospital have at the end of the week?
4. Alicia was 45 inches tall at the beginning of the first grade and 53 inches at the end of the third grade. How many inches did she grow during this time?
5. The Jones farm has 35 pigs and 18 chickens. How many more pigs are there than chickens?

Now go back to these five problems and answer the following questions:

- What action words describe what is happening in these four problems?
- In what ways are the problems different? In what ways are they similar?
- What does subtraction mean? For example, what words come to mind when you think of subtraction?

Contexts for Subtraction

One way in which the problems are different is connected to the two contexts we discussed for addition—that is, whether we are subtracting discrete sets or measured amounts.

Problems 1, 2, and 5 are discrete problems—we can represent the problems directly with pictures.

Problems 3 and 4 are measurement (continuous) problems—ounces and inches are measured amounts, representing how many of a certain unit we are talking about.

These problems differ in another way, and this difference is connected to the idea of parts and wholes that we find in all four operations. In “take-away” problems, a part is taken away from a whole. In “comparison” problems, two amounts are being compared. In some cases, the comparison is between two subsets of a large set (two parts of a whole); in other cases, the comparison is between two sets (wholes). Can you determine which of the five problems above are take-away and which are comparison? Think and then read on. . . . 



CHILDREN

Children often have trouble seeing the comparison model as subtraction. Do you see why? Think before reading on. . . .



Part of the reason is that many students will determine the answer by counting up from the smaller number. For example, when determining how many inches Alicia grew, many children see this as “45 plus what makes 53?”—that is, $45 + ? = 53$. For this reason, some teachers refer to the comparison model as the **missing addend model**.

Even with larger numbers, many children prefer to count up rather than to subtract. They will subtract if they are forced to, but it frequently does not jibe with their common sense. Recall that one of the lines in Standard 3 is that students should believe that mathematics makes sense.



CHILDREN

I recall reading, several years ago, a paper about children with mathematical learning disabilities that speaks to the need for learners to see the connectedness of mathematical ideas. The author was called in to diagnose a third grader who was having difficulties with mathematics. The author discovered that the child did not see addition and subtraction as related, was not able to see the operations as parts and wholes, and was not able to use the commutative property of addition. That is, she saw the four number sentences on the right as four unrelated number sentences. It was no wonder that she was having difficulty remembering her addition and subtraction facts!

Problems 1 and 3 are take-away problems, whereas the other three problems fit the comparison model. Figure 3.6 shows the subtraction problem $7 - 2$ in the two contexts.

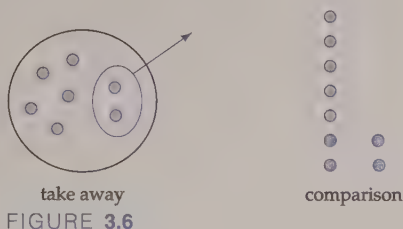


FIGURE 3.6

A Pictorial Model for Subtraction

Now look at all five subtraction problems, and put aside the categories (discrete measurement, take away, and comparison). Is it possible, as we did with addition, to represent all of these problems with a single model? Write your thoughts and then read on. . . .

One way to express the commonality is that in all cases, we have a large amount and two smaller amounts whose sum is equal to the larger number. Recall the pictorial model for addition: If we invert that diagram, we have a pictorial representation of a general model for subtraction (see Figure 3.7).

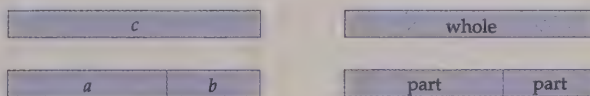


FIGURE 3.7

It is important to note that *whole* and *part* refer to the numerical values rather than the contexts. For example, in Problem 2, we are comparing two wholes and saying that one whole contains 9 more marbles than the other.

We define **subtraction** formally in the following manner:

$$c - b = a \text{ iff } a + b = c$$

That is,

The difference between two numbers c and b is a if and only if c is the sum of a and b .

In mathematical language, c is called the **minuend**, b the **subtrahend**, and a the **difference**.

This model also highlights the connections between addition and subtraction problems (see Figure 3.8):

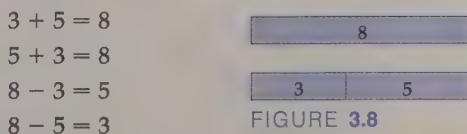


FIGURE 3.8

Referring to the composition and decomposition notion, we can join 3 and 5 to make 8, and we can decompose 8 into 5 and 3. We will use this awareness of the ability to break amounts into convenient parts when we examine mental computation and estimation in Section 3.3.

Subtraction with Number Lines

Many measurement situations, such as time and distance, naturally lend themselves to a number line representation. Let's use an example to explore this model. Joanne had 8 feet of rope and used 3 feet to stake a tent. How much rope does she have left? Try to represent this problem with a number line and then read on. . . . 

As with addition, we can use a number line to solve the problem in either of two ways (see Figure 3.9).

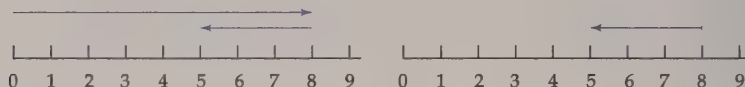


FIGURE 3.9

Do both ways make sense? Does one way feel more comfortable to you? As with the addition number lines, the left diagram more closely resembles the physical representation of the problem.

Properties of Subtraction

Think back to the properties of addition—identity, commutativity, associativity, and closure. Do those same properties hold for subtraction? Think and then read on. . . . 

Some students think that subtraction has an identity property, because if we take away zero from a number, its value does not change; that is, $a - 0 = a$. This is true; however, if we reverse the order, the result is not true—that is, $0 - a \neq a$. Therefore, we generally say that the operation of subtraction does not have an identity property. In higher mathematics, we say that the operation of subtraction has a right-identity but not a left-identity. After examining a few cases, you can see that the operation of subtraction does not possess the commutative property or the associative property. The commutative property is not immediately understood by children. I recall one first grader arguing that $3 - 5$ was 0 because “you can’t have less than nothing” and another arguing that it was 2 because “you just turn them around.” Finally, the operation of subtraction is not closed for the set of natural numbers because the difference of two numbers can be a negative number.

INVESTIGATION 3.2 How Far Is It?

This problem, which actually happened to me, provides a good opportunity for you to apply your developing problem-solving tools and your understanding of subtraction and number line models to a real problem. I had to go to a meeting one day in Concord, New Hampshire. It was a business meeting, so I was to be reimbursed for my mileage. When I left Keene, I noted that my odometer read 26,688. Unfortunately, I forgot to check my odometer in Concord. However, on the way home, I saw a sign that said I was 36 miles from Keene. At that point, my odometer said 26,768. I now had enough information to determine the round-trip distance between Keene and Concord. What did I come up with? Try to solve this problem on your own before reading on. If you get stuck, what problem-solving tools might be useful? 

DISCUSSION

Of course, I could have waited until I had gotten home, taken the mileage then, and subtracted 26,688 from that number. However, that would not be a challenging problem, and I did this actual problem because I was aware that there was a reasonable probability that I would forget to note my mileage when I got home!

This problem practically begs for a number line diagram. However, as many of my students have told me, creating good diagrams is easier said than done. To make this point more concrete, many students tell me that when they see a good diagram, it often makes sense, but that they don't know how to create the diagram themselves. In this problem, a useful diagram often emerges in two stages. For example, students respond in different ways to the diagram in Figure 3.10. Some students don't immediately understand the diagram, others understand it but don't find that it leads to the answer, whereas still others see it and "Eureka!" the solution to the problem literally appears right before their eyes. If you find that this diagram is not a eureka experience, what might be added to the diagram or how it could be modified? Please try to modify it yourself before reading on. . . .

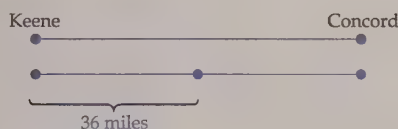


FIGURE 3.10

There are different ways in which students find this diagram useful. For some students, the diagram points to a further step in the problem. For example, they now know what to do with 26,688 and 26,768. For other students, the diagram tells them that they have enough information to find the round-trip distance. Yet other students use algebraic notation and see two equations. Depending on how you see the problem, your modified diagram might look like either of the ones in Figure 3.11. Using either of these diagrams, or one of your own making, can you solve the problem now? Try to do so before reading on. . . .

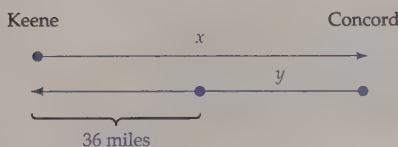


FIGURE 3.11

One key to this problem is to use given information to produce more information. If we subtract 26,688 from 26,768, we get 80; what does that tell us? It tells us that I had traveled 80 miles at that point; adding this 80 to 36, we know the round-trip distance. A student using algebra would have an

equation, $x + y = 80$, and would come to the same realization: The round-trip distance is 116 miles.



BEYOND THE CLASSROOM

In Investigation 3.2, we found the mathematical solution to the problem. Had I remembered to check my odometer in Concord or in Keene, however, it very well might not have shown that I drove 58 miles to Concord or 116 miles round trip. Do you see why? Think and then read on. . . . 

The 36-mile sign does not mean that I am 36 miles away from my house. It means that I am 36 miles from Keene. But that begs the question "36 miles

from *where* in Keene?" For example, think of a sign telling you that you are 36 miles from New York City. What does that mean? Does it mean 36 miles to the center of New York City or to the city limits? In general, it means that you are 36 miles from what the map makers consider to be the geographical center of the city, not the city limits. Do you see why mileage signs are made this way? How they determine the center is another question. What do you think is the center of the town closest to you?

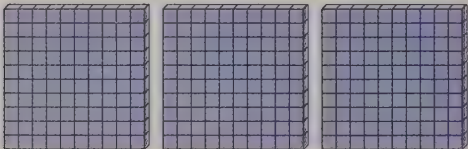
Understanding Subtraction in Base 10

Let us now examine some standard and nonstandard algorithms for subtraction. Again, recall the Alphonatian problems; for a child, subtracting $32 - 14$ can be as overwhelming and confusing as many students originally found $DA - BC$ in the Alphonatian system.

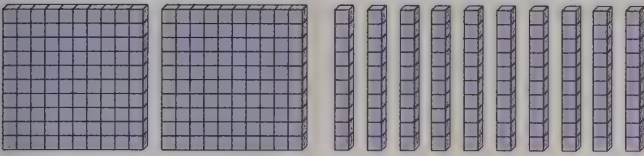
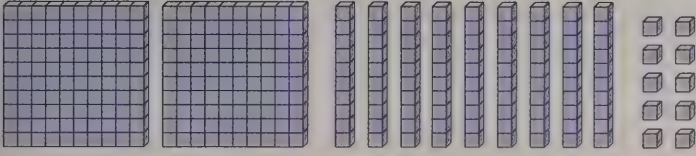
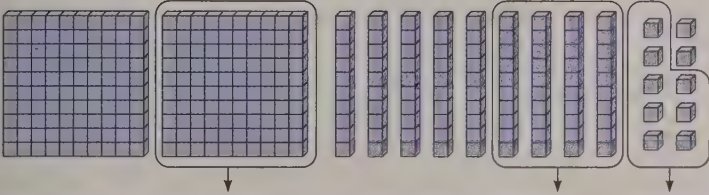
When nothing remains, put down a small circle so that the place be not empty, but the circle must occupy it, so that the number of places will not be diminished when the place is empty and the second be mistaken for the first.³

This quote comes from a text written by Al-Khowarizmi in A.D. 825 as he tried to explain the procedure for subtracting in the new (Hindu) numeration system. Do you understand what he is saying? What function does the circle serve?

Researchers have found that when subtraction problems have zeros, the success rate for most third graders goes down drastically. However, this need not be so. Let us examine the following problem: $300 - 148$. Although there is no single procedure that all students use to solve this problem (unless they are forced to by their teacher), we will examine here how this problem might be solved going from left to right.

Using manipulatives	Using words	Using symbols
	We need to regroup 300 so that we can "take away" 148.	$\begin{array}{r} 300 \\ - 148 \\ \hline \end{array}$

³Karl Menninger, *Number Words and Number Symbols: A Cultural History of Numbers* (Cambridge, Mass.: MIT Press, 1969), p. 413.

Using manipulatives	Using words	Using symbols
	We can trade one of the 3 hundreds for 10 tens, giving us 2 hundreds and 10 tens, which has the same value as 3 hundreds.	$\begin{array}{r} 2 \ 1 \\ \cancel{3} \ 0 \ 0 \\ - 1 \ 4 \ 8 \\ \hline \end{array}$
	Next, we can trade one of the tens for 10 ones. We now have 2 hundreds, 9 tens, and 10 ones, which still has the same value as 3 hundreds.	$\begin{array}{r} 2 \ 9 \ 1 \\ \cancel{3} \ \cancel{0} \ 0 \\ - 1 \ 4 \ 8 \\ \hline \end{array}$
	Now we can take away 1 hundred, 4 tens, and 8 ones.	$\begin{array}{r} 2 \ 9 \ 1 \\ \cancel{3} \ \cancel{0} \ 0 \\ - 1 \ 4 \ 8 \\ \hline 1 \ 5 \ 2 \end{array}$

Justifying the Standard Algorithm

It is beyond the scope of this book to prove formally every algorithm, procedure, and theorem. What is essential is that elementary students and teachers understand the algorithms that they use—that they understand the whys of the algorithm. Does the explanation above help you to understand the left-to-right subtraction procedure? If not, please read it again or refer to your notes from the Alphabitia explorations, or work with a friend so that it does make sense.

Some students find that representing the problem in expanded form is helpful. A key to understanding this algorithm is to understand the equivalence of 3 hundreds and 2 hundreds, 9 tens, and 10 ones. Although they look different, and the numerals are different (300 versus 29¹0), the amounts are equal. Another key to understanding this algorithm is to know why we needed to do the regrouping. This is easier to see at the physical level; without these regroupings, we cannot literally “take away” 1 flat, 4 longs, and 8 units.

3 hundreds + 0 tens + 0 ones - 1 hundred + 4 tens + 8 ones Cannot subtract the tens or the ones	→ 2 hundreds + 10 tens + 0 ones - 1 hundred + 4 tens + 8 ones Cannot subtract the ones	→ 2 hundreds + 9 tens + 10 ones - 1 hundred + 4 tens + 8 ones Subtraction possible
---	--	--

An Alternative Algorithm

It is important to note that the standard algorithm (in the discussion above) is more closely connected to the take-away context for subtraction than to the comparison or missing addend context. For example, consider the problem



LEARNING

It is important to reemphasize that “standard” algorithms are not the “right” ones, or even the “best” ones, but rather the ones that, for various reasons, have become most widespread. Many educators believe that more harm than good is done by forcing all students to learn the “standard” algorithms.

802 – 238 in a missing addend context: Joan wants to buy a computer for \$802; if she has \$238, how much more does she need? If we are solving this problem in the missing addend context, we are asking what number combined with 238 gives us 802. One possible algorithm would begin with the smaller amount and add to it, beginning with the hundreds place and finishing with the ones place. Both of my children invented this procedure when I asked them to tell me how they determined the difference before they were taught the standard algorithm.

WHAT IS THOUGHT	WHAT IS WRITTEN	
	“Answer” column	“Working” column
How many hundreds can we add to 238 without going over 802? We can add 500, which brings us to 738.	500	738
Now, how many tens can we add to 738 without going over 802? We can add 60, which brings us to 798.	60	798
How many ones do we need to add to 798 to get to 802? We need 4 ones.	4	802

The answer of 564 is obtained by adding the numbers in the answer column. Though many college students find this algorithm initially awkward, when elementary children are encouraged to develop their own means for answering subtraction problems, we see algorithms like this one much more often than the more standard ones that many textbooks contain. Exploration 3.5 illustrates the fact that the standard algorithm is one of many possible algorithms and is not necessarily the one that children find easiest to understand.

As with addition, the composition and decomposition of numbers enables us to understand better both the standard subtraction algorithms and other algorithms for subtraction. When you explored subtraction in Alphabitia in Exploration 3.3, you may have found that not all students subtracted in the same way; some went left to right, some went right to left, and for some it depended on whether there were zeros in the subtrahend. Regardless of the procedure, subtracting with confidence requires that we be comfortable with the idea of decomposing and recomposing the number.

“Carrying” and “Borrowing”

Traditionally, elementary teachers have tended to use the word *carry* when regrouping for addition and *borrow* when regrouping for subtraction. Many current elementary teachers do not use these words. Can you guess why? Think and then read on. . . . 

One problem is that the term *borrow* is misleading. When I borrow an egg from a neighbor or when I borrow a tissue, that is different from “borrowing” a ten.

Another problem with carry and borrow is that the terms imply two different processes, when in fact the same process is involved: trading (regrouping). We trade up when carrying, and we trade down when borrowing. For example, consider the related addition and subtraction problems $36 + 28 = 64$

and $64 - 36 = 28$. When we add the digits in the ones place, we trade in 10 of the ones for 1 ten, and so 14 ones becomes 1 ten and 4 ones. In the subtraction problem, we go the other way: We trade a ten for 10 ones, and so 1 ten and 4 ones become 14 ones.

(14 ones $\rightarrow$ 1 ten and 4 ones) (1 ten and 4 ones $\rightarrow$ 14 ones)

$$\begin{array}{r} 1 \\ 36 \\ + 28 \\ \hline 64 \end{array}$$

$$\begin{array}{r} 51 \\ \cancel{6}4 \\ - 36 \\ \hline 28 \end{array}$$

Many adults do not realize that these processes are virtually identical; they have never stopped to think about it. When learning the algorithms for the first time, young children find it much easier if there is a reason for what they do. The connection between these two processes is further examined in Exploration 3.3.

Summary

In this section, we have examined the operations of addition and subtraction. You now know that addition can mean to combine or join two sets; it can also mean to increase a set by a certain amount. You know that subtraction can mean to take away an amount from a set, it can mean to ask how much bigger one set is than another, and it can also mean to ask how much a set must increase in order to get to a certain amount. Especially in the case of subtraction, you have seen that these meanings are not always obvious to children.

Addition and subtraction problems arise from discrete or measurement (continuous) contexts. We can represent these problems in various ways: with circles, with part-whole rectangles, or with number lines. The choice of model sometimes depends on personal preference and sometimes depends on the nature of the problem being solved.

These representations help us to see connections between addition and subtraction. In one sense, addition consists of adding two parts to make a whole. In one sense, subtraction consists of having a whole and a part and needing to find the value of the other part. You are becoming more comfortable with the idea of composition and decomposition, which appears repeatedly throughout mathematics. In one sense, addition is an act of composition, putting together two parts to make a new whole. In one sense, subtraction is an act of decomposition, breaking a whole into a given part and a part we need to find.

We have examined patterns in the addition table. Recognizing these patterns helps children to become more comfortable with adding. We have seen that describing the patterns that we see requires some thinking and that mathematical vocabulary and definitions, developed over the centuries, can make communication easier and reduce ambiguity.

You have learned that there are certain properties that hold for addition of whole numbers but not for subtraction: identity, commutativity, associativity, and closure. The identity property may seem obvious now, but it will be seen as an important part of the whole set of properties in Section 3.2.

Finally, we have examined both standard and alternative algorithms for whole-number addition and subtraction. You have learned that the algorithms we currently use in the United States are not the only possible procedures for computing. The ability not only to compute with these algorithms but also to explain why they work is a crucial part of developing mathematical power.

In Section 3.2, we will do a similar examination of multiplication and division, and then we will examine the relationships among all four operations.

EXERCISES 3.1

1. Refer to the base 10 addition table. If we look at any of the diagonals going in a top-left-to-lower-right direction, the value of each successive term increases by 2 each time. Explain this pattern.

2. In many textbooks, students are encouraged to learn their fact families. This is the fact family for 10: $1 + 9$, $2 + 8$, $3 + 7$, $4 + 6$, $5 + 5$, $6 + 4$, $7 + 3$, $8 + 2$, $9 + 1$. Transform each fact pair to an ordered pair and graph the pairs on graph paper. Why does the line look the way it does?

3. Below is an addition algorithm from an old text. Explain why it works.

$$\begin{array}{r} 36 \\ +48 \\ \hline 14 \\ 7 \\ \hline 84 \end{array}$$

4. Solve the following addition problems with the lattice algorithm:

$$\begin{array}{r} \text{a. } 5568_{10} \\ +2745_{10} \end{array} \quad \begin{array}{r} \text{b. } 322_5 \\ +234_5 \end{array} \quad \begin{array}{r} \text{c. } 764_8 \\ +215_8 \end{array}$$

5. Pretend you are a young child again. Which algorithm do you think would be easiest to learn first: the standard algorithm, the partial sums algorithm, or the lattice algorithm? Explain your reasoning.

6. Finish the following addition problem, which has been done in expanded form.

$$\begin{array}{r} 345 = 3 \text{ hundreds} + 4 \text{ tens} + 5 \text{ ones} \\ +268 = 2 \text{ hundreds} + 6 \text{ tens} + 8 \text{ ones} \end{array}$$

7. Identify and explain the errors in each of the problems below, all of which have occurred in real classrooms.

$$\begin{array}{r} \text{a. } 36 \\ +28 \\ \hline 91 \end{array} \quad \begin{array}{r} \text{b. } 36 \\ +28 \\ \hline 514 \end{array} \quad \begin{array}{r} \text{c. } 365 \\ +287 \\ \hline 742 \end{array}$$

8. Place the digits 1, 2, 3, 6, 7, and 8 in the boxes to obtain

$$\begin{array}{r} \text{a. The greatest sum} \\ \text{b. The least sum} \end{array} \quad \begin{array}{r} \square \square \square \\ +\square \square \square \\ \hline \end{array}$$

9. Choose among the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 to make the sum equal 500. You may use each digit only once. How many different ways can you make 500?

$$\begin{array}{r} \square \square \square \\ + \square \square \\ \hline \end{array}$$

10. Determine the values of N and P that make this a true statement. Explain how you arrived at the answer.

$$\begin{array}{r} N \\ N \\ N \\ + P \\ \hline PN \end{array}$$

11. How many different ways can you make this a true statement, using each of the whole numbers 1 through 9 only once?

$$\begin{array}{r} \square \square \square \\ \square \square \square \\ +\square \square \square \\ \hline 9 \ 9 \ 9 \end{array} \quad \text{For example, } \begin{array}{r} 152 \\ 368 \\ \hline 479 \end{array}$$

12. Make up a subtraction story problem for each of the following contexts. Briefly *explain* why the story problem is an example of the particular model.

- Take-away
- Missing addend
- Comparison

13. a. Explain why the operation of subtraction is not commutative.

- b. Explain why the operation of subtraction is not associative.

14. A student's work on a subtraction problem is shown here. Explain the numerals in the top row, 3, 9, 10, as though you were talking to a third grader who is having difficulty with the process.

$$\begin{array}{r} 3 \ 9 \ 1 \\ \cancel{4} \ 0 \ 0 \\ -1 \ 3 \ 5 \\ \hline 2 \ 6 \ 5 \end{array}$$

15. Examine the problem at the right.

- a. Explain, in mathematical terms, what the student did wrong. Where are the problems in the student's thinking?

$$\begin{array}{r} 3 \ 1 \ 1 \\ \cancel{4} \ 0 \ 0 \\ -2 \ 3 \ 6 \\ \hline 1 \ 7 \ 4 \end{array}$$

- b. How would you help the student if the quote below represented her explanation:

"You can't take 6 from 0 and you can't take 3 from 0, so you have to borrow from the four. Cross out 4 and write 3, cross out the zeros and write 10."

16. This is a true story. A second-grade teacher asked her students to explain how they had solved the problem below. One student said the following: "If you take 6 away from 2, you get minus four. When you take 50 from 70, you get 20. Then minus 4 plus 20 is 16." What do you think of this student's thinking? Was he just lucky on this problem, or could it qualify as an algorithm? Does it extend to bigger numbers?

$$\begin{array}{r} 72 \\ -56 \\ \hline -4 \\ \hline 20 \\ \hline 16 \end{array}$$

17. Identify and explain the errors in each of the problems below, all of which have occurred in real classrooms.

$$\begin{array}{r} 76 \\ -48 \\ \hline 32 \end{array}$$

$$\begin{array}{r} 76 \\ -48 \\ \hline 22 \end{array}$$

$$\begin{array}{r} 76 \\ -48 \\ \hline 38 \end{array}$$

$$\begin{array}{r} 70 \\ -48 \\ \hline 38 \end{array}$$

$$\begin{array}{r} 70 \\ -48 \\ \hline 32 \end{array}$$

$$\begin{array}{r} 700 \\ -482 \\ \hline 228 \end{array}$$

18. Place the digits 1, 2, 3, 6, 7, and 8 in the boxes to obtain

a. The greatest difference

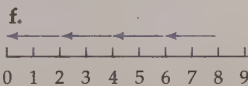
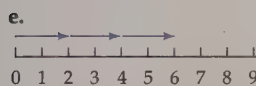
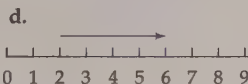
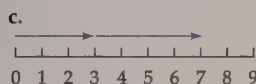
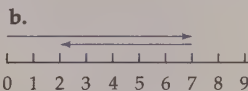
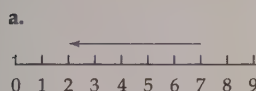
b. The least difference

$$\begin{array}{ccc} \square & \square & \square \\ -\square & \square & \square \\ \hline \end{array}$$

19. Choose among the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 to make the difference 234. You can use each digit only once. How many different ways can you make 234?

$$\begin{array}{ccc} \square & \square & \square \\ -\square & \square & \square \\ \hline \end{array}$$

20. For each number line problem below, identify the computation it models and briefly justify your answer.



21. Represent the following problems on a number line. Explain each problem as though you were talking to someone who is not taking this class.

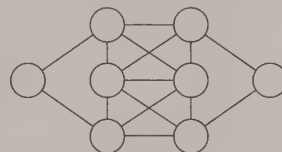
a. $5 + 4$

b. $8 - 3$

22. Respond to the following as though you were talking to a principal who is familiar with the NCTM standards.

You want to convince the principal to hire you. The principal has just said that she has only recently realized that the words *borrowing* and *carrying* are no longer used by effective math teachers. Rather, they use alternative terms like *renaming* or *regrouping*. The principal asks whether you agree with the change in language and, if so, why.

23. With three boys on a large scale, it read 170 pounds. When Adam stepped off, the scale read 115 pounds. When Ben stepped off, the scale read 65 pounds. What is the weight of each boy?
24. A mule and a horse were carrying some bales of cloth. The mule said to the horse, "If you give me one of your bales, I shall carry as many as you." "If you give me one of yours," replied the horse, "I will be carrying twice as many as you." How many bales was each animal carrying?
25. Place the numbers 1 through 8 in the circles so that no two consecutive numbers are in circles that are adjacent to each other (that is, are directly connected).



26. Briefly describe what you learned about addition and subtraction in this section. What do they mean, how are they connected, and why do the algorithms work. Then select the most important learning and describe what it was that you learned and why this is the most important learning. For example, "I always had trouble subtracting when there were zeros in the top number (subtrahend). Now I understand why." Of course, you would include an example (i.e. $800 - 134$) and explain how it works.

SECTION

3.2

UNDERSTANDING MULTIPLICATION AND DIVISION

Do you remember learning about multiplication and division in elementary school? If so, what do you remember? When I ask my own students this question, most of their answers have to do with trying to learn the multiplication table and with the frustration of learning the long-division procedure. Admittedly, multiplication and division are more complex concepts than addition and subtraction. However, students who develop a good understanding of the structures of multiplication and division and their connections with each other and with addition and subtraction report being amazed to find that multiplication and division can actually make sense!

As in Section 3.1, we will first examine the multiple meanings of each operation and models for representing them. Examination of patterns and properties will lay the groundwork for understanding how the algorithms

WHAT DO YOU THINK?

- What other meanings are there for multiplication beyond “repeated addition”?
- Why do we move over when we multiply?
- Are there more even or more odd numbers in the multiplication table?
- Do you think our multiplication and division algorithms would work with the Roman or Egyptian numeration system?
- When we do long division, we “bring down” the next number. What does *bring down* mean?



CHILDREN

Many young children encounter the concept of multiplication even before school; for example, if the tooth fairy gives me fifty cents for each tooth, how much will I get for all my teeth?

work. Then we will examine the connections among the various operations. This, in turn, paves the way for a strong operation sense, which enables you to solve the nonroutine and multistep problem situations that pervade everyday and work settings.

Multiplication

Many elementary methods textbooks group addition and subtraction in one chapter and then multiplication and division in another chapter. This is not simply because children learn addition and subtraction first but is also because multiplication and division are more complex and abstract concepts. In fact, many of my students report that although they learned *more* about addition and subtraction in this course, they learned quite a bit of *new* material about multiplication and division. Mathematics starts to crumble for many students when they get to multiplication, especially learning (or memorizing) the multiplication tables. In fact, many adults do not know all the single-digit multiplication facts, such as 8×7 . Given how available calculators are, this is not a huge problem, although it is sometimes embarrassing. However, more problematic is the fact that few adults understand the different meanings of multiplication, and this lack of knowledge limits their ability to solve real-world problems. One of the goals of the following examination of multiplication is for you to understand the different meanings of multiplication and why it is useful to know these different meanings.

Assume that you are a young child who has not yet learned multiplication. How might you solve these problems—using manipulatives or diagrams, or using common sense and other mathematical knowledge? Note your thoughts before reading on. . . . 

Four multiplication problems

1. One piece of candy costs 4¢. How much would 3 pieces cost?
2. If Jackie walks at 4 mph for 3 hours, how far will she have walked?
3. A carpet measures 4 feet by 3 feet. What is the area of the carpet?
4. Carla has 4 blouses and 3 skirts. How many combinations can she wear? (Assume that all possible combinations go together!)

Contexts for Multiplication

Now go back to these four problems and answer the following questions:

- In what ways are the problems different? In what ways are they similar?
- What does multiplication mean? For example, what words come to mind when you think of multiplication?

If we represent the four problems with diagrams, as in Figure 3.12, the differences appear to be tremendous, even though all four problems are solved as 3 times 4.

Problem 1 is like the discrete sets we encountered with addition and subtraction. It is also literally **repeated addition**; it is solved by adding $4 + 4 + 4$.

Problem 2 involves a certain number of measured units and can be represented with a number line that shows repeated addition of the measures.

Problem 3 involves measures, but it involves repeated addition less than it involves counting the number of new units. This problem illustrates the area context of multiplication.

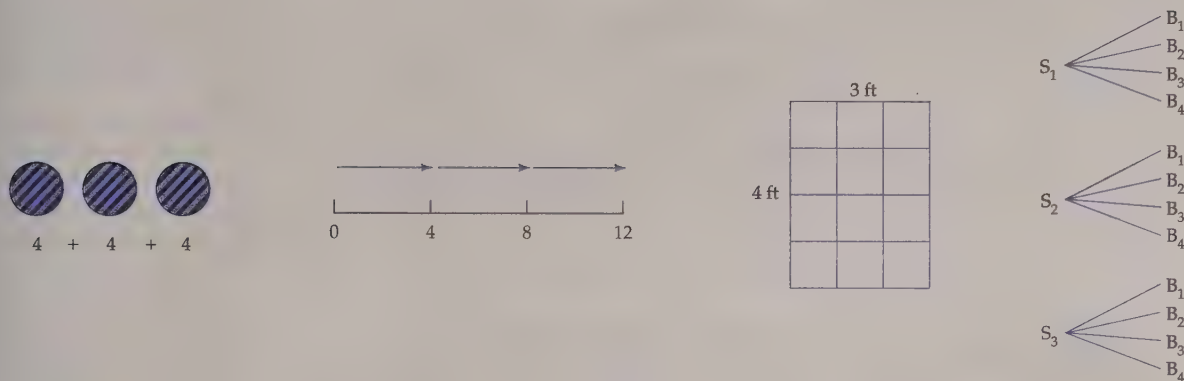


FIGURE 3.12

Problem 4 involves discrete objects, but it relies on repeated addition less than it relies on putting combinations in an array. In set language, what we did in this problem was to examine two sets and look at all possible ways of pairing the elements of those sets. We will see later in this section that we can represent this context with the *Cartesian product model of multiplication*.

A General Model for Multiplication

As you can see, multiplication is a more complex concept than addition and subtraction, and it is more than simply repeated addition. Can you describe the four problems in such a way that they all have something in common, as we did for addition and subtraction? Work on this question and then read on. . . .

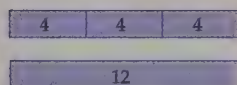


FIGURE 3.13

Our general model for addition can be extended to multiplication. Figure 3.13 shows one diagram for $3 \cdot 4$. Do you see the resemblance between the addition and multiplication models? As with addition and subtraction, the *general model of multiplication* can be cast in part-whole language: The product (the whole) is built from parts that are equal in size or amount.

Traditionally, multiplication is represented as

$$n \text{ times } a = \overbrace{a + a + a + \cdots + a}^{n \text{ times}}$$

We generally use a dot to denote multiplication (since x is used in algebra to denote a variable). If $n \cdot a = b$, then n is called the **multiplier**, a is called the **multiplicand**, and b is called the **product**. Furthermore, n and a can be said to be **factors** of b ; b is a **multiple** of both n and a .

Using boxes, we can represent a **general model for multiplication** (Figure 3.14):

$n \text{ times } a =$ the amount a added n times

That is, a is the value of whatever is in each box.

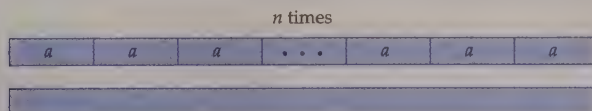


FIGURE 3.14

The general model shown in Figure 3.14 works well for Problems 1 and 2. However, the last two problems do not really fit the model. We can force-fit them, but repeated addition is not the essence of those models. Therefore, we look for other models that can represent these contexts better.

The need for this is not readily apparent when we are working with whole numbers. Thus, a sneak preview: In Chapter 5, we will see that the repeated addition model of multiplication just doesn't work well for $3/4$ times $2/3$; that is, adding three-fourths two-thirds of a time simply doesn't make sense as repeated addition. Let us now examine more closely the area model and the Cartesian product model of multiplication.



FIGURE 3.15



FIGURE 3.16

Area Model for Multiplication

The **area model for multiplication** is an important one that we will see throughout the book. Figure 3.15 is a discrete representation for $4 \cdot 3$.

If we can change each dot to a square, as in Figure 3.16, many students can better see the connection between the repeated addition model and the area model.

As you may recall, the formula for the area of a rectangle is length times width. Thus, any multiplication problem $a \cdot b$ can be represented as a rectangle whose length is a units and whose width is b units. We will use this context in the next section when we examine the multiplication algorithm and in Chapter 5 when we examine multiplication of fractions and decimals.

Cartesian Product Model for Multiplication

As we saw in Problem 4, the Cartesian product of any two sets A and B is the set consisting of *all* possible ways of combining elements of the first set with elements of the second set. Using more formal language, we say that the Cartesian product of any two sets A and B consists of all possible ordered pairs such that the first element is from set A and the second element is from set B . In mathematical notation, we write

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

In Problem 4, the Cartesian product of the set S of skirts and the set B of blouses is

$$\{(S_1, B_1), (S_1, B_2), (S_1, B_3), (S_1, B_4), (S_2, B_1), (S_2, B_2), (S_2, B_3), (S_2, B_4), \\ (S_3, B_1), (S_3, B_2), (S_3, B_3), (S_3, B_4)\}$$

The definition of multiplication as Cartesian product is similar to our definition of addition using set language:

If $n(A) = a$ represents the number of elements in set A , and if $n(B) = b$ represents the number of elements in set B , then the number of elements in the Cartesian product of sets A and B is equal to the product of a and b .

In notation,

$$a \cdot b = n(A \times B)$$

That is, the value of a times b is equal to the number of elements in the Cartesian product of set A and set B .

There are many real-life situations in which the Cartesian product is a useful problem-solving tool. In Chapter 7, we will find that this concept can help us examine probability situations.

Properties of Multiplication

Which of the properties that hold for addition also hold for multiplication? Write your thoughts and then read on. . . 

When adding, we found that “zero doesn’t do anything.” However, when we multiply any number by zero, the product is zero. This property of multiplication, which is not intuitively obvious to many children, is known as the **zero property of multiplication**:

$$a \cdot 0 = 0 \cdot a = 0$$

Many children are either intrigued or confused by the fact that when adding, “zero doesn’t change your answer,” but that when multiplying, “one doesn’t change your answer.” Stated in everyday language, “when we multiply any number by 1, we get the same number.” This is called the **identity property of multiplication**:

$$a \cdot 1 = 1 \cdot a = a$$

Just as with addition, when we multiply any two numbers, we get the same product regardless of the order; that is, $a \cdot b = b \cdot a$. This property is known as the **commutative property of multiplication**.

Just as with addition, grouping doesn’t matter when we multiply several numbers. For example, $(5 \cdot 4) \cdot 7 = 5 \cdot (4 \cdot 7) = 140$, although the problem is much easier to do mentally in the first way. Why is this? This grouping property is known as the **associative property of multiplication**:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

There is another property, called the **distributive property**, that connects multiplication to the operations of addition and subtraction. Consider the following problem given to third-grade students: How many cans of soda are in three 24-can cases? When students are given freedom to invent their own solutions to these problems rather than “discover” the teacher’s way, they will solve this problem in a variety of ways, three of which are discussed below.

Some will simply add $24 + 24 + 24$.

Others will represent the problem with base 10 blocks, as in Figure 3.17. They will count 6 tens and 12 ones, convert to 7 tens and 2 ones, and give the answer of 72.

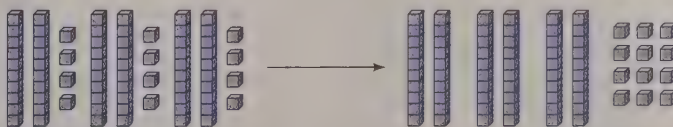


FIGURE 3.17

In this case, the students are using the **distributive property of multiplication over addition**:

$$a(b + c) = ab + ac$$

They have transformed $3 \cdot 24$ into $3 \cdot (20 + 4) = 3 \cdot 20 + 3 \cdot 4 = 60 + 12 = 72$. That is, 3 groups of 24 can be decomposed by breaking the 24 into 2 tens and 4 ones, which is equivalent to 3 groups of 20 and 3 groups of 4 (see Figure 3.18).

Others will solve the problem with money: 24¢ is 1 penny less than a quarter; do this three times and you will have 3 quarters take away 3 pennies, that

MATHEMATICS

The distributive property is a powerful one that we will encounter later in this chapter when we learn why the multiplication algorithm works. We will also see it in Chapter 5 when we examine multiplication with fractions and decimals, and it occurs throughout algebra. It is also widely misunderstood; hence many students do not have this property in their toolbox. This is like a mechanic’s toolbox not containing a crescent wrench!



is, $75 - 3 = 72$. In this case, the students are using the **distributive property of multiplication over subtraction**:

$$a(b - c) = ab - ac$$

The students have transformed $3 \cdot 24$ into $3 \cdot (25 - 1) = 3 \cdot 25 - 3 \cdot 1 = 75 - 3 = 72$

The Multiplication Table

We discussed patterns in the multiplication table in Exploration 3.7. Let us further examine the multiplication table to see how patterns can help children learn the table's 100 "multiplication facts." What patterns do you see in Table 3.2? Write your observations and then read on. . . . 

TABLE 3.2

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

INVESTIGATION 3.3 A Pattern in the Multiplication Table

 Let us examine one pattern. A student noted that every odd number in the table is "surrounded" by even numbers but that the reverse is not the case. Let us investigate the first observation. Why do you think every odd number in the table is surrounded by even numbers? Work on this problem before reading on. . . . 

DISCUSSION

A key to unlocking this mystery comes from the following generalizations: The product of two even numbers is always an even number, the product of two odd numbers is always an odd number, and the product of an odd number and

an even number is always an even number. We will explore odd and even numbers further in Chapter 4. If you didn't make much progress on this question before, try to use this information and see whether you can explain this phenomenon now. Then read on. . . . 

Let us look first at a specific case and then a general case. The diagram below shows 35 (from $7 \cdot 5$) and the numbers that surround it in the table.

		6	7	8
		.	.	.
		.	.	.
		.	.	.
4	. . .	24	28	32
5	. . .	30	35	40
6	. . .	36	42	48

More generally, we can write

		Even	Odd	Even
		.	.	.
		.	.	.
		.	.	.
Even	. . .	Even	Even	Even
Odd	. . .	Even	Odd	Even
Even	. . .	Even	Even	Even

Thus, we see that each of the eight numbers surrounding an odd number in a multiplication table is either the product of two even numbers or the product of an odd number and an even number. Thus, none of them can be an odd number.

Looking back Because of space constraints, we cannot “look back” after every problem-solving situation in the textbook. However, we do so occasionally to illustrate the value of this step. Stop for a moment to reflect on the following questions: What did you learn from exploring this pattern? How might this investigation help one with multiplication facts? Why aren't even numbers surrounded by odd numbers? 

Some students find that Investigation 3.3 illustrates several problem-solving strategies: breaking a problem into smaller parts and looking for other ways to represent a problem. Some students highlight the odd numbers on a multiplication table and “see” the pattern in a new way as a result of the highlighting.

Using this pattern I will present two ways in which this knowledge can help with multiplication facts. First, a brief reflection on Investigation 3.3 yields the deduction that $3/4$ of the multiplication facts are even numbers; that is, only one in four multiplication facts is an odd number. Second, nowhere in the

multiplication table do we have two odd numbers in a row. Thus, for example, if I know that $7 \cdot 7$ is 49, then $8 \cdot 7$ can't be 55 or 57. The generalizations about the products of odd and even numbers would help, for example, a student who wasn't sure whether $7 \cdot 7$ was 48 or 49.

Ultimately, the most important outcome of looking for and examining patterns is that the student comes to see that mathematics is more than just a bunch of facts and rules and that these "relationships" not only are sometimes intriguing but also make us more mathematically powerful.

Changes in Units

There is an important difference between multiplication and addition (or subtraction). When we add or subtract two amounts, the labels do not change: oranges plus oranges = oranges. However, this is not true with multiplication. Look back to the four multiplication problems presented at the beginning of this section. Do you see this? 

- We are multiplying 3 *pieces* by 4 *cents per piece* and getting 12 *cents*.
- We are multiplying 4 *miles per hour* by 3 *hours* and getting 12 *miles*.
- We are multiplying 4 *feet* by 3 *feet* and getting 12 *square feet*.
- We are multiplying 4 *blouses* by 3 *skirts* and getting 12 *outfits*.

This matter of changing units is a crucial one and will develop over the course of the book.

Now let us apply some of this knowledge about multiplication to a real-life setting.

INVESTIGATION 3.4 Using Various Strategies in a Real-Life Multiplication Situation

A warehouse has 50 bays (places to stack pallets). Four pallets can be stacked in each bay, each pallet can hold 24 cartons, and each carton holds 12 boxes. How many boxes can the warehouse contain? If each box sells for \$4, what is the value of the merchandise in a full warehouse? Work on this problem yourself before reading on. . . . 

DISCUSSION

As with many of our problems, there are several strategies that will lead to a solution. Many of my students tell me that they can understand the problem when they read my solutions but that they have trouble coming up with a strategy on their own. Therefore, I urge you not to read the discussions of the investigations until after you have attempted the problem (on your own or with a friend), and when you do read the discussion, read "actively."

STRATEGY 1: Draw a diagram

There are many possible ways to represent this problem with a diagram. The diagram in Figure 3.19 is not "the right diagram" but rather an example of a useful diagram. If you didn't solve the problem or if you didn't solve it with a diagram, take a few moments to look at this diagram. Does it help? How? Does it connect to or stimulate your thinking about what operation(s) might be involved? 

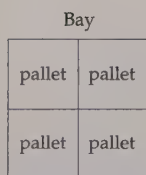


FIGURE 3.19

STRATEGY 2: Use smaller numbers

What if there were 3 bays, 4 pallets in each bay, 2 cartons in each pallet, and 5 boxes in each carton?

Many researchers have found that if they give two problems that are mathematically identical, but one of which has big or messy numbers, the success rate can be dramatically different. Do the smaller numbers help you to see the problem more clearly so that you can deduce what operation(s) to use? Some students find that this strategy makes it easier to draw a diagram that helps them figure out what to do.

STRATEGY 3: Use dimensional analysis

Recall this tool, which was introduced in Investigation 1.2 (How Much Will the Patio Cost?). Using dimensional analysis, you cancel the larger units to concentrate on the aspect of the problem you want to solve. In this problem, we have 50 bays. However, we don't want to know the amount in terms of bays, we want to know it in terms of a smaller unit—boxes.

We can multiply 50 bays by 4 pallets per bay. Using dimensional analysis, this looks like

$$50 \text{ bays} \cdot \frac{4 \text{ pallets}}{\text{bay}} = 200 \text{ pallets}$$

That is, if we use pallets as our unit of measurement, we have 200 pallets. Recall the *Changes in Units* discussion on p. 142. Because we do not want the answer in terms of pallets, we continue to change units:

$$200 \text{ pallets} \cdot \frac{24 \text{ cartons}}{\text{pallets}} = 4800 \text{ cartons}$$

Translating to an even smaller unit, we find that

$$4800 \text{ cartons} \cdot \frac{12 \text{ boxes}}{\text{carton}} = 57,600 \text{ boxes}$$

The value of the merchandise is

$$57,600 \text{ boxes} \cdot \frac{\$4}{\text{box}} = \$230,400$$

A student more confident with using dimensional analysis can do the entire problem in one step:

$$50 \text{ bays} \cdot \frac{4 \text{ pallets}}{\text{bay}} \cdot \frac{24 \text{ cartons}}{\text{pallet}} \cdot \frac{12 \text{ boxes}}{\text{carton}} \cdot \frac{\$4}{\text{box}} = \$230,400$$

I recall being shown this tool in a high school chemistry class, and I have used it in all kinds of school and nonschool problems since then. If this tool is new to you, does dimensional analysis make sense to you now? If it does not, you might want to work on this with a classmate or with your instructor. You might do Exercise 42 at the end of this section to see if you can use this tool on your own.

The Multiplication Algorithm in Base 10

From this section and Explorations 3.6 and 3.9, you now have a better understanding of different models of multiplication and an understanding of the importance of the distributive property. We will now examine the standard multiplication algorithm, which is perhaps the most difficult of the

algorithms to understand at the conceptual level. It rests on an understanding that any multiplication problem can be represented as a rectangle and on an understanding of the distributive property. We will examine this algorithm with $56 \cdot 34$.

The rectangular model of the product Without base 10, we are left with an imposing problem! Figure 3.20 shows 34 rows of 56 units. The answer is there, but who wants to count them all?

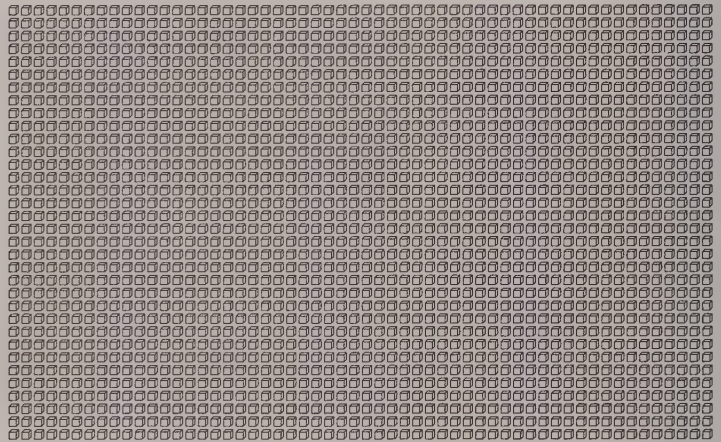


FIGURE 3.20

However, if we apply our knowledge of base 10, we can find alternatives to counting each single unit.

Using base 10 to model the product One first step is to see what the problem would look like if we represented it using our knowledge of base 10 (see Figure 3.21). That is, we would have 56 (5 tens and 6 ones) 34 times. This makes our job only slightly easier. Who wants to add 56 thirty-four times?

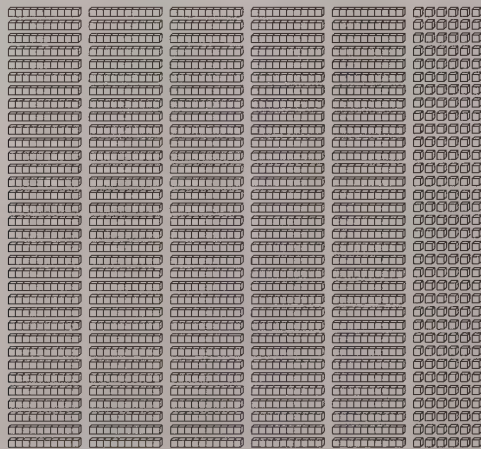


FIGURE 3.21

Once again, our knowledge of base 10 can make the counting process less tedious.

Using our knowledge that 10 ones make a ten, and that 10 tens make a hundred, we can turn units into longs and longs into flats (see Figure 3.22). Do you see the connection between these processes and the following diagram? Do you see that this diagram is another representation of 56×34 ? Do you see connections between this diagram and how you would compute 56×34 ?

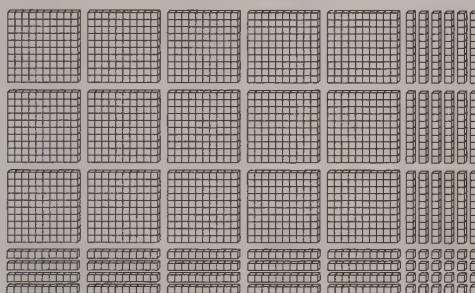


FIGURE 3.22

Counting the new groups When we represent the problem using the rectangular model of multiplication, we see that the standard algorithm systematically does the regrouping for us. Let us first look at how we can quickly count the amount in each of the four regions in Figure 3.22. Can you use your knowledge of multiplication of units to do so? Try to do this on your own before reading on. . . 

- The flats region consists of $3 \cdot 5 = 15$ flats, which have a value of 1500.
- The longs region above the units region consists of $3 \cdot 6 = 18$ longs, which have a value of 180.
- The longs region to the left of the units region consists of $4 \cdot 5 = 20$ longs, which have a value of 200.
- The units region consists of $4 \cdot 6 = 24$.

Adding these four regions, we have

$$\begin{array}{r} 1500 \\ 180 \\ 200 \\ 24 \\ \hline 1904 \end{array}$$

Thus, when we use the algorithm to determine the product $56 \cdot 34$, we determine four products (in this case, $4 \cdot 6$, $4 \cdot 5$, $3 \cdot 6$, and $3 \cdot 5$), and we know what to do with the products. These products are called **partial products**.

Do you see how the four regions in Figure 3.22, the four sums in the addition problem above, and the four partial products are related to one another?

When we represent the problem in expanded form, we can see how the four partial products connect to the algorithm.

$$\begin{array}{r} 50 + 6 \\ \times 30 + 4 \\ \hline 200 + 24 \\ 1500 + 180 \\ \hline \end{array}$$

The standard multiplication algorithm Now look at the algorithm in its most concise form. Can you justify each step of this algorithm?

$$\begin{array}{r} 56 \\ \times 34 \\ \hline 224 \\ 1680 \\ \hline 1904 \end{array}$$

When we move up to larger multiplication problems, the manipulatives and visual representations become more cumbersome. However, if the students understand the process, they can apply that understanding to the larger problems. This understanding also lays the foundation for estimation strategies, which we shall explore in Section 3.3. For example, when we represent 324 times 6 in expanded form, we have three partial products. The value of the partial product $3 \cdot 6$ (which represents 300 times 6) is much larger than the value of the other two partial products (120 and 24). Therefore, 300 times 6 (1800) is a quick and reasonable estimate for this problem.

$$\begin{array}{r} 324 \\ \times 6 \\ \hline \end{array} \qquad \begin{array}{r} 300 + 20 + 4 \\ \times \qquad \qquad \qquad 6 \\ \hline 1800 + 120 + 24 \end{array}$$

As with addition and subtraction, we come to understand the algorithm by decomposing the numbers (using expanded form) and looking at how the ideas of base and place value help us to determine the total product. Recall again the statement by Lauren Resnick, “(p)robably the major conceptual achievement of the early school years is the interpretation of numbers in terms of part and whole relationships.” When one understands what the operations mean and has the tools (expanded form, place value, and so forth), then one can make full sense of the operations, and that forms the basis for what the NCTM standards have called “mathematical power.”

Division

Division is probably the least understood of the four basic operations. In workshops with elementary teachers, I have found that although most of them can perform division computations, many of them find word problems requiring division to be baffling. Many of the classic difficulties people have with division—understanding the long-division algorithm, knowing how to deal with remainders, and knowing when to divide and when to use other operations—go back to a lack of understanding of what division means. To develop that understanding, I ask you to examine the following division problems and then answer the following questions in your own words.

Contexts for Division

Assume that you are a young child who has not yet learned division. How might you solve these problems, using manipulatives or diagrams, or using common sense and other mathematical knowledge? Note your thoughts before reading on. . . . 

Four division problems

1. Applewood Elementary School has just bought 24 Apple computers for its 4 fifth-grade classes. How many computers will each classroom get?

2. Jeannie is making popsicles. If she has 24 ounces of juice and each popsicle takes 4 ounces, how many popsicles can she make?
3. Melissa, Vanessa, Corissa, and Valerie bought a bolt of cloth that is 24 yards long. If they share it equally, how many yards of cloth does each person get?
4. Carlos has 24 apples with which to make apple pies. If it takes 4 apples per pie, how many pies can he bake?

Now go back to these four problems and answer the following questions:

- In what ways are the problems different? In what ways are they similar?
- What does division mean? For example, what words come to mind when you think of division?

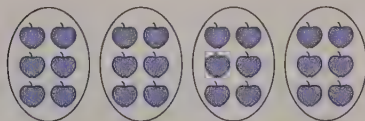
As with the other operations, some of these problems involve discrete objects (Problems 1 and 4), whereas other problems involve measured (continuous) amounts (Problems 2 and 3).

Another difference emerges when we solve these problems as young children do. Left to themselves, most children will solve Problems 1 and 3 in a way that is very different from the way they will solve Problems 2 and 4. Do you see why?

Models for Division

Let us solve Problems 1 and 4 simultaneously to understand two different models of division.

To solve the first problem, many children will give one computer to each class, dealing them out one by one: one for our class, one for your class, and so on. Some students use multiplication facts or guess and check and start out bigger: two for our class, two for your class, and so on. They then keep dealing until all the computers have been distributed. Visually, we have



Each classroom gets 6 Apple computers.

This model is called the **partitioning model** because we solve the problem by first setting up the appropriate number of groups (partitions), which we then fill.

To solve the fourth problem, most children do something very different. They make groups of 4 (each representing one pie) until there are no more apples left.

Visually, we have



We have enough apples to make 6 pies.

This model is called the **repeated subtraction model** because we repeatedly subtract the given amount until we can do so no more. Some books call this the measurement model of division.

Now let us look at how the models are related. First we need some terminology.

Division terminology In both cases, the problem would be written as

$$24 \div 4 = 6 \text{ or as } 4 \overline{)24}$$

The division of the number b by the nonzero number n is formally defined as

$$b \div n = a \text{ iff } a \cdot n = b$$

The number b is called the **dividend**, n is called the **divisor**, and a is called the **quotient**.

Comparing the two division models With this terminology, we can compare the two models in Table 3.3. What patterns or connections between the two models do you notice? Think and then read on. . . . 

TABLE 3.3

	Dividend		Divisor	Quotient
Partitioning	24 computers	divided into	4 classes yields	6 computers per class
Repeated subtraction	24 apples	divided into	4 apples yields yields per pie	6 pies

We can also represent both division contexts with the general part-whole box model we have developed for the other three operations. At the most general level, we have

	Dividend	Divisor	Quotient
Partitioning	Whole	Number of parts (groups)	Number in each part (group)
Repeated subtraction	Whole	Number in each part (group)	Number of parts (groups)

Stop for a moment to reflect on these models. Does this representation connect with the problems we have done here and the problems you did in your explorations? How are these two models of division alike, and how are they different? 

We can represent the models visually:

PARTITIONING MODEL

$24 \div 4 = 6$



4 classes
6 per class

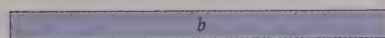
REPEATED SUBTRACTION MODEL

$24 \div 4 = 6$

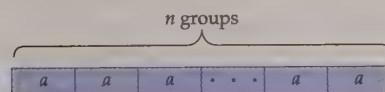


4 apples per pie
6 pies

At the most general level, we have

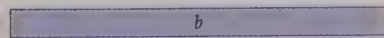


The whole is partitioned into n groups:

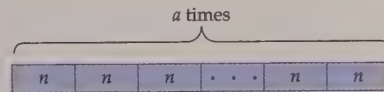


The answer is a (in each group).

At the most general level, we have



We repeatedly subtract n at a time until we can subtract no more.



The answer is a (the number of times that you subtracted).

As with the other operations, we can “see” division in part-whole language: We begin with a whole that is divided into equal parts. There are two different ways in which we can divide the whole: by making partitions of equal size or by repeatedly subtracting parts of equal size. In both cases, we have a whole that we are decomposing in different ways.

The Missing Factor Model of Division

Going back to the formal definition of division, $b \div n = a$ iff $a \cdot n = b$, we can understand another model for division.

Just as some children interpret and solve some subtraction problems by finding the missing addend, some children interpret and solve some division problems by finding the missing factor. For example, consider the two problems we have just analyzed. Some children will solve either or both of these problems by asking themselves: 4 times what is equal to 24?

This model for division, called the **missing factor model** for division, gives us another tool for working with division, one that we will use often in this and future chapters.

Division and Number Lines

Although the number line is not commonly used to represent division problems, it can be used. What problem is represented in the number line in Figure 3.23? Does the number line fit both models or just one model? Think and then read on. . . .

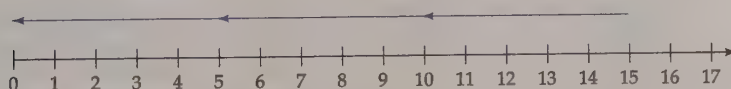


FIGURE 3.23

The problem is $15 \div 5 = 3$. The number line fits only the repeated subtraction model. In other words, we cannot use the number line to model a partitioning problem. Why is this?

Properties of Division

Of the properties we have investigated for the other operations, identity, commutativity, associativity, and closure, which do you think hold for division? Think and then read on. . . .

If you have been making connections among the operations, you probably have realized that addition and multiplication are alike in that each operation has an identity and that the commutative and associative properties hold for those operations. You saw that those properties did not hold for subtraction, although subtraction does have a right-identity. Therefore, you might be inclined to say the same for division. Division does have a right-identity; that is, $a \div 1 = a$ for all numbers. The commutative and associative properties do not hold for division. Similarly, the set of whole numbers is not closed under division. Why is this?

The definition of division enables us to see the connection between division and multiplication, and the models of division enable us to understand how division with smaller numbers works. As you may have already realized, just as we noted earlier that multiplication is more complicated conceptually than addition and subtraction, so is division. There are two complexities that pose problems for most children: division by zero and division with remainders. Let us examine them now.

Division by Zero

Our definition for division implies that “you can’t divide by zero.” Can you explain why? Think about this and then read on. . . . 

There are several ways in which we can investigate this problem. We can use inductive reasoning: Make a table and look for patterns. Do you understand Table 3.4? What pattern do you see that can help you to explain why “you can’t divide by zero”?

TABLE 3.4

Computation	Dividend	Divisor	Quotient
$5 \div 1$	5	1	5
$5 \div 0.1$	5	0.1	50
$5 \div 0.01$	5	0.01	500
$5 \div 0.001$	5	0.001	5000
$5 \div 0.0000001$	5	0.0000001	?

We can also approach the problem of dividing by zero by seeing whether either context fits the model. What if we had 24 computers and we wanted to distribute them to zero classrooms? How many computers would each classroom get? That doesn’t make much sense. Let’s try repeated subtraction. What if we had 24 apples and each pie took zero apples? How many pies could we bake? This doesn’t make sense either. From a practical (contextual) level, division by zero just doesn’t make sense. Thus, we say that division by zero is undefined.

A third approach to this problem is to assume that it is possible; this is called **indirect proof**. If division by zero were possible, then a nonzero x divided by zero would be equal to some number k (that is, $x \div 0 = k$). However, if this is true, then according to our definition of division, $k \cdot 0 = x$, but this is not possible because we know that $k \cdot 0 = 0$.

Division with Remainders

There is a wonderful children’s book called *The Doorbell Rang*. In this story, the mother has a plate of cookies that are to be shared equally among the children. Before the children eat the cookies, however, the doorbell rings and another



HISTORY

Many people do not realize how recent much of our mathematical notation is. The symbols we use for addition and subtraction, $+$ and $-$, first appeared in Germany in the late 1400s. Historical records indicate that these symbols were first used to indicate sacks that were surplus or minus in weight. Michael Stiffe was the first person to use $+$ and $-$ as symbols of operation as well as of abbreviation. He did this in the mid-1500s. In 1631, William Oughtred first used the letter x to represent multiplication (as in $3 \times 4 = 12$). Italian merchants introduced the symbol for division ($\div$) in the 1400s to indicate a half. For example, they wrote $4 \div$ to indicate $4\frac{1}{2}$. The equal sign first appeared in the late 1500s in a book by Robert Recorde: "I will setteas I doe often in woorke vse, a pair of paralleles, or . . . lines of one lengthe, thus: $=$, bicause noe .2. thynges can be moare equalle."⁴ It was more than 100 years later that the use of this sign became widespread.

$$\begin{array}{r} 6 \\ 8 \overline{)5742} \\ \underline{48} \\ 9 \end{array}$$

person comes. Now they have to figure out how many cookies each person gets, and then another person comes, and another, and. . .

Let's say there are 12 cookies and 3 children—each child gets 4 cookies. What if another child comes?—now each child gets 3 cookies. What if another child comes?—now each child gets 2 cookies, and there are two left over. We can now say that $12 \div 5$ is 2 with 2 cookies left over. As we saw above, the set of whole numbers is not closed under division; in everyday English, this means that when we divide one whole number by another, the answer is not always a whole number. However, the *division algorithm* assures us that a solution involving whole numbers will exist for any whole-number division problem, other than dividing by zero.

The **division algorithm** states that for any whole numbers a and b , where $b \neq 0$, there are absolutely going to be two numbers, which we will designate as q and r , that enable us to know the answer to $a \div b$. That is,

$$a = bq + r$$

There is one other stipulation, that r be a whole number less than b . Do you see why?

Without this stipulation, we lose the uniqueness of r and q . For example, if we divide 33 by 4, we could say $33 \div 4 = 8 \text{ R } 1$ or $33 \div 4 = 7 \text{ R } 5$.

Because communication is one of the NCTM process standards, I want to express the division algorithm in its most concise form:

$$\forall a, b \in W, (b \neq 0) \exists q, r \in W \exists a = b \cdot q + r, 0 \leq r < b$$

A direct translation in English is: For any whole numbers a and b , except that b cannot be equal to zero, there exist two whole numbers q and r such that a is equal to b times q plus r , where r is less than b and could possibly be equal to zero.

Translated into everyday English, we have the following: For any two whole numbers a and b (as long as b is not equal to zero), we can find two whole numbers q and r that make this equation true ($a = bq + r$), and r must be greater than or equal to zero but less than b .

As we have seen before, mathematical notation and shorthand was introduced to make communication easier and to reduce the possibility of ambiguity and confusion.

Again, some readers will say, "So what?" There are two responses to this question. The first is that many aspects of our numeration system and operations on numbers that seem so obvious to you as adults are neither obvious nor simple to young children: Recall your experiences with Alphabitia. Many of these concepts were developed over hundreds or even thousands of years. Second, understanding some of the formalities of mathematics will help you to recognize and address problems that children often have. For example, when faced with division problems with remainders, some children actually wonder whether there is a solution to the problem. And some children, whose multiplication facts are poor, make mistakes when dividing that cause huge problems. For example, consider the example at the left. In this case, the student does not own the understanding that r must be less than b , and thus is doomed to failure—and to more frustration. Exploration 3.14 also addresses the idea of remainders when dividing.

Division Algorithms

At the beginning of each course, I invite my students to write math autobiographies in which I ask them to tell me math-related memories, both positive

⁴ H. A. Freebury, *A History of Mathematics* (New York: Macmillan, 1961), p. 89.

and negative, from early childhood through high school. A common sentence in these autobiographies goes something like “I liked mathematics until long division.” To this day, many adults think that understanding why the standard division algorithm works is like understanding the structure of quantum physics—that is, something beyond the ability of “ordinary people.” They know what to do, but not why. For example, faced with 252 divided by 4, most people say something like “4 gazinta 25 6 times,” but “gazinta” is a mystery. The purpose of Investigations 3.5 and 3.6 is to convince you that “gazinta” is not as esoteric as quantum physics.

INVESTIGATION 3.5 Understanding Division Algorithms

Solve the two division problems as though you didn’t know any algorithms; you may use manipulatives or draw diagrams or use reasoning. Then read on. . . .

- A. Warren has 252 guests coming to his wedding. Each table holds 4 guests. How many tables will he need?
- B. Mickey has 252 marbles that he wants to distribute equally into 4 piles. How many marbles are in each pile?

DISCUSSION

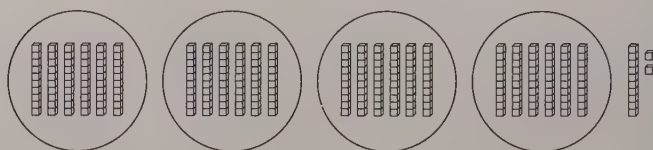
A If you represent these problems with base 10 blocks, you may have found that they weren’t very useful for Problem A because it essentially asks how many groups of 4 there are in 252. Some students solve this problem by using the missing factor model: 4 times what makes 252?

Guess	Result	Thinking
50 tables	200 people	Too low, try a bigger number, let’s say 60 tables.
60 tables	240 people	We need 3 more tables for the 12 remaining people.
63 tables	252 people	

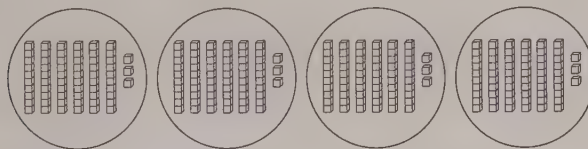
B The base 10 blocks are more useful for Problem B. Recall the janitor trying to determine how many desks went into each room in Exploration 3.12. In this case, Mickey’s problem is to divide (distribute) 252 into four groups equally. Using base 10 manipulatives, he can figure out how to do this most efficiently. Because he cannot give 100 to each person, he has to trade the 2 hundreds (flats) for 20 longs. He still has 252, but physically he has 25 longs and 2 units.



Now he can place 6 longs in each group.



He is left with 1 long and 2 units, representing 12; thus he trades the long for 10 units. This lets him place 3 ones in each group.



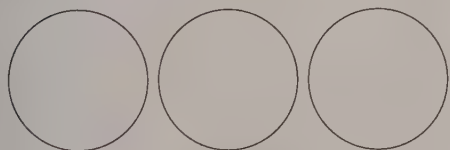
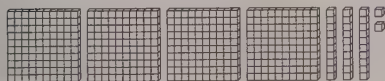
INVESTIGATION 3.6 Understanding Long Division at the Symbolic Level

Because the long division algorithm is intimidating to many people, let us examine another division problem. What is $432 \div 3$? First, try to do this yourself, justifying each step. Then read on. . . 

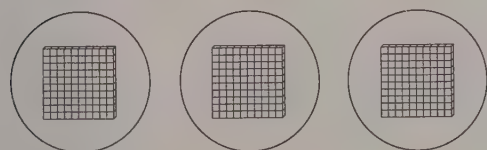
DISCUSSION

We want to divide 432 into 3 equal groups.

At the physical level, we have



Dividing flats We can place 1 flat (hundred) into each group. Then we have to change the remaining hundred into 10 tens. We have 132 left.



At the symbolic level, we have

$$3 \overline{)432}$$

Dividing in the hundreds column 4 divided by 3 is 1, and 1 times 3 tells us how much has been distributed.

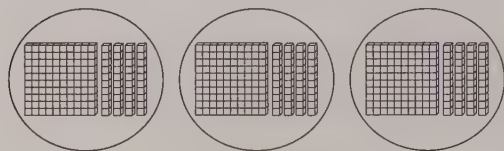
We place the 1, representing 1 hundred, in the hundreds column.

We then subtract 3 from 4 to find out how many hundreds we have left to distribute (1).

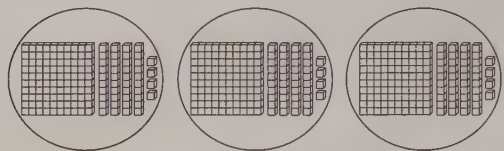
Last, we “bring down” the next number (representing 3 tens). We now interpret the “13” as 13 tens.

$$\begin{array}{r} 1 \\ 3 \overline{)432} \\ \underline{3} \\ 13 \end{array}$$

Dividing longs We can place 4 longs (tens) in each pile. The remaining ten is converted into ones. We have 12 ones left.



Dividing units We can place 4 units (ones) into each pile. We are finished.



Dividing the tens column 13 divided by 3 is 4, and 3 times 4 tells us how much has been distributed.

We place the 4, representing 4 tens, in the tens column.

We subtract the 12 tens from the 13 tens to find out how many tens we have left to distribute (1).

We bring down the 2 ones. We interpret the 12 as 12 ones.

$$\begin{array}{r} 14 \\ 3 \overline{)432} \\ \underline{3} \\ 13 \\ \underline{12} \\ 12 \\ \underline{12} \\ 0 \end{array}$$

Dividing in the ones column When we divide 12 ones by 3, we get 4 ones with no remainder.

$$\begin{array}{r} 144 \\ 3 \overline{)432} \\ \underline{3} \\ 13 \\ \underline{12} \\ 12 \\ \underline{12} \\ 0 \end{array}$$

Looking back Having examined long-division problems in these two investigations, write your response to the following questions before reading on. . . . 

1. What is happening when we say "gazinta"?
2. What is happening when we multiply?
3. What is happening when we subtract after we multiply?
4. What is happening when we bring down the next number?

Let us look at the answers to these questions.

1. When we say "gazinta," we are simply determining the first step in distributing the amount in the dividend into equal groups. For example, in the problem $36 \overline{)864}$, when we say that 36 "doesn't go into 8," mathematically, this means that when we divide 864 into 36 equal groups, we do not have enough to have 1 hundred in each group, and therefore we have to look at the next place. When we say that 36 goes into 86 two times, mathematically this means that when we distribute 86 tens into 36 groups of equal size, we have enough for 2 tens in each group, and thus we place the 2 over the 6, that is, in the tens place.
2. When we multiply, we determine how much has been distributed, or, in other words, how much of our original amount has been used up. In the case of $36 \overline{)864}$, we find that we have used up 720.

HISTORY

The common division algorithm evolved from a method called a *danda*, which is derived from the same root as the word giving. The "giving" part is referred to by most people as the "bringing down" part. That is, when you have finished subtracting, you "bring down" the next digit in the dividend (the number you are dividing). From another perspective, you are giving that number to the remainder. The earliest version of this algorithm dates to the fourteenth century.

 HISTORY

The earliest book on arithmetic is the papyrus of Ahmes, who was an Egyptian scribe in about 1700 B.C. It is believed that the book is a compilation of what was known at the time, some of which goes back to 3000 B.C. The title of the papyrus was "Directions for Obtaining the Knowledge of All Dark Things." The first arithmetic book printed in the West was printed in Treviso, Italy, in 1478. The first one in the United States was printed in 1729; it was written by Isaac Greenwood, a professor of mathematics at Harvard College.⁵

3. We subtract after we multiply so that we can find out how much of the dividend is left after the first distribution. In this case, we have 14 tens left over. That is, these tens cannot be distributed equally into 36 equal-size groups, and so they have to be traded in for ones.
4. When we bring down the next number, we are determining how much we have to distribute. In this case, we have the 14 tens and the 4 ones that we bring down—that is, a total of 144—to distribute equally into 36 groups.

As with the three other operations, a deep understanding of this division algorithm requires one to be comfortable with the ideas of base and place value and also to be comfortable with the ways in which a number can be decomposed into various parts.

It is important to emphasize that although our numeration system was invented over a thousand years ago, the division algorithm that you learned in elementary school is only one of many algorithms for division that were invented over this time. In Exploration 3.13 you explored two alternative algorithms for division.

INVESTIGATION 3.7 Missing Digits

Determine the three missing digits:

$$4\square\square2 \div 8\square = 48$$

How can you apply your understanding of multiplication and division to solve this problem? What problem-solving strategies seem more helpful? Work on this yourself and then read on. . . . 

DISCUSSION

One of the key themes in this book is the idea of multiple representations, which is like looking at an object from different perspectives—from above, from below, at ground level, from the side, and so on. Sometimes, one representation sheds more light on a problem than another. If you were stuck, look at the following representation of the problem. Does this help?

$$\frac{4\square\square2}{8\square} = 48$$

If you are still stuck, try using a multiplication representation:

$$8\square \cdot 48 = 4\square\square2$$

What does this tell you now about the ones digit of the first number? Think and then read on. . . . 

It must be a 4 or a 9. Do you see why? Test your intuition—which one do you think it is? 

Now you can finish the problem. When you have come up with a solution, don't forget to look back at your work. What did this investigation tell you

⁵ Louis Charles Karpinski, *The History of Arithmetic* (New York: Russell & Russell, Inc., 1965), p. 81.

about operations? How did your problem-solving toolbox—looking for patterns, working backwards, multiple representations, and so on—grow as a result of doing this problem?

INVESTIGATION 3.8 Connecting Algorithms to Contexts and Models

C Consider the four computations below, which illustrate the standard algorithms.

$$\begin{array}{r}
 \overset{1}{4}63 \\
 + 854 \\
 \hline
 1317
 \end{array}
 \qquad
 \begin{array}{r}
 \overset{6}{7}\overset{9}{0}\overset{1}{3} \\
 - 476 \\
 \hline
 227
 \end{array}
 \qquad
 \begin{array}{r}
 46 \\
 \times 87 \\
 \hline
 322 \\
 368 \\
 \hline
 4002
 \end{array}
 \qquad
 \begin{array}{r}
 875 \\
 3\overline{)2625} \\
 \underline{24} \\
 22 \\
 \underline{21} \\
 15 \\
 \underline{15}
 \end{array}$$

How is each algorithm related to a context or model for an operation? Write your ideas and then read on. . . . 

DISCUSSION

The standard addition algorithm connects nicely to both joining and increasing contexts. The standard subtraction algorithm, however, is an enactment (acting out) of the take-away context. The standard multiplication algorithm represents the area model. The standard division algorithm connects to both the partitioning model and the repeated subtraction model; in this text, we have demonstrated the connection to the partitioning model.

Algorithms and the Hindu-Arabic Numeration System

We have just examined connections between algorithms and specific operations. Let us look at another important connection—between algorithms and our numeration system. These algorithms could not be used with Roman or Egyptian numerals. Do you see why? How does each of these algorithms use base 10 concepts? In each of the algorithms, we encounter rules that tell us where to place the number in question, whether it be a regrouped amount in addition and subtraction, a partial product in multiplication, or a number in the quotient in division. Each of these algorithms represents the culmination of years of efforts to find more and more effective ways to combine the brilliance of the Hindu-Arabic numeration system with the essential concepts of the four operations. As noted before, these are not the only algorithms for the operations (and not everyone agrees that they are the easiest to learn or use). Rather, they are the ones that are most commonly used in the United States at the present time.

Operation Sense

Now that we have examined each of the four operations, let us explore some problems in which we can apply our knowledge. Someone who knows which operation to perform in a problem is said to have good **operation sense**, a term that represents an important skill but is hard to define. Operation sense is more than just knowing what operation to use in what situation. If we want mathematics problems to model real-life problems, then, as you have seen in this

book, many problems will not be routine, one-step problems. In the context of solving more complex problems, students with good operation sense have the following “knowledge”:

- They can see the relationships among the operations—for example, multiplication is the inverse of division.
- They can apply their understanding of the properties of the operations—for example, they can use the distributive property.
- The diagrams that they draw connect the problem to the models of the operations.
- They have a sense of the effects of each operation—for example, when dividing, the larger the divisor the smaller the quotient.

One of the goals of Investigation 3.9 and Explorations 3.15 and 3.16 is for you to come to appreciate the need for operation sense and to develop this ability. That is, the successful completion of these problems requires you to see and use various connections between these four operations.

INVESTIGATION 3.9 Applying Models to a Real-Life Situation

A teacher is going to do a project with the two fifth-grade classes in the school. One class has 21 students, and the other has 15 students. She has decided that all of these students will be divided into 12 groups and that each group will need 72 inches of string. She has a roll of string that is 882 inches long. Does she have enough string, or will she need to buy more? Do this problem yourself without using a calculator before reading on. . . . 

DISCUSSION

You can use either multiplication or division to solve this problem. Using multiplication, $72 \cdot 12 = 864$ tells you that you will have enough string for each group, with 18 inches of string to spare. If you use division, the problem can be interpreted either as a partitioning problem or as a repeated subtraction problem. Do you see why?

- If you divide 882 inches by 72 inches per group, this says that you are seeing the problem in terms of repeated subtraction. What does your answer of 12 with a remainder of 18 tell you?
- If you divide 882 inches by 12 groups, this says that you are seeing the problem in terms of partitioning. What does your answer of 73 with a remainder of 6 tell you?

In the repeated subtraction model, the 12 says that if you repeatedly cut 72 inches, you can do this 12 times, and you will have 18 inches of string left over. In the partitioning model, the 73 says that if each group is to have an equal amount of string, you have enough string to make 12 pieces each of which is 73 inches long and you will have 6 inches left over. In either case, you have to understand what your quotient means, and you have to think about what the remainder means.

There are two other aspects of this problem that are worthy of emphasis.

First, although the mathematical answer is that the teacher has enough string, the teacher might actually conclude from her computations that she needs more string. Why is this?

What if the students make mistakes? Because $72 \cdot 12 = 864$, the teacher has only 18 inches of string in reserve. Thus, although the mathematical answer is that the teacher has enough string, the real-life answer would depend on the nature of the project. For example, if the students were to cut the pieces into specified lengths and I anticipated that some groups would mismeasure the lengths, I would not feel confident that I had enough string.

Second, compare this question to the following problem: A teacher is going to have her students do a project in which each group of students will need 72 inches of string. If the teacher has 882 inches of string, how many groups can she have in her class?

This problem is a one-step, routine problem, the kind the NCTM says should be deemphasized. The student simply has to decide whether to add, subtract, multiply, or divide; if the student just guesses, there is a 25 percent chance of being right. This is what many students do. Larry Sowder⁶ examined children's strategies for solving word problems and reported these kinds of strategies: "Look at the numbers; they will tell you which operation to use. Try all the operations and choose the most reasonable answer. Look for key words or phrases to tell which operation to use."

Furthermore, the actual problem has been sanitized so that the student simply has to use the numbers given. In the first version of the problem, the number of students in each class (21 and 15) is *irrelevant to the question being asked*. These numbers were given not to try to trick you, but because in real-life settings, we generally have lots of information. The first part of the problem is to decide whether we have enough information to solve the problem and to decide what information is relevant to the question we are asking.

Order of Operations

Complex computations that use more than one operation pose a potential problem. To see what I mean, do this problem with pencil and paper and then on your calculator: $3 \cdot 4 - 8 \div 2$. What did you come up with? Stop, think, and then read on. . . . 

Unless you have a very inexpensive or antique calculator, you will get 8. However, if you do each operation in the order in which it appears in the problem, you will get 2. Why did the calculator give a different answer?

The fact that we can get two different answers from the same problem has led to rules called the **order of operations**. These rules tell you the order in which to perform operations so that each expression can have only one value. Most calculators are programmed to obey the order of operations.

What did the calculator actually do to get 8? Write the process in words and see whether a friend can use your explanation to see how to get 8. Then read on. . . . 

In the absence of parentheses, multiply 3 times 4 to get 12, then divide 8 by 2 to get 4. Now subtract 4 from 12 to get 8. You may remember order of operations from school as "Please Excuse My Dear Aunt Sally," which is a mnemonic to remind you of the order in which a computation is done: start inside the parentheses, then do multiplication and division as they occur from left to right, and finally do addition and subtraction as they occur from left to right. (The *e* from "Excuse" refers to exponents, which we will discuss in Chapter 5.)

⁶Larry Sowder, "Children's Solutions of Story Problems," *Journal of Mathematical Behavior*, 7 (1988), pp. 227–228.

If we insert parentheses so that more people would understand this problem, it would read as: $(3 \cdot 4) - (8 \div 2)$. How many different answers can you get depending on different placements of parentheses?

Operations and Functions

Each of the operations in this chapter is a function because for each input (an ordered pair of two numbers), there is one and only one output. For example, the input $(6, 2)$ produces a sum of 8, a difference of 4, a product of 12, and a quotient of 3. Although awareness of this connection between operations and functions does not necessarily make you a stronger problem solver, it is important in addressing the realization that mathematics is much more like a road map or a connected network than like a set of unrelated formulas and procedures, an image many adults have of mathematics.

Summary

There are actually two parts to this section, first our examination of multiplication and division, and then our examination of the connections among all four operations.

We have found that multiplication can be represented as repeated addition, as the area enclosed by the multiplier and the multiplicand, or as the number of combinations (Cartesian product). Similarly, division problems can be seen as partitioning, repeated subtraction, or as missing factor. Being able to interpret and represent a problem goes a long way toward being able to solve problems.

Visual models help us to see connections between multiplication and division. In one sense, multiplication consists of adding two or more equal-size parts to make a whole. Similarly, division can be seen as decomposing a whole into parts that are all the same size.

You have learned that there are certain properties that hold for multiplication of whole numbers but not for division: commutativity, associativity, and closure. Just as we have the identity property of addition, we have the identity property of multiplication. Thus, the numbers 0 and 1 hold a certain importance, and they also have a rich history, as you will find in Chapter 4. We also learned of a property that connects operations: the distributive property. This property is a key to understanding how the multiplication algorithm works.

We have seen that the multiplication and division algorithms we currently use in the United States are not the only possible procedures for computing. Your ability not only to compute with these algorithms but also to be able to explain why they work is a crucial part of developing mathematical power.

We have also examined the various connections among the four operations, including order of operations, and applied our understanding of those connections so that our operation sense becomes more powerful.

A key goal of this section is for you to better understand how these four operations are related to one another. What similarities do you see among the operations? What connections do you see among them? Take a few minutes to draw a diagram that shows the relationships among the four operations. Then read on. . . . 

Addition and subtraction are inverse operations. Multiplication and division are inverse operations.

One model of multiplication is repeated addition, and one model of division is repeated subtraction. We have a missing addend model for subtraction, and we have a missing factor model for division.

The set of whole numbers is closed under the operations of addition and multiplication and is not closed under the operations of subtraction and division. The commutative and associative properties hold for addition and multiplication, but not for subtraction and division. We have identity properties for addition and multiplication; because subtraction and division are not commutative, we have only right-identities for these operations. We can speak of the distributive property of multiplication over addition and of the distributive property of multiplication over subtraction.

With addition and subtraction, the units in each part are always identical; for example, 3 *sheep* plus 5 *sheep* equals 8 *sheep*. However, with multiplication and division, the units are different; for example, 3 *miles per hour* times 4 *hours* yields a product of 12 *miles*.

In many addition and multiplication situations, we are joining or combining parts to make a whole. In many subtraction and division situations, we are taking a whole apart.

These statements represent a verbal description of the operations' similarities and connections. One of many visual representations of the similarities and connections is shown in Figure 3.24.

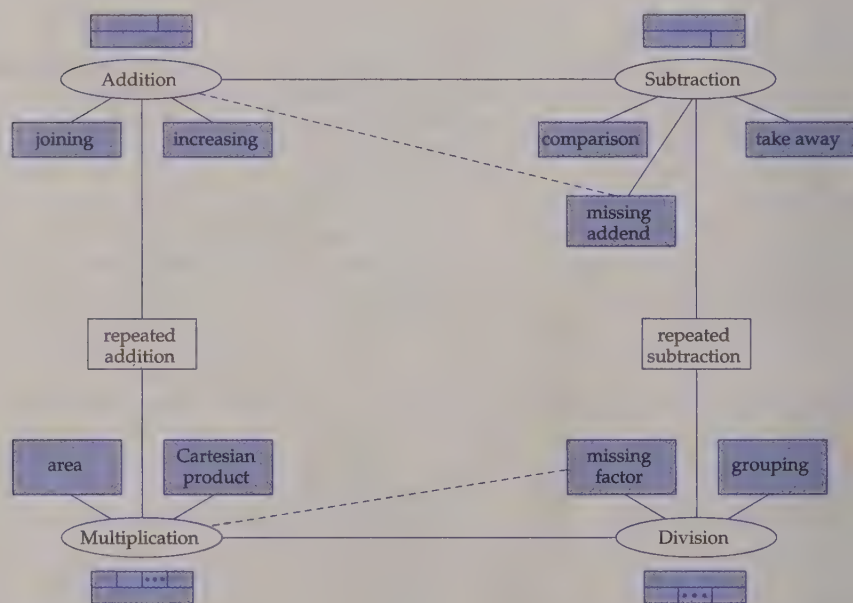


FIGURE 3.24

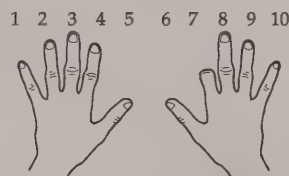
We have seen that the standard algorithms are related to some of the operations' contexts and models. These algorithms rely on our base 10 system. Again, it is important to reemphasize that these standard algorithms are not the only ones, nor are they necessarily the best ones. In Explorations you saw other algorithms, some of which are currently used in other parts of the world.

A statement from the beginning of the chapter is worth repeating: Understanding these four operations is central to knowing mathematics. Many of my students have been amazed to find that a solid understanding of these operations enabled them to solve multistep and nonroutine problems that initially seemed very imposing and complex. In Chapter 5, when we examine three extensions of the number line—integers, fractions, and decimals—you will find that you will have to modify your understanding of these operations to accommodate the problems that arise with those kinds of numbers.

EXERCISES 3.2

- Make up story problems and diagrams to illustrate $5 \cdot 3$ in the four different contexts shown in Figure 3.12.
- Write a realistic story to represent each problem below.
 - $12 \cdot 4$
 - $35 \cdot 3$
 - $5 \cdot (3 + 7)$
- $V = \{a, e, i, o\}$ and $E = \{a, b, c, d, e\}$.
 - Name one element of $V \times E$.
 - Name one element of $E \times V$.
 - How many elements are in $V \times E$?
- $A = \{p, l, a, y\}$ and $B = \{b, a, l\}$.
 - How many elements are in $A \times B$?
 - Is $(p, a) \in A \times B$?
- If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, find $A \times B$.
- If $A \times B = \{(6, x), (6, y), (6, z), (7, x), (7, y), (7, z)\}$, find A and B .
- There are 12 elements in $E \times F$. Four of them are $(1, 6)$, $(3, 2)$, $(4, 2)$ and $(5, 5)$. List the rest.
- Do you think we could speak of a distributive property of addition over multiplication? How would you determine whether it held or not?
- For each of the following questions, refer to the base 10 multiplication table.
 - The top-left-to-lower-right diagonal divides the table into three sets: diagonal, upper part, and lower part. What do you notice about the upper part and the lower part? What property is illustrated here?
 - Look at the top-right-to-lower-left diagonal. Describe the patterns you see. Justify the patterns.
 - The products of the diagonals of any 2 by 2 matrix are equal. Explain why.
 - In any row, the sum of the x th number and the y th number is equal to the $(x + y)$ th number. Suppose someone did not understand this terminology. Express this pattern in another way. Explain why the pattern occurs.
 - Look at the 9s column in the multiplication table. The sum of the digits is always 9. Can you explain why?
 - Look at the 9s column in the multiplication table. As we go down the column, the ones digit decreases by 1 each time and the tens digit increases by 1 each time. Can you explain why?
- Let's say you forgot what $9 \cdot 7$ is. How would you figure out the product, using other multiplication facts? There are many, many different ways!
- Fingers are a set of manipulatives that most people have, and they can be used to help students remember their multiplication facts for 9. This is how the method works. Suppose you want to know $9 \cdot 7$. Bend down finger number 7. The number of fingers to the left of

the bent finger gives you the value of the tens place of the product, and the number of fingers to the right of the bent finger gives you the value of the ones place of the product. Why does this work?



- Before the Russian Revolution in 1917, most of the people in Russian villages did not use base 10 but still used the Roman system. The Egyptian method of multiplication had somehow become transformed into a different process, known today as the Russian peasant algorithm. It works by simultaneously doubling one of the numbers and halving the other. Fractions are simply disregarded; for example, half of 19 is $9\frac{1}{2}$ —drop the $\frac{1}{2}$. We find the product by first looking in the “Halve” column. Whenever you see an odd number in that column, circle that row. To find the answer, add the circled numbers in the “Double” column. The example below shows how it works.

Problem: $25 \cdot 19$.

Double	Halve
25	19
50	9
100	4
200	2
400	1

Circling the appropriate numbers, we add:

25
50
400
475

Solve the following problems using the Russian peasant algorithm.

- $25 \cdot 17$
 - $48 \cdot 39$
 - $120 \cdot 42$
- Explain why this algorithm works.
- Another ancient method of multiplication is shown below with the problem $102 \cdot 96$.
Find the average: 99
Take half the difference of the numbers: 3
Look up the square of 99: 9801
Look up the square of 3: 9
Subtract and you have your answer: 9792
 - Use this method to determine the product of 36 and 54.
 - Explain why it works.
 - Draw a diagram to represent $34 \cdot 28$. Explain how to obtain the product from the diagram.

15. Look at this multiplication problem, solved using the standard algorithm:

$$\begin{array}{r} 42 \\ \times 36 \\ \hline 252 \\ 126 \\ \hline 1512 \end{array}$$

- a. In this problem, we multiply $2 \cdot 6$ and carry the 1. What is it that we are carrying? What is the mathematical meaning of this carrying?
- b. Before we do the second set of multiplications (that is, $3 \cdot 2$ and then $3 \cdot 4$), we move over. Why do we move over instead of just putting the product (126) directly below the previous product (256)?
16. When we multiply larger numbers, we are used to moving over one column each time we move through the process. Why do we move over?

$$\begin{array}{r} 645 \\ \times 324 \\ \hline 2580 \\ 1290 \\ 1935 \\ \hline 208980 \end{array}$$

17. a. Write directions for multiplying $46 \cdot 37$ using the lattice algorithm.
- b. Explain why this method works.
- c. What mathematics do you have to know in order to be able to use this method?
- d. Discuss some advantages and disadvantages of this algorithm.
18. Below is the start of the solution of a multiplication problem using a new algorithm. Explain the meaning of the numbers 20, 15, 8, and 6. Then finish the problem, showing and explaining your work. *Note:* You need to explain *why* the four numbers are placed where they are and then explain *why* the algorithm works. You may, but are not required to, use a diagram in your explanation.

$$\begin{array}{r} 4\ 3 \\ \times 2\ 5 \\ \hline 20\ 15 \\ 8\ 6 \\ \hline \end{array}$$

19. Identify and explain the errors in the problems below, all of which have occurred in real classrooms.

a.

$$\begin{array}{r} 46 \\ \times 28 \\ \hline 404 \\ 101 \\ \hline 1414 \end{array}$$

b.

$$\begin{array}{r} 46 \\ \times 28 \\ \hline 3248 \\ 812 \\ \hline 4060 \end{array}$$

c.

$$\begin{array}{r} 46 \\ \times 28 \\ \hline 368 \\ 92 \\ \hline 460 \end{array}$$

d.

$$\begin{array}{r} 46 \\ \times 28 \\ \hline 92 \\ 368 \\ \hline 3772 \end{array}$$

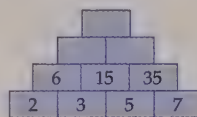
20. Place the digits 0, 1, 3, 5, 7 in the boxes below to obtain

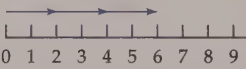
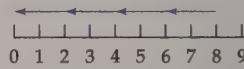
a. The greatest product

b. The least product

$$\begin{array}{ccc} \square & \square & \square \\ \times & \square & \square \\ \hline \end{array}$$

21. Place the numbers 2, 3, 5 and 7 in the boxes at the bottom to make the largest product at the top. The figure at the left shows one possibility that has been started.



22. Let's say you have an inexpensive calculator that has only an eight-digit or nine-digit display, and thus you cannot get the answers to these computations directly from your calculator. Determine the products by a means other than doing them out longhand, assuming that you have only these resources: your knowledge of multiplication and base 10 and a calculator with an eight- or nine-digit display. Justify your method; that is, why will it give you the desired product?
- a. $999,999,999 \cdot 56$
- b. $987,654,321 \cdot 9$
- c. $11,111,111,111 \cdot 45$
- d. $34,000 \cdot 56,000$
23. Select any two-digit number and multiply it by 99. Repeat this process for two or more two-digit numbers. Look for patterns that would enable you to compute the product of any two-digit number and 99 in your head.
- a. Describe how you can determine the product of any two-digit number and 99.
- b. Describe how you can determine the product of any two-digit number and 999.
24. Take your house number or room number and double it, add 5, multiply by 50, add your age, and subtract 250. What do you get? Can you explain why this trick works?
25. Four children want to share 168 jelly beans. How many does each get? Which model of division is operative here? Model the problem with a diagram.
26. a. Make up a division story problem for the partitioning model. Make a diagram and explain why this problem is an example of the partitioning model.
- b. Make up a division story problem for the repeated subtraction model. Make a diagram and explain why this problem is an example of the repeated subtraction model.
27. For each number line problem below, identify the computation it models and briefly justify your answer.
- a. 
- b. 
28. Represent the following problems on a number line. Explain each problem as though you were talking to someone who was not taking this class.
- a. $6 \cdot 3$
- b. $12 \div 3$
29. You are teaching fifth grade and you are "reviewing" division. Let's say that Lionel has a hard time with

division and uses the scaffolding method, but he forgot to write the answer. What is the answer? Explain Lionel's method.

$$\begin{array}{r}
 7 \\
 10 \\
 60 \\
 100 \\
 400 \\
 12 \overline{)6924} \\
 \underline{4800} \\
 2124 \\
 \underline{1200} \\
 924 \\
 \underline{720} \\
 204 \\
 \underline{120} \\
 84 \\
 \underline{84} \\
 0
 \end{array}$$

30. Identify and explain the errors in the problems below, all of which have occurred in real classrooms.

a. $\begin{array}{r} 54 \\ 7 \overline{)3528} \\ \underline{35} \\ 28 \\ \underline{28} \end{array}$	b. $\begin{array}{r} 540 \\ 7 \overline{)3528} \\ \underline{35} \\ 28 \\ \underline{28} \end{array}$	c. $\begin{array}{r} 79 \text{ R}26 \\ 8 \overline{)658} \\ \underline{56} \\ 98 \\ \underline{72} \end{array}$
--	---	---

31. Place the digits 2, 4, 7, and 9 in the boxes below to obtain:

- a. The greatest quotient b. The least quotient

$\square \overline{) \square \square \square}$

32. If you divide a number that is larger than 75 by a number that is less than 3, what can you tell about the relative value of the quotient? Explain your choice.

- It must be greater than 25.
- It must be less than 25.
- It will be between 3 and 75.
- There is not enough information to conclude any of the above.

33. When 96 is divided by x , a quotient of $y \neq 1$ and a remainder of 5 result. Find x and y .

34. a. How many buses are needed if each bus can take 25 passengers and there are 334 students?

- b. How many buses are needed if we also place two adults on each bus?

35. Wei is having a party. She is expecting 120 people and wants enough juice so that each person can have two glasses. If the juice comes in 32-ounce containers, how many cans of juice should she buy? If each can costs 89 cents, how much will the juice for her party cost?

36. Telephone directories are now available on CD-ROMs, which can hold an enormous amount of information. Let's say that there was a master directory that contained the names and addresses of 275 million people,

roughly the population of the United States. If we were to print the names and addresses of all these people on paper, how many pages would that require? First, you will need to determine how many names will fit on one page.

37. Joe had planned to sell 10 pencils at 20¢ each. However, his little brother broke two of the pencils. If he wants to make the same amount of money, how much should he charge for the pencils now?

38. Orange soda comes in six-packs that are packed four to a carton. To make sure that each of the 247 students in the sixth grade will have a soda for the class picnic, how many cartons should we order? How could you figure this out mentally?

39. Janice has decided to buy a computer for her son. So far, she has \$450 in the bank. The computer costs \$1234. If she can afford \$15 per week, when will she have the money?

40. Melanie recorded a tape of her guitar playing. It cost her \$1200 to record the master tape, and it costs \$2.50 to make each copy. If she sells the tapes for \$10 each, how many tapes must she sell to break even?

41. Your elementary school is having a fund-raiser. The school has a deal with a company from which it buys wrapping paper at the following rate: 100 rolls for \$150. The school plans to sell the paper in packs of 4 for \$9. How many rolls should the school buy to make a profit of \$500?

42. A factory can make 250 gizmos every hour. If the factory operates 24 hours per day, how many gizmos can they make in one year?

43. Mars is 34 million miles away. Presently, spaceships go about 16,000 miles per hour. How long would it take a spaceship to get to Mars?

44. In 2000 the U.S. federal debt was \$5,600,000,000,000. Let's say a decision was made to pay off that debt in much the same way that an individual pays off a debt, such as a car loan or a home mortgage: by paying back a specified amount of money at regular time intervals.

- a. Let's say that the decision was made to reduce the debt by \$10 million per day. How long would it take to pay off the debt at this rate?

- b. Before doing the computations, note any assumptions that you make.

45. This problem is over two thousand years old! A dog is chasing a rabbit that has a 150-foot head start. If the dog runs 9 feet for every 7 feet jumped by the rabbit, in how many leaps will the dog catch up with the rabbit?

46. Mathematics abounds in children's literature. Consider this age-old children's rhyme:

As I was going to St. Ives
I met a man with seven wives.
Every wife had seven sacks;

Every sack had seven cats;

Every cat had seven kits.

Kits, cats, sacks, and wives:

How many were going to St. Ives?

47. Find the five-digit number that has the following pattern: If you put a 1 after it, the number is three times as large as it would be if you had put a 1 before it.

48. a. How many cubes would it take to make a staircase like the one on the left if the last stair was 99 cubes high?
b. How many cubes would it take to make a staircase like the one on the right if the last stair was 99 cubes high?



49. A 5 by 5 grid has 16 squares on the border.
a. How many squares will be on the border of a 10 by 10 grid?
b. How many squares will be on the border of an n by n grid?
50. Using a calculator, Ralph multiplied by 10 when he should have divided by 10. The display read 300. What should the correct answer be?

51. Determine the numbers in the empty boxes.

a.

a	b	$a + b$	$a \cdot b$
65		131	
234		1000	
	72		86976

b.

a	b	$a + b$	$a \div b$
153			17
		20	19

c.

a	b	$a - b$	$a \cdot b$
		20	1056

d.

a	b	$a \cdot b$	$a \div b$
		20	5
	37		925

52. Find the missing digits below

a. $74 \cdot 8\Box = 6\Box\Box4$

b. $\Box\Box4 \cdot 64\Box = 489,346$

c. $18 \cdot (\Box\Box4 + 29\Box) = 10,116$

d. $(\Box3) \cdot (\Box) \cdot (\Box7) = 1547$

e. $1530 = \Box\Box \cdot 48 + 42$

53. In the following puzzle, each letter has a value between 0 and 9. Find the value of each symbol so that the following computations are all accurate.

A	C	B	F	A	B
$\times A$	$\times C$	$\times E$	$\times D$	$\times A$	$\times F$
B	DE	B	F	G	HF
C		J		K	
$\times A$		HDB		AHD	
DC					

54. For each of the following, determine possible whole numbers a , b , and c for which the statement is true.

a. $(a \div b) \div c = a \div (b \div c)$

b. $a(b - c) = ab - ac$

c. $(a - b) - c = a - (b - c)$

55. The square on the left is a multiplication magic square. How is a multiplication magic square similar to an addition magic square? How is it different? Can you make another multiplication magic square?

14	39	6	74
111	4	26	21
26	21	111	4
6	74	14	39

56. Place the numbers 1 through 9 in the boxes below so that all three equations work. You cannot use any number twice.

$\Box + \Box = \Box$

$\Box - \Box = \Box$

$\Box \div \Box = \Box$

57. Briefly describe what you learned about multiplication and division in this section. Discuss what they mean, how they are connected to other operations, and why the algorithms work. Then select the most important learning and describe what it was that you learned and why this is the most important learning. For example, "I never realized that multiplication can be represented as a rectangle. This has helped me to understand the multiplication algorithm better. For example, 14×5 can be seen as 1 ten and 4 ones 5 times."

SECTION

3.3

MENTAL ARITHMETIC AND ESTIMATION

WHAT DO YOU THINK?

- With the prevalence of calculators, why is mental computation still important?
- Why might people use rounded numbers rather than exact numbers?
- What numbers on the front page of today's newspaper are really estimates?

The availability of inexpensive calculators has profoundly changed the way we compute. Most people now rely on calculators when exact answers are needed. However, estimating skills are still very important in cases where an exact answer is not needed and to check the reasonableness of results obtained on the calculator. Estimation, in turn, requires good mental arithmetic skills, which come from an understanding of the nature of the operations, a firm understanding of place value, and the ability to use various properties (for example, the associative and distributive properties).

The NCTM Standards not only emphasize the need for students to understand how to estimate but also reinforce the importance of their being more actively involved in the construction of knowledge, as opposed to simply receiving procedures from teachers. Thus, the standards note that students need to “develop, analyze, and explain procedures for computation and techniques for estimation” (*Curriculum Standards*, p. 94).

Estimation

When are numbers estimates and not exact numbers? Before we examine some methods of estimation, we need to understand when numbers represent estimates and when they represent exact numbers. All of the following numbers are estimates. What ways can you see to group them according to why they are estimates? For example, the age of a dinosaur bone is an estimate because present dating methods do not allow us to get an exact number; in other words, the exact age is unknown.

- A certain dinosaur bone is 65 million years old.
- The number of hungry children in the United States is 12 million.
- The area of the Sahara Desert is 3,320,000 square miles.
- The mean July temperature in Tucson, Arizona, is 86 degrees.
- Jane lives 55 miles from the nearest airport.
- My office is 12 feet by 9 feet.

Numbers are estimates when:

1. The exact value is unknown—for example, predictions and numbers that are too large or difficult to determine.
2. The value is not constant—for example, population and barometric air pressure.
3. There are limitations in measurement—for example, the length of a desk and the area of a pond.

Note that these are not the only instances when a number is actually an estimate, but rather three common categories. The purpose here is not to memorize categories, but to realize that many numbers we see around us are actually estimates.

Rounding Just as many numbers in everyday life are estimates rather than exact numbers, many numbers are rounded. For example:

- It took Jackie 10 hours to get from Boston to Buffalo.

- Anna gets 34 miles per gallon with her new car.
- Rosie put 2100 miles on her car last month.
- Fred paid \$12,000 for his new car.
- The population of Sacramento, California, is 370,000.

The preceding examples bring another question to mind: Why do we round? Stop for a minute to think of your own response. Then read on. . . . 

Below are two of many possible reasons:

1. Rounding makes comprehension easier. Which of the following sentences would you prefer to read in a newspaper? Why?

The school budget for the 34,168 students was \$153,167,458.

The school budget for the 34,000 students was \$153 million.

2. Rounding makes computation easier. Let's say you are trying to determine the cost of buying new textbooks for your fifth-grade class. There are 23 students, and the textbooks cost \$19.75. If you wanted to get a sense of the cost, you would probably round 19.75 to 20 and multiply 23 by 20.

LEARNING

Every year in my middle school mathematics methods course, my students develop a series of questions about a topic, and then they go into the schools and ask actual students these questions in order to learn more about what the students know and what they don't know. Every year my students make up estimation problems, and every year my students are quite stunned to find that many of the middle school students have few or no estimation and mental math strategies in their repertoire. This is all the more amazing in view of the fact that in their everyday lives, people require estimates more often than exact answers.

When do we use estimation and when do we use exact computation?

Before you read this section, close the book and write down your thoughts about this question. Then read on. . . . 

Following are several examples of when people generally estimate:

- Making a budget—cost of college, cost of food per month
- Determining the cost of a trip or vacation—ski trip, camping trip, trip to Europe
- Deciding which to buy—whether to buy a new car or a used car
- Determining time—how long to get to . . .
- Determining whether we have enough money—being at the grocery store when short on cash
- Deciding how much the tip should be (at a restaurant)
- Determining how long the paper or project will take

Estimation methods As you will find in this chapter, we do not estimate in the same way in all situations. The method(s) we use to estimate depend partly on how close the estimate has to be and on whether we deliberately want to over- or underestimate (for example, underestimating when budgeting can cause serious problems). Finally, people usually prefer certain methods to other methods. In many cases, different methods will produce estimates that are nearly identical, and in some cases, certain methods are easier than other methods.

Before we develop our estimation toolbox, we will first develop our ability to compute exact answers mentally. This may surprise you, but the development of mental math skills will result in your coming to a deeper understanding of the four operations and will, in turn, make your estimation tools stronger.

Mental Arithmetic

Occasionally, I will note some ways in which the philosophy of this book and that of traditional instruction differ. In a more traditional setting, this book (and your teacher) would “teach” you the “best” mental arithmetic strategies, probably by demonstration, and then have you practice those strategies. A more progressive approach would be to “teach” you the “best” strategies but have you actively involved in understanding and making sense of each strategy. If you have done Exploration 3.18, you have experienced a teaching approach consistent with a philosophy called constructivism. That is, rather than receive the teacher’s best strategies, you construct your own strategies by connecting knowledge and information you already know and by explaining your ideas and listening to others. This section enables you to examine strategies that you might have heard but only partially understood or strategies that did not come up in your class or group. If you have not done the explorations—or even if you have!—I encourage you to read this section actively.

Mental Addition Strategies

Do the following ten computations in your head. Briefly note the strategies you used, and try to give names to them.

Note: One mental tool all students have is being able to visualize the standard algorithm in their heads. For example, for the first problem, you could say: “ $9 + 7 = 16$, carry the 1, then $5 + 3 = 8$ + the carried 1 makes 9; the answer is 96.” However, because you already know that method, I ask you not to use it here but to try others. There are actually quicker ways to do this problem in your head than using the traditional algorithm. See whether you can discover any of them.

- | | | | | |
|---|---|---|---|--|
| 1. $\begin{array}{r} 39 \\ +57 \\ \hline \end{array}$ | 2. $\begin{array}{r} 38 \\ +46 \\ \hline \end{array}$ | 3. $\begin{array}{r} 68 \\ +35 \\ \hline \end{array}$ | 4. $\begin{array}{r} 66 \\ +19 \\ \hline \end{array}$ | 5. $\begin{array}{r} 686 \\ +140 \\ \hline \end{array}$ |
| 6. $\begin{array}{r} 545 \\ +228 \\ \hline \end{array}$ | 7. $\begin{array}{r} 186 \\ +125 \\ \hline \end{array}$ | 8. $\begin{array}{r} 4,265 \\ +436 \\ \hline \end{array}$ | 9. $\begin{array}{r} 245 \\ +138 \\ \hline \end{array}$ | 10. $\begin{array}{r} 580 \\ +423 \\ \hline \end{array}$ |

The right answer to each of these sums isn’t really what’s at issue. Rather, the hope is that you will use these strategies as a jumping-off point for understanding mental addition more clearly. Now that you have done the computations on your own, let us examine several strategies. How are they different from, or similar to, the strategies you used?

Leading digit One strategy that works nicely with most addition problems is **leading digit**. Some people refer to it as **front end** because we add the “front” of the numbers first. For example, we can look at Problem 1 as $80 + 16 = 96$; that is, we add the numbers in the tens column ($3 + 5$), add the numbers in the ones column ($9 + 7$), and then add $80 + 16$ to get 96.

This strategy can be used with larger numbers. For example, try this strategy with Problem 6 and then read on. . . . 

$$700 + 60 + 13, \text{ or } 773$$

The leading digit method is new to most students, and some find it awkward at first. However, in the long run, it is generally faster than doing the algorithm in your head. For example, doing Problem 6 in your head using the standard algorithm looks like this: $5 + 8 = 13$, carry the 1; $4 + 2$ + the carried 1 is 7, so that’s 73; $5 + 2 = 7$, so it’s 773.

Try some of the other problems using the leading digit method. Make up problems for additional practice (with a partner if possible). It is important that you become familiar with this strategy because it is commonly used in estimating.

Compensation Another powerful mental math strategy is called **compensation**. Look at Problem 1 to see how this strategy works. Instead of trying to add $39 + 57$, we could have added $40 + 56$. Do you see how we transform $39 + 57$ into $40 + 56$?

By adding 1 to the first number and subtracting 1 from the second number, we have created a new problem that is much easier to solve mentally. Why is this? Mathematically, we used the following transformation:

$$39 + 57 = (40 - 1) + 57 = 40 + (57 - 1) = 40 + 56$$

Which other problems lend themselves to this strategy?

Break and bridge We can use the **break and bridge** strategy in Problem 3. To add $68 + 35$, break 35 into $30 + 5$ and do the following:

$$(68 + 30) + 5 = 98 + 5 = 103$$

Which other problems lend themselves to this strategy?

Compatible numbers Another powerful strategy is creating **compatible numbers**. This often involves seeing pairs of digits whose sum is 10. Just as children quickly learn the pairs of digits whose sum makes a long (for example, 1 and 9, 2 and 8), most people can sense which combinations of numbers will make multiples of 100. Thus, in Problem 7, $186 + 125$, we can see that the 180 and the 120 combine to make 300, and then we have 11 ($6 + 5$) more.

Do you see how place value and base 10 are used in this strategy? If you can see Problem 7 as 18 tens and 12 tens, then you have 30 tens—that is, 300. Look at Problems 3 and 10. What ways can you see to get multiples of 10 and 100 in those problems?

The compatible numbers strategy is especially useful when adding several numbers mentally. Try it with the following problem and then read on: $16 + 35 + 28 + 15$. 

We can obtain the exact sum by adding the 35 and the 15 together and the 28 and the 16 together. Next, we add the results:

$$50 + 44 = 94$$

Choosing a strategy Which strategy you use is often a matter of preference. For example, Problem 3 ($68 + 35$) may be done mentally in at least four different ways, each of which is the easiest way for some students. Although in general no one way is inherently superior to another, it is better to be proficient at several methods than at just one.

Leading digit:	$68 + 35 = 60 + 30 + 8 + 5$
----------------	-----------------------------

Compensation:	$68 + 35 = 70 + 33$
---------------	---------------------

Break and bridge:	$68 + 35 = 68 + 30 + 5$
-------------------	-------------------------

Compatible numbers:	$68 + 35 = 65 + 35 + 3$
---------------------	-------------------------

Justifying strategies How do these strategies use base 10, expanded form, and the commutative and associative properties? Think and then read on. . . . 

Let's look at the compatible numbers strategy for Problem 3 in detail:

The action	Justification
$68 + 35 = (65 + 3) + 35$	Substitution
$= 65 + (3 + 35)$	Associative
$= 65 + (35 + 3)$	Commutative
$= (65 + 35) + 3$	Associative
$= 100 + 3$	Addition
$= 103$	Addition

Estimation Strategies for Addition

Explorations 3.19 and 3.20 challenged you to apply your understanding of base 10 and your mental math abilities to determine estimates in two settings outside the classroom. Here we will analyze some estimation problems to understand better the application of base 10 concepts and mental math strategies.

INVESTIGATION 3.10 What Was the Total Attendance?

In each of the following estimation problems, first obtain a rough estimate (5–10 seconds), then obtain the best estimate you can, and then read on. Use mental arithmetic strategies in your estimates. 

- A.** Approximately what was the total attendance for the following three football games at Foxborough Stadium? 75,145 34,135 55,124

DISCUSSION

Remember that one of the main goals of the investigations is for you to develop a repertoire of strategies. Check your estimates against the ones obtained below with different strategies. Do you find you like certain strategies better than others? Are there any that you find you have a hard time understanding? In this situation, there are several estimation strategies that work well:

- Leading digit: $7 + 3 + 5 = 15$; that is, 150,000
The leading-digit method will always give you an estimate that is lower than the actual sum. Why is this?
- Refined leading digit: $150,000 + (5 + 4 + 5 = 14 \text{—that is, } 14,000)$
 $= 150,000 + 14,000 = 164,000$
- Rounding: $80,000 + 30,000 + 60,000 = 170,000$
- Compatible numbers: $75 + 35 + 55 = 110 + 55 = 165$, which represents 165,000

- B.** Approximately what was the total attendance for three baseball games at Wrigley Field in Chicago? Make your own rough estimate and then a refined estimate before reading on. . . . 

32,425 31,456 34,234

DISCUSSION

Are you getting quicker at estimating? Are you finding that you are applying the mental addition techniques? Each of the four methods above could have been used here. You may well have come up with another strategy called **clustering**, because all three numbers are relatively close together. In this case, a very quick, rough estimate would be

$$30,000 \times 3 = 90,000$$

What strategy or combination of strategies would yield an estimate that was closer to the exact attendance? If we use, for example, a refined leading digit strategy, we can get $90,000 + 7000$, and looking at the 425, 456, and 234, we can see that this is about 1000. Thus, a more refined estimate is about 98,000.

Looking back In this investigation, how close were your rough estimates to your best estimates? Can you make up a situation in which someone would need only a rough estimate? How about a situation in which someone would want an estimate that was more refined? 

There are two points to keep in mind when estimating and doing mental mathematics:

1. The method you use is often partially determined by the problem itself. If you want only a rough estimate, you might use leading digit or rounding. If you want a more refined estimate, you might use compatible numbers. If you want to make sure you have enough money, you might round everything up so that the estimated sum is definitely greater than the actual sum.
2. It is not so much a matter of there being right or wrong ways to estimate or there being better or poorer methods. There is a certain amount of latitude for preferences, just as some tennis players will like a certain type of racket and others will like another type of racket. However, if you have a large repertoire of estimating and mental math techniques in your toolbox, you will be more skillful.

A Refined Technique for Rounding

Most textbooks offer the following guidelines for rounding:

If the appropriate digit is less than 5, round down. Otherwise round up.

For example, if we estimate $35 + 57$, we round to $40 + 60 = 100$. There is a problem with this method, though. Suppose we want to estimate the sum of the following numbers:

Problem	Traditional rounding
35	40
42	40
76	80
45	50
35	40
<u>85</u>	<u>90</u>

If we have a problem in which many of the numbers end in a 5, this method will give an estimate that is too high.

A more refined set of guidelines for rounding makes the following modification:

If the digit to the left of the 5 is odd, round up.

If the digit to the left of the 5 is even, round down.

When we encounter problems with columns of addition, it is likely that the numbers of evens and odds will be close, thus canceling each other out. Let us see how this affects the estimate with the preceding problem:

Problem	Standard rounding	Refined rounding
35	40	40
42	40	40
76	80	80
45	50	40
35	40	40
85	90	80
318	340	320

If we use this refined method, our estimate will generally be closer to the actual amount. Why won't it always be closer?

INVESTIGATION 3.11 Estimating by Making Compatible Numbers

Using compatible numbers is an effective strategy to estimate the following sums. Try this on your own and then check below. . . .

A. 38 **B.** 23

72 359

89 177

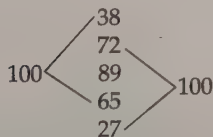
65 675

27 162

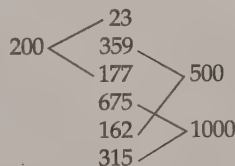
 315

DISCUSSION

A. In this case, a quick glance shows us that 38 and 65 will make a sum close to 100, and so will 72 + 27. If we see this, our estimate of $200 + 89 = 289$ is quite close to the actual answer of 291.



B. Using compatible numbers, we can estimate $200 + 500 + 1000 = 1700$.



INVESTIGATION 3.12 How Much Money Did Sandy Spend?

Sandy wrote checks for the following amounts. Approximately how much did she just spend? 

\$173	Credit card
\$97	Electricity
\$63	Gas credit card
\$212	Insurance
\$4	Parking ticket
\$36	Phone

- A. As before, first make a very quick, rough estimate.
- B. See how close you can get to the exact answer, but without using pencil-and-paper or a calculator.

DISCUSSION

A. To find a rough estimate, we could use the leading digit method; this would give us 3 hundreds, but then we would have to remember the 3 hundreds while adding the tens. A quicker way to make a rough estimate is to use compatible numbers: $173 + 212$ make about 400; 97 and 4 make another 100, and then $63 + 36$ make another 100. Our rough estimate is \$600.

B. There are many ways to obtain a more accurate estimate. One way might begin by rounding:

$$170 + 100 + 60 + 210 + 40$$

Then we could add these rounded numbers to obtain an estimate of \$580.

Again, it is not a matter of which strategies are better or right, but of having a sense of how accurate an estimate we need and then being able to employ a strategy that can get us close to the actual answer.

What would be a scenario in which Sandy would be satisfied with a rough estimate? What would be a scenario in which Sandy would want her estimate to be “pretty” close to the actual amount?

Looking back Before we move on to subtraction, take a few moments to look back over the strategies for mental addition and estimation that we have explored. Do you find that you prefer some over others? Do you see that some strategies work better with some numbers than with others? Identify those places in which these strategies use properties of addition or important base 10 structures.

Mental Subtraction Strategies

Not surprisingly, fewer people estimate confidently when subtracting than when adding. In fact, one of the ways that I assess how well a student understands the various models for subtraction and the relationship between addition and subtraction is by seeing the extent to which a student can develop a repertoire of mental strategies for subtraction.

Do the following computations in your head. Briefly note the strategies you used, and try to give names to them.

Note: One mental tool all students have is being able to visualize the standard algorithm in their heads. For example, for the first problem, you could say,

"Cross out the 6, and now subtract $15 - 8 = 7$ and then $5 - 2 = 3$; the answer is 37." However, because you already know that method, I ask you not to use it here, but to try others.

- | | | | | |
|---|---|---|--|--|
| 1. $\begin{array}{r} 65 \\ -28 \\ \hline \end{array}$ | 2. $\begin{array}{r} 62 \\ -29 \\ \hline \end{array}$ | 3. $\begin{array}{r} 184 \\ -125 \\ \hline \end{array}$ | 4. $\begin{array}{r} 132 \\ -36 \\ \hline \end{array}$ | 5. $\begin{array}{r} 1000 \\ -648 \\ \hline \end{array}$ |
|---|---|---|--|--|

As with addition, there are a variety of subtraction strategies. As you read the discussion below, reflect on the strategies. Do they make sense to you? If not, ask yourself why. Is it because you don't understand how the strategy works? Is it because you don't feel your mental subtraction skills are strong enough to use it? Again, it's not so much a matter of right or better strategies as it is of having a repertoire. Almost all of the strategies work better with certain kinds of numbers than with others.

In Problem 1, one alternative is to **add up**—that is, to ask how we get from 28 to 65. One student's actual thinking could look like this: " $28 + 40 = 68$, that's 3 too much, so the answer is $40 - 3 = 37$."

Another strategy for Problem 1 is to find the nearest ten and add up: " $28 + 2 = 30$, then we need 35 more to get to 65, and $35 + 2$ is 37."

Some students understand this problem better with a **number line**, as in Figure 3.25. In one sense, we are going from 28 to 30 to 65.

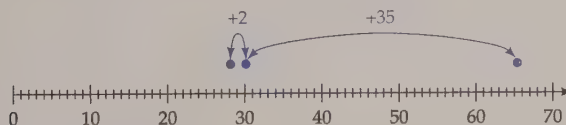


FIGURE 3.25

In Problem 2, compensate: Add 1 to both numbers. This transforms the difficult $62 - 29$ to the relatively simple $63 - 30$.

Some students better understand the compensation strategy by using a number line (see Figure 3.26). On a number line, $62 - 29$ can be interpreted as how far apart the two numbers are. In other words, how long is the road from 29 to 62? If we move the beginning and the ending of the road 1 unit, we haven't changed the length of the road.

Thus $62 - 29$ and $63 - 30$ have the same difference.

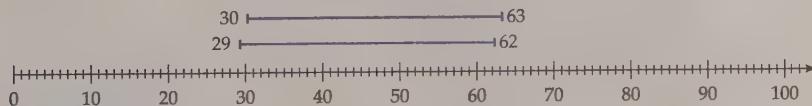


FIGURE 3.26

In Problem 3, add up: $125 + 60 = 185$ and then take off 1. The answer is $60 - 1 = 59$.

In Problem 4, try the compensation strategy: $132 - 36$ is close to $136 - 36 = 100$, but we must take away 4 to get 96. In other words, we added 4 to 132, but then we had to remember to take it off again.

Problem 4 can also be solved by using compatible numbers: $36 + 64 = 100$, and then 32 more to get to 132, so the answer is $64 + 32 = 96$.

In Problem 5, use compatible numbers: 48 and 52 are compatible numbers; that is, their sum is 100. Therefore, $648 + 52$ makes 700. Then we add 300 to get to 1000. The answer is 352.

Problem 5 can also be solved by adding up: $648 + 300 = 948$. We need 52 more. The difference is $300 + 52 = 352$.

Do you understand all of the strategies discussed here? Try to make a problem in which one strategy would be more appropriate than another. Most students find that they retain these strategies if they make up some problems on their own and practice.

INVESTIGATION 3.13 Mental Subtraction in Various Contexts

Do these two problems on your own before reading on. . . 

- A. Mental math with money: How much change do you get if you give a \$10 bill when the cashier tells you \$4.76? How would you do this problem mentally? Try it and then read on. . . 
- B. Mental math with non-base 10 numbers: You left home at 3:25 and got to the meeting at 6:15. How long did it take you to get to the meeting? Try it and then read on. . . 

DISCUSSION

A. Before we had electronic cash registers, cashiers had to determine the change, and most could do it mentally. What strategy does the following language imply: “4.76, 4.80, 5 dollars, 10 dollars”? The cashier is adapting the add-up strategy: 4 pennies, 2 dimes, and 5 dollars makes \$5.24.

B. Even when using non-base 10 numbers, adding up is an effective strategy: $3:25 + 35 \text{ minutes}$ makes 4 o’clock, 2 more hours gets you to 6 o’clock, and then 15 more minutes. In other words you are adding $2 \text{ hours} + 35 \text{ minutes} + 15 \text{ minutes} = 2 \text{ hours} + 50 \text{ minutes}$.

Once again a number line is instructive (see Figure 3.27).

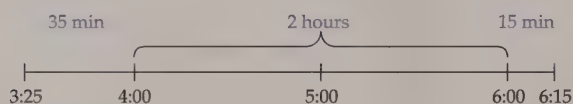


FIGURE 3.27

Alternatively, you might reason that 3:25 to 6:25 is 3 hours and you need to “back off” 10 minutes; that is, the answer is 2 hours and 50 minutes.

INVESTIGATION 3.14 Rough and Best Estimates with Subtraction

Now let us apply these skills to problems where we are likely to estimate with subtraction. Remembering that one of the main goals of the investigations is to develop a repertoire of strategies, check your estimates against the ones discussed below. In each of the problems below, first obtain a rough estimate (5 to 10 seconds). Then obtain the best estimate you can. Do you find that you like certain strategies better than others? Are there any that you find you have a hard time understanding? 

- A. In 1995, Acstead State College had an enrollment of 4234, whereas Milburn College had an enrollment of 3475. How many more students did Acstead have than Milburn?

$$\begin{array}{r} 4234 \\ -3475 \\ \hline \end{array}$$

- B. The attendance at the Yankees game was 73,468, whereas the attendance at the Mets game was 46,743. How much greater was the attendance at the Yankees game?

$$\begin{array}{r} 73,468 \\ -46,743 \\ \hline \end{array}$$

DISCUSSION

- A. You can get a quick estimate by rounding: $4200 - 3500 = 700$

To get a more accurate estimate, you could use adding up:

$$3475 + 800 = 4275, \text{ then back off } 40 \text{ to get } 4235. \text{ The estimate is } 800 - 40 = 760.$$

Do these strategies make sense to you? Did you do something different? How would you describe a scenario in which someone would be satisfied with a rough estimate? How would you describe a scenario in which someone would want the estimate to be pretty close to the actual amount?

- B. Leading digit: $7 - 4 = 3$, giving an estimate of about 30,000.

Rounding: $70,000 - 50,000 = 20,000$.

Rounding and adding up: $47,000 + 3000$ makes 50,000. Adding up to 73,000 gives 23,000. Our estimate is $3000 + 23,000 = 26,000$.

Looking back Before we move on to multiplication, take a few moments to look back over the strategies for mental subtraction and estimation that we have explored. Do you find that you prefer some over others? Can you connect these strategies to different models or contexts of subtraction—for example, take-away, comparison, missing addend, number line? Can you explain which strategies make powerful use of structures of our base 10 numeration system—for example, the leading digit strategy? Do you see that some strategies work with some numbers better than with others?

Mental Multiplication

Many of my students are surprised to find that with some thought and practice, they can do many more multiplication problems mentally than they would have thought possible. Doing so successfully generally requires us to apply the various properties of multiplication, often creatively!

Do the following computations in your head. Briefly note the strategies you used, and try to give names to them.

- | | | | | | |
|--|---|---|---|---|---|
| 1. $\begin{array}{r} 64 \\ \times 5 \\ \hline \end{array}$ | 2. $\begin{array}{r} 16 \\ \times 25 \\ \hline \end{array}$ | 3. $\begin{array}{r} 35 \\ \times 12 \\ \hline \end{array}$ | 4. $\begin{array}{r} 849 \\ \times 2 \\ \hline \end{array}$ | 5. $\begin{array}{r} 60 \\ \times 30 \\ \hline \end{array}$ | 6. $\begin{array}{r} 65 \\ \times 24 \\ \hline \end{array}$ |
|--|---|---|---|---|---|

As you read these strategies, once again monitor your own thinking. Do you understand how the strategies work? If not, do you simply need to reread the discussion, or would it be helpful to make up and do some similar problems, or would it be more helpful to practice with a friend?

BEYOND THE CLASSROOM

In Part B of Investigation 3.14, a Yankee fan would be more likely to do the first estimate, a Mets fan the second estimate, and a baseball announcer or executive the third estimate.

In Problem 1, use **multiples of 10** as a reference point:

$$64 \times 5 = \frac{1}{2} \text{ of } 64 \times 10 = \frac{1}{2} \text{ of } 640 = 320$$

In Problem 2, the **halve and double** method gives

$$16 \times 25 = 8 \times 50 = 400$$

Do you see the use of the commutative and associative properties here?

For Problem 3, try the distributive property: 35×12 can be seen as twelve 35s, which can be broken apart to ten 35s + two 35s, or

$$(10 \times 35) + (2 \times 35) = 350 + 70 = 420$$

In Problem 4, try compatible numbers in conjunction with the distributive property:

$$849 \times 2 = (850 - 1) \times 2 = 1700 - 2 = 1698$$

For Problem 5, use multiples of 10:

$$60 \times 30 = 6 \times 3 \times 10 \times 10 = 18 \times 100$$

What people often think in this case is “18 with two zeros,” or 1800.

In Problem 6, try using **expanded notation** and then determining the four **partial products**:

$$\begin{array}{r} 65 \\ \times 24 \\ \hline \end{array} \qquad \begin{array}{r} 60 + 5 \\ 20 + 4 \\ \hline 240 + 20 \\ 1200 + 100 \end{array}$$

Adding the four partial products, starting from left to right, one might say, “1200 + 100 is 1300, then add 240 to get 1540, and then add 20 to get 1560.”

Because looking back or reflecting is such an important learning tool, and because so few students bring that tool into this course, I am risking overemphasizing it. Now that you have examined various mental multiplication strategies, do you see advantages and limitations of the various strategies? Do you understand the names given to the strategies, or can you come up with names that you like better?

INVESTIGATION 3.15 Developing Estimation Strategies for Multiplication

In each of the problems below, first obtain a rough estimate; then obtain the best estimate you can before reading on. Can you describe a scenario in which the rough estimate would be appropriate? Can you describe a scenario in which a more refined estimate would be appropriate? 

- A. Chip rode 78 miles last week. At this rate, how many miles will he ride this year?

DISCUSSION

STRATEGY 1: Find a *lower* and *upper bound* for the answer

Because we want to estimate the product of 78 and 52, if we round both numbers down to the nearest 10, we have $70 \times 50 = 3500$. Similarly, if we round both numbers up to the nearest 10, we have $80 \times 60 = 4800$. The actual product of 78 and 52 lies between these two numbers. The estimate from rounding

down is called a **lower bound**, and the estimate from rounding up is called an **upper bound**. Using this technique, we can quickly say that Chip will ride between 3500 and 4800 miles this year.

STRATEGY 2: Round one number up and round the other number down

Thus 78×52 becomes $80 \times 50 = 4000$.

STRATEGY 3: Use expanded form and estimate the sum of the four partial products

$$\begin{array}{r} 70 + 8 \\ 50 + 2 \\ \hline 140 + 16 \\ 3500 + 400 \end{array}$$

One thought process might go like this: " $70 \times 50 = 3500$, 50×8 is 400, and 70×2 is more than 100. So the answer will be 3500 plus more than 500, let's say 4050."

Determine the actual answer. How close were the different estimates?

- B.** Jane plans to start graduate school in September. She figures that she can save \$345 per month for the next 9 months. How much will she have saved? Work on this and then read on. . . . 

DISCUSSION

Use the distributive property:

$$345 \times 9 = (345 \times 10) - (345 \times 1) = 3450 - 345 \approx 3100$$

- C.** There were 47,752 Americans killed or missing in Vietnam. The number of Americans killed or missing in World War II was about 6 times that number. Approximately how many Americans were killed or missing in World War II? Work on this and then read on. . . . 

$$\begin{array}{r} 47,752 \\ \times \quad 6 \\ \hline \end{array}$$

DISCUSSION

We can get a rough estimate using rounding:

$$50,000 \times 6 = 300,000$$

We can get a more refined estimate by rounding and using mental math (double and halve):

$$48 \times 6 = 96 \times 3 = 288, \text{ or } 288,000$$

Looking back Before we move on to division, take a few moments to look back over the strategies for mental multiplication and estimation that we have explored. Do you find that you prefer some over others? Can you identify those places in which these strategies use properties of multiplication (for example, the distributive property)? Can you see which strategies make use of the multiplication algorithm (for example, partial products)? Can you see which strategies make use of base 10 structures (for example, expanded notation)? Do you see that some strategies work with some numbers better than with others?

Mental Division Strategies

Mental division is new territory for almost all of my students. Even the division idea of **canceling zeros** is one that many students feel apprehensive about before this course. Therefore, examine each of the problems below carefully. How might you determine the exact answer, applying what you know about division? Briefly note the strategies you used, and try to give names to them.

1. $20\overline{)6000}$ 2. $400\overline{)20,000}$ 3. $8\overline{)152}$ 4. $5\overline{)345}$

In Problem 1, we will examine two different ways to use **canceling**.

One method of canceling transforms the problem from $6000/20$ to $(60 \cdot 100)/20$, which can then be simplified to $3 \cdot 100$ or $60 \cdot 5$, because 20 divides both 60 and 100 without remainder.

$$\frac{6000}{20} = \frac{60 \cdot 100}{20} = \frac{\overset{3}{\cancel{60}} \cdot 100}{\underset{1}{\cancel{20}}} = \frac{60 \cdot \overset{5}{\cancel{100}}}{\underset{1}{\cancel{20}}}$$

Another way to cancel in Problem 1 is to transform it from $6000/20$ to $600/2$, which is equal to 300.

In this case, we are actually applying the idea of **equivalent fractions**, which we will examine in Chapter 5. That is, we are dividing both numerator and denominator by the same amount, 10.

Can you use either of these strategies to solve Problem 2? If so, which? Problem 3 lends itself to compatible numbers; for example, $160 \div 8 = 20$. We can use the distributive property to get an exact answer, since

$$\frac{152}{8} = \frac{160 - 8}{8} = 20 - 1 = 19$$

We can use compatible numbers in Problem 4: $350 \div 5 = 70$.

We can also use the distributive property in Problem 4:

$$\frac{345}{5} = \frac{350 - 5}{5} = 70 - 1 = 69$$

In Problem 4, we can also use the idea of equivalent fractions and multiply both numerator and denominator by 2. The resulting fraction $690/10$ easily simplifies to 69.

INVESTIGATION 3.16 Estimates with Division

Now let us apply these mental division skills to problems in which we are likely to use division to estimate. In each of the problems below, obtain an estimate before reading on. . . 

Do you find that you like certain strategies better than others? Are there any that you have a hard time understanding? Can you describe a scenario in which a rough estimate would be adequate? A scenario in which a more refined estimate would be more appropriate?

- A.** Mr. & Mrs. Smith pay \$9100 a year for their son's college. Translate this into a monthly payment that they can put into their budget.

DISCUSSION

Using the **missing factor model**, ask yourself, "12 times what is closest to 91?" $12 \times 7 = 84$, $12 \times 8 = 96$. Because 91 is just about in the middle (of 84 and 96),

it is reasonable to conclude that $12 \times 7\frac{1}{2}$ will be close to 91, and so our estimate is $7\frac{1}{2}$ hundred a month, or, more conventionally, \$750 per month.

- B.** Pierre just bought a new van, and he wants to see what kind of mileage he gets. He filled up with gas after going 489 miles, and the car took 19 gallons of gas. Estimate the mileage.

DISCUSSION

One strategy is to round both the divisor and the dividend up.

$\frac{489}{19}$ will be close to $\frac{500}{20}$, and this is equivalent to $\frac{50}{2}$, and so he is getting about 25 mpg.

This strategy is interesting because it is not identical to the one used in multiplication, where we get more accurate estimates if we round one number up and the other down. In the case of division, we obtain more accurate estimates if we round both numbers up or both numbers down to manageable numbers. Can you explain why? Work on this for a while and then read on. . . . 

Hint: It might be helpful to make up some problems and check this out. For example, try $3750 \div 13$. Decrease both: to 3600 and 12, which are compatible numbers. Increase both: to 3900 and 13. Then decrease one and increase the other—for example, 3750 to 4000 and 13 to 10. Compare the results. Make up some other problems. Can you use ideas developed thus far to help you?

- C.** A large business ran 5432 copies in the last 7 business days. How many copies is it averaging per day?

DISCUSSION

A rough estimate could be gotten by making compatible numbers, increasing both, and solving the problem. For example, $5600 \div 7$ gives an estimate of 800.

A more refined strategy would be to use multiplication:

$$7 \times 700 = 4900$$

$$7 \times 800 = 5600$$

Because 5432 is closer to 5600 than it is to 4900, a rough estimate would be just under 800.

Do any of these strategies reflect the work you did in Exploration 3.20 with the risograph machine?

- D.** Employees put 25¢ in a can for each cup of coffee they drink. Last week the can contained \$29.50. How many cups of coffee were drunk?

DISCUSSION

STRATEGY 1: Solve the problem in dollars

That is, 25¢ per cup means 4 cups per dollar.

You can estimate $\$30 \times 4$ cups per dollar = 120 cups.

Using this strategy, you could actually use mental math and get the exact answer in two ways:

$$29.50 \times 4 = (29 \times 4) + \left(\frac{1}{2} \times 4\right) = 116 + 2 = 118$$

or

$$29.50 \times 4 = (30 \times 4) - \left(\frac{1}{2} \times 4\right) = 120 - 2 = 118$$

STRATEGY 2: Solve the problem in cents

That is, \$29.50 is about \$30, which is equivalent to 3000¢.

$$\frac{3000}{25} = \frac{2500 + 500}{25} = 100 + 20 = 120$$

Looking back Take a few moments to look back over the strategies for mental division and estimation that we have explored. Have you added to your repertoire? Do you understand how the distributive property and expanded notation are applied? Do you see that some strategies work with some numbers better than with others? Can you connect these strategies to different models or contexts of division—for example, missing factor or repeated subtraction?

Summary

Take a few moments to look back upon this section and your notes. In this section and its *Explorations*, we have applied many of the ideas that were developed in the previous three sections—that is, base 10 and place value from Section 3 of Chapter 2 and the concepts and procedures for adding, subtracting, multiplying, and dividing from the first two sections of this chapter. In one sense, your ability to do mental arithmetic and estimation depends on how well you understand the ideas of numeration and the four operations. Many of the mental arithmetic and estimation strategies make use of place value and of expanded form. Many of these strategies are directly connected to certain models and properties of the operations: for example, subtraction can be interpreted as the distance between two numbers. We frequently used properties of the operations, especially the distributive property. Many of the strategies in this section apply aspects of the algorithms in Sections 3.1 and 3.2—partial sums and partial products, for example.

As stated throughout this chapter, it is crucial to recognize numbers as compositions of parts and to be able to decompose numbers according to the needs of the problem. When we examine fractions in Chapter 5, composition and decomposition will be essential if you are to understand what fractions mean and be able to understand the connection between procedures for computing with whole numbers and with fractions.

In this section and its *Explorations*, we explored more real-life problems than we have elsewhere. To see why, look at the front page of any newspaper and examine the numbers in each of the stories. Which numbers are exact numbers and which are not? Which of the nonexact numbers are estimates and which are rounded? How can you tell? Why do you think the writer or editor chose to estimate or round?

This last question is another way of assessing the extent to which the ideas of this section are owned as opposed to rented: If someone asked you why we need to learn mental math and estimation when a calculator is generally nearby, what would you say?

EXERCISES 3.3

Note: In all of the exercises, briefly explain your reasoning.

1. Determine the following sums mentally. Briefly describe how you obtained the sum.

a. 47	b. 69	c. 88	d. 56	e. 577
+55	+77	+77	+19	+126

f. 735	g. 269	h. 75	i. 24	j. 583
+248	+ 87	36	53	+423
		187	387	
		65		

2. Estimate the following sums. Briefly describe how you obtained your estimate.

a. 473	b. 6963	c. 88,865	d. 536,455
345	3286	32,565	94,352
+355	+7147	+23,784	+659,346

e. 473	f. 69,655	g. 234,345,343
345	32,438	325,345,689
134	90,432	587,247,678
565	24,684	56,345,865
943	79,833	+234,764,874
+355	+71,347	

3. From each of the following lists, select three numbers whose sum will be closest to the target sum. Briefly explain your reasoning—how did you determine those three numbers?

Numbers				Target
a. 37	83	56	74	150
b. 73	32	94	53	200
c. 24	39	47	97	170
d. 115	175	164	153	400
e. 185	372	153	274	650
f. 352	423	439	583	1200

4. Determine the following differences mentally. Briefly describe how you obtained your estimate.

a. 87	b. 70	c. 82	d. 500	e. 502
-29	-23	-34	-134	-206
f. 473	g. 625	h. 808	i. 506	j. 4000
-258	-475	-707	- 29	- 555

5. Estimate the following differences. Briefly describe how you obtained the difference.

a. 4473	b. 65,963	c. 73,463	d. 43,433
-2355	-29,147	-28,543	-16,328
e. 413,082	f. 320,283	g. 3,133,543	h. 71,234,033
-285,876	-184,438	-1,903,253	-32,753,962

6. From each of the following lists, select two numbers whose difference will be closest to the target difference.

Numbers Target

a. 315	475	764	300
b. 185	372	953	650
c. 382	723	793	350

7. Determine the products mentally.

a. 16	b. 16	c. 66	d. 849	e. 60	f. 450
$\times 16$	$\times 25$	$\times 5$	$\times 2$	$\times 30$	$\times 20$
g. 35	h. 65	i. 736	j. 35	k. 632	
$\times 12$	$\times 42$	$\times 4$	$\times 16$	$\times 4$	

8. Estimate as closely as you can.

a. 41×68	b. 56×74	c. 62×83
d. 57×38	e. $34,345 \times 48$	f. 417×23

9. By estimating, determine which of the following is wrong. Explain your reasoning.

a. $3312 \times 13 = 43,056$
b. $23 \times 874 = 30,102$
c. $563 \times 86 = 48,418$

10. Which two numbers will have a product closest to 2000?

43	34	65	83	111
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11. Which of the following might represent the cube of 156?

a. 3,796,416	b. 4,251,528
c. 3,944,312	d. 3,581,577

12. Determine the quotients mentally.

a. $\frac{6000}{20}$	b. $\frac{4800}{60}$	c. $\frac{10,000}{200}$
d. $\frac{9000}{30}$	e. $\frac{3600}{90}$	f. $\frac{20,000}{400}$

13. Estimate as closely as you can.

a. $8 \overline{)5432}$	b. $7 \overline{)29,123}$	c. $20 \overline{)342,354}$
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14. By estimating, determine which of the following equation is wrong.

a. $75,296 \div 16 = 4706$
b. $34,272 \div 48 = 714$
c. $38,844 \div 498 = 780$

15. In each of the following, use the information from the computed problem to compute the other problems mentally.

a.	$587 \times 40 = ?$
$\begin{array}{r} 587 \\ \times 345 \\ \hline 2935 \\ 23480 \\ 176100 \\ \hline 202,515 \end{array}$	$587 \times 45 = ?$
	$587 \times 305 = ?$
	$87 \times 45 = ?$
	$870 \times 45 = ?$
	$587 \times 300 = ?$

b.

$$\begin{array}{r}
 742 \\
 36 \overline{)26,712} \\
 \underline{25,200} \\
 1510 \\
 \underline{1440} \\
 72 \\
 \underline{72} \\
 0
 \end{array}$$

$$\begin{aligned}
 36 \times 400 &= ? \\
 742 \times 36 &= ? \\
 25,200 \div 360 &= ? \\
 742 \times 30 &= ? \\
 26,712 - 1510 &= ?
 \end{aligned}$$

In Exercises 16–22, determine the answer mentally.

16. Peter bought a house for \$116,000 and sold it for \$145,000. How much profit did he make?
17. I left my house at 3:34 P.M. and arrived at 6:15 P.M. How long did the trip take me?
18. Washington College has 4132 full-time students, whereas Lincoln College has 2824 students. How much larger is Washington than Lincoln?
19. A concert hall has 36 seats in a row, and there are 25 rows. How many people can be seated?
20. A fund-raiser has a goal of \$55,000. So far, \$34,854 has been collected. How much more is needed?
21. A camping show rents booth space for \$1.50 per square foot. How much would it cost to rent a 10 foot by 15 foot space?
22. I went shopping for socks. Assuming that the two brands were of comparable quality, which is the better deal: 9 pairs for \$6.99 or 4 pairs for \$3.49?

In Exercises 23–42, do each of the following. First, determine a rough estimate in 5 to 10 seconds. Then determine your best estimate. “Best estimate” means no pencil and paper and less than 30 seconds or so. Otherwise, we defeat the purpose of estimating! Then determine the exact answer.

23. Several years ago, the flat roof over my kitchen was leaking. My friend at the hardware store told me of a new product that would fix my roof. Each can of roof sealant would cover 100 square feet and cost \$26. My roof was 23 feet 8 inches long and 15 feet 6 inches wide. How many cans of roof sealant did I need to buy?
24. Mindy rode her bike 3 times last week. At the beginning of the week, the odometer read 302.4, and at the end of the week it read 315.2. At this rate, how many miles will she ride this year?
25. Mr. and Mrs. Smith will pay \$8400 this year for their daughter’s college tuition. Approximately how much is this per month?
26. Jackie Adams bought a piece of property for \$28,543 and sold it for \$73,463. Approximately how much profit did she make?
27. a. It is 450 miles to home. How long will it take at 55 mph?

b. It is 450 miles to home. How long will it take at 65 mph?

28. A traveling saleswoman is adding up her miles this week. How many miles did she travel?

$$\begin{array}{r}
 165 \quad 345 \\
 78 \quad 142 \\
 57
 \end{array}$$

29. Sandy wants to get an idea of the cost of lumber for a project. She has determined that she needs a total of about 450 feet of boards, and the cost is 23¢ per foot. Approximately how much will they cost?
30. Two college students are riding across the United States on bicycles. They are averaging 112 miles per day.
 - a. If it is 2987 miles from San Francisco to New York, will they complete the trip in 1 month?
 - b. Approximately how far would they travel in 30 days if they averaged 167 miles per day?
31. In driving to Peterborough, it took me 27 minutes to go 18 miles. What was my average speed?
32. When someone turns 21, how many days old is that person?
33. A soft drink manufacturer produces 3240 cans in an 8-hour day. Cans are packaged 24 to a case. How many cases are produced each week? each month?
34. We bought a new van in June 1990. In December 1991, we had logged 18,345 miles. Our plan was to keep the van until 100,000 miles. At the rate at which we were putting miles on the van, approximately when would the van hit 100,000 miles?
35. Julie is working out on a computerized ski machine. She starts at 12:05 and will exercise for 30 minutes, and her goal is to burn 400 calories. At 12:17, the machine says that she has burned 148 calories. At this rate, will she reach her goal?
36. It takes the average person 6 seconds to read this sentence. In that time, 24 people will be born on the earth. At this rate, how many people are born each year? How long will it take to equal the number of people in your state?
37. A student lives in Hanover, which is 62 miles from her school. How many miles per semester does she travel if she drives to school twice a week? How much does she spend for gas? Assume a 15-week semester.
38. Fund-raisers have a goal of raising \$55,000 in 1 month. After 5 days, the amount that has been raised is \$9345. Are they on track?
39. The Alvarez family has decided that camping will be more fun if they buy the following items. Approximately how much will the family spend?

Lantern	\$35.95
New stove	\$59.99

- | | |
|---|--------------|
| Four new air mattresses at | \$14.95 each |
| A screen house to put over the picnic table | \$57.50 |
| A new set of camping pots and pans | \$24.95 |
40. If a dripping faucet wastes water at the rate of 75 ounces per day, how many gallons would this be per year?
 41. Take your pulse for 15 seconds. On the basis of this measurement, how many times would you say it has beaten since you got up this morning? This week? Since you were born?
 42. Given the increasingly fast advances in technology, it is not far-fetched to think that someday, every person in the United States could have an Internet account. Imagine that 275 million Americans each had an Internet account and that an Internet directory was available. If we printed that directory, how many pages would it contain?
 43. How many people are listed in the phone book in the town or city in which your college or university is located? First, describe and justify your plan for answering this question.
 44. Determine which number goes where: 7 90 20
An adult dolphin is about ____ feet long and has ____ teeth. Dolphins live about ____ years.
 45. A student of a teacher with whom I was working made up the following problem.
Determine which number goes where: 30 48 1961 20 20,000
The Berlin wall was set up in _____. The wall was approximately ____ miles long or ____ kilometers long. More than _____ people fled before the wall was erected. The wall was about ____ feet tall.
 46. Make up your own problem similar to the one above.
 47. In 1993, a newspaper reported that nearly 3,000,000 crimes occur each year in U.S. schools. What do you think was meant by crimes? Do you believe this figure? Do you think it is too small? too big? Why?
 48. A newspaper story said that 7 percent of all Americans eat at McDonald's each day. Is this true or another case of journalistic exaggeration? Write your report. One fact: There are approximately 9000 McDonald's restaurants.
 49. How many Cheerios are in a box? First, define the problem better. Devise a strategy to estimate the number; then devise another strategy. Are your results close? Which strategy do you think is more accurate?
 50. What base 10 concepts and/or properties are we making use of when we employ the leading digit method of estimation?
 51. An article in the January 1995 issue of *Teaching Children Mathematics* (p. 269) reported the following statistic: Every 57 minutes an under-age drinker is involved in a traffic fatality.
 - a. Estimate the number in 1 year.
 - b. What additional information would you like to have to clarify the question?
 52. Find an article from a newspaper or magazine that demands from the reader knowledge of numeration, basic operations, or estimation. Describe the mathematical knowledge you used in analyzing the article.
 53. Develop a question for which you will need to gather data. For example, how much water do you use when taking a shower? How much time do you spend studying? What percent of the students on your campus smoke?
 - a. Write down the question.
 - b. Write down the numbers that you will need to find or estimate in order to answer the question.
 - c. Write down any assumptions you are making in order to determine the estimate.
 - d. Find or estimate the numbers, do the computations, and give the answer to your question. Show your work.
 54. A computer company has 3765 copies of a hot-selling program and wants to ship the same number to each of its 18 distributors. Approximately how many programs does each distributor get? Francie estimated this way: She rounded 3765 to 4000 and 18 to 20 and then divided 4000 by 20 in her head; her estimate was 200. However, Nadine says that this way of estimating isn't good because, if you are going to round, you should round one number up and one number down. Nadine rounded 18 up to 20 and 3765 down to 3600, and so her estimate was $3600 \div 20 = 180$. Francie responds that although the latter method works for multiplication (round one number up and one number down), it doesn't work for division. With whom do you agree? Why?

CHAPTER SUMMARY

1. Many students have said that really understanding base 10 and the four operations, was, for them, the beginning of a new attitude toward mathematics. We will continue to examine new and important mathematical ideas throughout this book, but the foundation for much of elementary mathematics has now been laid.
2. Each operation has multiple meanings.

3. Many algorithms have been developed to enable us to compute more efficiently.
4. The standard algorithm for each operation does not connect equally well to each meaning of the operation.
5. Being able to make sense of algorithms requires:
 - The ability to apply base 10 and place value concepts
 - The ability to compose and decompose the numbers (for example, to use expanded form)
6. Patterns enable us to understand the operations more deeply.
7. In many real-life problems, the answer depends on knowing how to interpret one's computation.
8. Being able to perform mental math and to estimate requires:
 - The ability to apply base 10 and place value concepts
 - The ability to compose and decompose the numbers (for example, to use expanded form)
 - The ability to apply properties of the operations, especially the commutative, associative, and distributive properties
9. Numbers in real-life settings are sometimes exact, sometimes rounded, and sometimes estimates.
10. In real-life problem-solving, one needs to know when to find an exact answer and when to find an estimate.
11. Real-life problem solvers need to know whether their estimates are reasonable.
12. People may use rounded numbers rather than exact numbers for a variety of reasons.

Basic Concepts

Section 3.1 Understanding Addition and Subtraction

Addition terminology:

sum 116 addends 116
addition 116

Addition contexts:

discrete 116
continuous 116

Addition models:

pictorial 117 number line 118
tables 119

Addition properties:

identity 119 commutativity 119
associativity 120 closure 120

Algorithms for addition:

common 124

Subtraction contexts:

take away 126 comparison 126
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Subtraction terminology:

subtraction 126 minuend 127
subtrahend 127 difference 127

Subtraction models:

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Algorithms for subtraction:

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Other terminology:

matrix 120 algorithm 122
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Section 3.2 Understanding Multiplication and Division

Multiplication contexts and models:

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Multiplication terminology:

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zero 139 identity 139
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Division contexts and models:

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Division terminology:

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Section 3.3 Mental Math and Estimation

Strategies for mental computation and estimation:

(Note that some of these strategies can be used both for mental computation and for estimation, some strategies are applicable to more than one operation, and some are alternative ways to solve the same problem.)

rounding 165	compensation 168
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Number Theory

- ◆ 4.1 Divisibility and Related Concepts
- ◆ 4.2 Prime and Composite Numbers
- ◆ 4.3 Greatest Common Factor and Least Common Multiple



HISTORY

The Pythagoreans were a brotherhood who believed that the whole universe could be explained using only natural numbers. They were not simply mathematicians, though; they believed that all areas of life could be explained in terms of numbers, including science, music, and philosophy. For example, if the ratio of the lengths of two otherwise identical strings is 2:1, the notes are exactly one octave apart.

The Pythagoreans were a secretive society. Because none of their accomplishments were attributed to individual members, we cannot say who discovered what; thus, all of their discoveries are simply attributed to Pythagoras. We also know that at least 28 women were classified as Pythagoreans and that Pythagoras' wife, Theono, was a teacher in his school.¹

Somewhere around the fourth century B.C. in Greece, a profound shift in the focus of mathematics took place—a shift from simply wanting to know *how* to wanting to know *why*. Until that time, numbers were primarily used for practical purposes. Probably the most recognizable example of this changed focus is the Pythagorean theorem: $a^2 + b^2 = c^2$. Long before the Pythagoreans, other people had known of this relationship. However, the Pythagoreans and other early Greek mathematicians were not content to know that this relationship held for certain right triangles; they proved that it *must* hold for *any* right triangle. The field of number theory was born during this time.

Number theory is a branch of mathematics that focuses primarily on natural numbers and their relationships.² “Number theory offers many rich opportunities for explorations that are interesting, enjoyable, and useful. These explorations have payoffs in problem-solving, in understanding and developing other mathematical concepts, in illustrating the beauty of mathematics, and [in] understanding the human aspects of the historical development of numbers” (*Curriculum Standards*, p. 91).

A curious reader might ask why future elementary teachers should study number theory. There are several reasons. Many number theory concepts, including factors and multiples, prime and composite numbers, and divisibility relationships, are firmly embedded in the elementary school curriculum. If elementary students understand these concepts, they can more easily compose and decompose mathematical expressions. This, in turn, makes the transition to high school mathematics less difficult.

Many students at all levels find numbers and number patterns fascinating. Being able to see patterns and to make and test generalizations is an important

¹ Cited in Lancelot Hogben, *Mathematics for the Millions* (New York: W. W. Norton & Co., 1937), p. 190.

² Technically, number theory deals with the larger set of integers; recall our discussion in Chapter 2 of various sets of numbers: natural numbers (N), whole numbers (W), and integers (I). Pedagogically, however, it works better to focus our investigations on the set of natural numbers, which is also called the set of counting numbers.

part of mathematical thinking, making number theory fertile ground for mathematically rich investigations. For thousands of years, people throughout the world have been fascinated by patterns in numbers and relationships between different sets of numbers. In this chapter, we will explore some of the more interesting and famous ones. Remember Carl Gauss, from Chapter 1, who found a way to add the first 100 natural numbers so quickly? As an adult, he once said, “Mathematics is the queen of the sciences, and arithmetic is the queen of mathematics.” By arithmetic he meant number theory.

Number theory connects with other mathematical concepts and fields. Some of the topics that number theory relates to include fractions (least common multiple and greatest common factor), probability (patterns in counting different combinations), decimals (patterns in repeating as opposed to nonrepeating decimals), algebra (modular arithmetic), and geometry (string art, star patterns).

SECTION

4.1

DIVISIBILITY AND RELATED CONCEPTS

WHAT DO YOU THINK?

- What do the terms *odd* and *even* mean?
- How can you tell whether one number is divisible by another without dividing?
- What connections do you see between divisibility and decomposition?

In this section, we will investigate the basic concepts that will help you to understand the various structures of natural numbers. In Chapter 3, we examined two different ways in which numbers can be decomposed: additive and multiplicative. For example, 12 can be decomposed into $8 + 4$, and it can be decomposed into $4 \cdot 3$. Depending on circumstances, either decomposition might be relevant. In this chapter, we will explore multiplicative decompositions of natural numbers. As you will discover, being able to decompose numbers and knowing which decomposition is relevant is a theme that recurs throughout elementary mathematics.

INVESTIGATION 4.1 Interesting Dates

We will begin our investigations with a playful question. Many mathematics teachers and mathematicians would smile when writing down December 8, 1996. Do you see why? Think before reading on. . . 

If we were to write this date in shorthand, we would write 12/8/96.

- A. Describe the relationship among these three numbers in words.
- B. Determine how many instances of this pattern occurred in the years 1900–1919.

DISCUSSION

 One way of describing the relationship among the three numbers is to say that the third number is the product of the first two: 96 is the product of 12 and 8.

There are several mathematical concepts and terms that come out of this relationship: *factor*, *multiple*, *divisible*, *divisor*, and *divides*. These terms are all related. Therefore, we will define one term formally and state how the other terms are related to it.

If a and b are two whole numbers ($b \neq 0$) and there is a third natural number c such that $a \cdot c = b$, then we say that a **divides** b , and we write $a \mid b$.

When one number does not divide another number, we write $a \nmid b$.

- If $a \mid b$, then a is a **factor** of b .
- If $a \mid b$, then b is a **multiple** of a .
- If $a \mid b$, then b is **divisible** by a .
- If $a \mid b$, then a is a **divisor** of b .

B Let us now explore the second question. There are no such dates in 1900. In 1901, there is only one such date: 1/1/01. Before you look at the table below, make your own table and then read on. . . . 

STRATEGY 1: Make a table

Table 4.1 is not the only possible table, nor is it necessarily the best table.

TABLE 4.1

1/1/01					
1/2/02	2/1/02				
1/3/03	3/1/03				
1/4/04	2/2/04	4/1/04			
1/5/05	5/1/05				
1/6/06	2/3/06	3/2/06	6/1/06		
1/7/07	7/1/07				
1/8/08	2/4/08	4/2/08	8/1/08		
1/9/09	3/3/09	9/1/09			
1/10/10	2/5/10	5/2/10	10/1/10		
1/11/11	11/1/11				
1/12/12	2/6/12	3/4/12	4/3/12	6/2/12	12/1/12
1/13/13					
1/14/14	2/7/14	7/2/14			
1/15/15	3/5/15	5/3/15			
1/16/16	2/8/16	4/4/16	8/2/16		
1/17/17					
1/18/18	2/9/18	3/6/18	6/3/18	9/2/18	
1/19/19					

In one sense, we are done—we have found all 53 possible dates. However, there is still a lot of interesting mathematics to be gleaned from looking for and examining patterns and relationships. Take a few minutes to examine the table above. What patterns and relationships do you observe in the table? 

STRATEGY 2: Make a table with a different organization

Before we discuss the patterns and relationships in Table 4.1, let us take a look at another representation of our data. As you may recall, one of the themes of this book is that different representations of information and data often enable us to see patterns that we did not see before, in the same way that a sculpture looks different as we walk around it. Look at Table 4.2. It presents the same 53 dates as Table 4.1 but is organized differently. Note any new insights, patterns, or observations before reading on. . . . 

TABLE 4.2

1/1/01					
1/2/02	2/1/02				
1/3/03	3/1/03				
1/4/04	4/1/04				2/2/04
1/5/05	5/1/05				
1/6/06	6/1/06	2/3/06	3/2/06		
1/7/07	7/1/07				
1/8/08	8/1/08	2/4/08	4/2/08		
1/9/09	9/1/09				3/3/09
1/10/10	10/1/10	2/5/10	5/2/10		
1/11/11	11/1/11				
1/12/12	12/1/12	2/6/12	6/2/12	3/4/12	4/3/12
1/13/13					
1/14/14		2/7/14	7/2/14		
1/15/15				3/5/15	5/3/15
1/16/16		2/8/16	8/2/16		4/4/16
1/17/17					
1/18/18		2/9/18	9/2/18	3/6/18	6/3/18
1/19/19					

One pattern is that many dates have a twin. For example, the dates in the first and second columns, the dates in the third and fourth columns, and the dates in the fifth and sixth columns are twins. Someone who sees this connection early can finish the problem more quickly—sort of a “2 for 1” deal. Mathematically, we can talk about symmetry in our set.

We can also see that some dates do not have twins. For example, 1/7/07 has a twin, but 1/13/13 doesn’t. Why not? 

If we stop for a moment, we realize that 13/1/13 is not a valid date.

Some valid dates have no twins: 2/2/04, 3/3/09, 4/4/16. In this case, the numbers 4, 9, and 16 are square numbers.

Odd and Even Numbers

One of the first ways in which children encounter number theory is with **odd numbers** and **even numbers**. Let me introduce this discussion with a story. One morning my five-year-old son, Josh, asked me at the breakfast table if 44 was an odd or an even number; at that time, my 45th birthday was a few days away. I asked him what he thought, and he said he thought it was even. I asked him what he thought an even number was. Before you read his response, think about this question yourself. Then read on. . . . 

Josh’s response was that even numbers were fair. I asked him what he meant by fair. His response was, “Well, it’s fair if two teams can get the same amount.”

Take a few moments to write your own definition of odd and even. Given our explorations in Alphabitia, can you make a definition that will work for that system too?

Below are several different ways in which even numbers have been defined. Note that the notion of *even* is closely tied to the notion of decomposition. That is, if a number can be decomposed in the following way, then it is even.

MATHEMATICS

The concept of evenness can be constructed in additive and multiplicative terms. For example, evenness can be thought of in additive terms: x is an even number if $a + a = x$; that is, it is the sum of two equal numbers. Evenness can also be thought of in multiplicative terms: x is an even number if $2a = x$ —that is, if x is the double of some number.

A number is an even number if:

- It has 2 as a factor.
- It is divisible by 2.
- Its ones digit ends in 2, 4, 6, 8, or 0; (this one is not true for all bases; do you see why?).
- It can be represented as the sum of two equal numbers.

A natural number x is even if there exists another natural number a such that $x = 2a$.

Take a minute to write the analogous ways in which odd numbers can be defined. 

Please don't skip over this. You will use this thinking in the next investigation.



CHILDREN

It is important to emphasize that most concepts in the elementary school curriculum can be represented concretely and that most, but not all, students need to experience these concepts concretely in order to understand them deeply. This is true with the concepts of factor, multiple, and divisible.

Cuisenaire rods are commonly used manipulatives in elementary schools that work nicely with many elementary mathematics concepts. They come in ten different lengths, each length being a different color (see Figure 4.1). The length of each of the rods is a multiple of the length of the shortest rod.

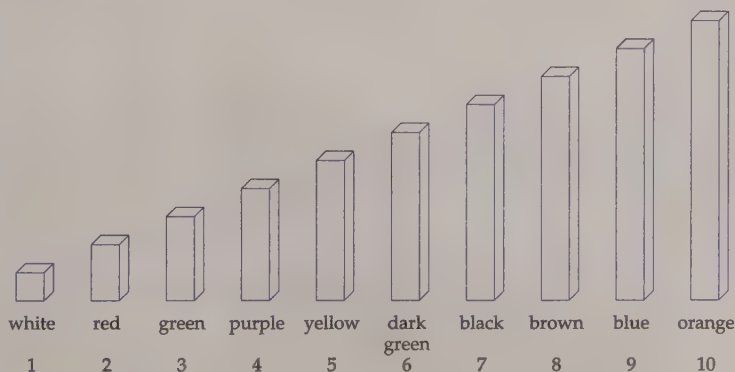


FIGURE 4.1

Children will spontaneously build “trains” with Cuisenaire rods. To see whether one number is a factor of another number, we can begin by making a large train using Cuisenaire rods of only one color. For example,

consider a 20 train (two orange rods). A child can readily test whether 4 is a factor of 20 by seeing whether the large train can be made using only purple (4) rods (see Figure 4.2).

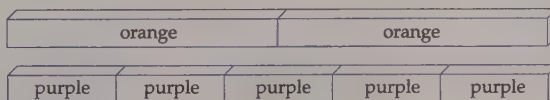


FIGURE 4.2

INVESTIGATION 4.2 Patterns in Odd and Even Numbers

In this investigation, we will examine and discover some of the many patterns that emerge when odd and even numbers are combined. There are several reasons for investigating odd and even numbers. One is that many people find it fun; I have seen the excitement on young children's faces as they discover these patterns. Another reason is that in order to see and then explain these patterns, one has to look more closely at the concepts; thus, we come out of the investigation with a deeper understanding of odd and even numbers and other mathematical ideas and tools. Finally, this and many other investigations in this book focus on going from noticing a pattern that seems to be true to being able to prove it is true everywhere. Once again, this ability to prove the "*why* behind the *what*," a habit of mind that we first see in the early Greeks more than 2000 years ago, represents one of the great advances in human thinking.

Begin the investigation by thinking about the following questions and noting your responses before reading on. . . . 

- A. What do you notice about adding two odd numbers? two even numbers? an odd and an even number?
- B. Can you explain the pattern you see in the sum of two odd numbers?

DISCUSSION

A. The patterns for adding odd and even numbers can most succinctly be represented as

$$\text{odd} + \text{odd} = \text{even}$$

$$\text{even} + \text{even} = \text{even}$$

$$\text{odd} + \text{even} = \text{odd}$$

B. Now that we have seen *what* the pattern is, how might we explain *why* it holds for all odd and even numbers? What tools do you have that might enable you to show *why* this is true?

STRATEGY 1: Make a drawing

Let us begin by examining the question from a geometrical perspective. See Figure 4.3, which represents any odd number. Why is this figure a valid representation of any odd number?

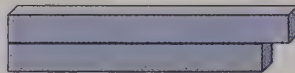


FIGURE 4.3

Think of two Cuisenaire rods, one of which is one unit longer than the other. Any odd number can be represented in this manner. Can you connect any of your definitions to this diagram? How?

If we think of *even* as meaning able to be separated into two groups that have the same amount, then *odd* must mean that we can't do that; in other words, one of the piles will be one unit longer. What if we combine two different odd numbers? Draw that diagram. What do you see? At this point, make another attempt to explain why the sum of two odd numbers must be an even

number. You may find that you can refine or extend your ideas from your first explanations, or you may find that the picture and explanation above are different from what you had written.

When we build a concrete model to show the combination of two odd numbers, the first odd number has one unit left over and the second odd number has one unit left over, as in Figure 4.4. When we combine the two numbers, the two units become a pair.

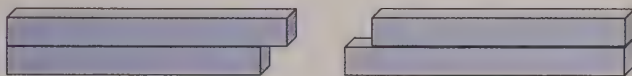


FIGURE 4.4

There are two ways in which this model connects with our idea of *even*. Some students might say that now there are no units left over, and thus we have an even number. Other students might refer to Figure 4.4 and, using language from our work with base 10 blocks, say that when we combine the two numbers, we have two sticks of equal length; thus, it is an even number.

STRATEGY 2: Use algebra

Let us now examine an algebraic explanation for why the sum of two odd numbers must be an even number. Because one of the characteristics of *all* even numbers is that they are divisible by 2, we can say that $2n$ represents *any* even number as long as n represents a natural number. Because one of the characteristics of *all* odd numbers is that they are not divisible by 2, we can say that $2n + 1$ represents any odd number. Why is this? Try to use these ideas to show with symbols why the sum of any two odd numbers is an even number. Then read on. . . 

Statement	Reason
$(2n + 1) + (2m + 1)$	This represents the sum of <i>any</i> two odd numbers, where m and n represent <i>any</i> natural numbers
$= 2n + 2m + 2$	We can change the original expression to this expression because of the associative and commutative properties.
$= 2(n + m + 1)$	We can change the previous expression to this expression because of the distributive property.

However, this form means that this number, $2(n + m + 1)$, has to be an even number. Why is this?

See whether you can prove the other two statements (even + even = even and odd + even = odd) in a similar fashion.

In Chapter 2, we stated that many relationships between two sets in mathematics are functional relationships. Is this relationship we have just examined (that is, whether the sum of two numbers will be odd or even) a functional relationship? Why or why not? Think and read on. . . 

Function language If we cast this notion of the sum of two numbers being odd or even in function language, we can see that the input set consists of all ordered pairs of natural numbers; that is,

$$\text{Input set} = \{(a, b) | a, b \in N\}$$



HISTORY

In a fascinating book with the not so fascinating title of *The Penguin Dictionary of Curious and Interesting Numbers*, David Wells describes the values that many people associated with the first several natural numbers. The Pythagoreans considered 2 and all even numbers to be female, and they considered 3 and all odd numbers to be male. Female numbers were considered to be human and earthy, whereas male numbers were considered to be divine and heavenly. Not surprisingly, they associated the number 5 with marriage, because it is the unity of the first male and female numbers. Some readers may be angered at the Greeks' characterization of male and female. I think it is important for people to realize that mathematics is more than just numbers and that mathematics has helped to shape our cultural values in many ways.

Similarly, the output set consists of two elements, even and odd; that is,

Output set = {even, odd}

We can illustrate this with a mapping diagram (Figure 4.5):

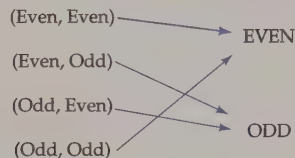


FIGURE 4.5

We see that every element of the input set is connected to exactly one element of the output set; for example, even + even always produces an even number. Therefore, this relationship is a functional one.

INVESTIGATION 4.3 Understanding Divisibility Relationships

There are a host of theorems that concern divisibility. We will examine two here, partly because we will need them later in the section and partly because this investigation provides practice in the kind of mathematical thinking originally developed by the early Greeks—that is, advancing from demonstrating that a statement is true in a particular case to proving that the statement is true for all numbers—what we call proving the general case.

- A. We know that $6 \mid 42$ and $6 \mid 72$. Does it necessarily follow that 6 divides their sum—that is, does 6 divide $(42 + 72)$? Before doing any computation, what do you think? 

DISCUSSION

In this particular case, it is true; 6 does divide $42 + 72$, because $6 \cdot 19 = 114$.

- B. In fact, pick any three natural numbers a , b , and c for which $a \mid b$ and $a \mid c$. Then $a \mid (b + c)$. Can you prove this?

BEYOND THE CLASSROOM

The question in Investigation 4.3 may seem to be silly to some people, but science is full of examples of products or medicines or mathematical principles that worked in many test cases but were later found not to work in all cases. In some instances, it was a matter of "Aw, shucks, we were wrong," but in other cases, it was a matter of lives being lost. Therefore, this ability to be able to prove that a hypothesis either is true in only some cases or is true in all cases is an important ability in many areas of life.

Let me restate this general question in English and in mathematical notation:

English: If a number divides two numbers, will it necessarily divide their sum?

Notation: If $a \mid b$ and $a \mid c$, is it always true that $a \mid (b + c)$?

How might we prove this? Work on this question and then read on. . . 

DISCUSSION

One proof uses the definition of divisibility and looks like this:

If $a \mid b$, then there is a natural number x such that $ax = b$.

Similarly, if $a \mid c$, then there is a natural number y such that $ay = c$.

The proof consists of transforming $b + c$ to show that it must be divisible by a :

$$\begin{aligned} b + c &= ax + ay && \text{Here we are substituting } ax \text{ for } b \text{ and } ay \text{ for } c. \\ &= a(x + y) && \text{This is true because of the distributive property.} \end{aligned}$$

We now have a valid proof. Do you see why? Think and then read on. . . . 

We have proved the general case, because if $b + c$ can be expressed as the product of a and some natural number, then by definition $(b + c)$ is divisible by a . Therefore, we have proved that if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

INVESTIGATION 4.4 Determining the Truth of an Inverse Statement

LEARNING

Because this book does not emphasize the more formal aspects of mathematics, we will generally be concerned with justifying the if-then statements that we encounter, rather than formally proving them. In the *NCTM Principles and Standards*, we find the term *justify* over and over again. If students in elementary and middle grades are used to justifying their conclusions, then the step to logical proofs is not a huge one. However, we will occasionally present proofs because this more formal aspect of mathematics becomes increasingly important in secondary and college mathematics. It is my assumption that helping children develop the intuitive and inductive parts of the mind is the best foundation for the more formal and deductive work that comes later. All three kinds of thinking (deductive, inductive, and intuitive) are crucial.

Suppose we turn things around and consider the inverse statement: What if a number a divides neither of two larger numbers? Can we say that it will never divide the sum of those two numbers? Think and then read on. . . . 

This question can be stated in English and in notation as follows:

English: If a does not divide b and a does not divide c , then will a divide their sum?

Notation: If $a \nmid b$ and $a \nmid c$, then $a \nmid (b + c)$?

DISCUSSION

Although this hypothesis seems reasonable, we can show that it is invalid by using a **counterexample**, an example that proves a hypothesis to be false.

For example, consider $a = 3$, $b = 7$, and $c = 2$.

Although $3 \nmid 7$ and $3 \nmid 2$, $3 \mid (7 + 2)$.

Thus, it is not true that if $a \nmid b$ and $a \nmid c$, then $a \nmid (b + c)$.

Note: A mathematical statement is considered to be true only if it is true 100 percent of the time. If there is even one exception (a counterexample), then the statement is considered to be mathematically false.

Divisibility Rules

Before the widespread use of calculators, knowing divisibility rules was quite useful. For example, simplifying $42/54$ is much easier for a student who immediately sees that both numbers are divisible by 6. In the context of the NCTM standards, the usefulness of the concept of divisibility is not so much in your knowing the rules as it is in the opportunity for investigations with this concept to deepen your understanding of whole-number relationships and to develop your problem-solving toolbox.

Think back to Investigation 4.1. Are there any dates in the year 1996 that fit that pattern? Work on this and then read on. . . . 

There are several ways to answer this question. One is simply random guess-check-revise. Another way is to be systematic: $1 \cdot ? = 96$, $2 \cdot ? = 96$, $3 \cdot ? = 96$, and so forth. Another way is first to find all the factors of 96. We will explore the second strategy now and the third strategy in the next section.

It turns out that there are four dates in 1996 that fit the pattern: 4/24/96, 6/16/96, 8/12/96, and 12/8/96. If we use a systematic approach, we are essentially asking, in each case, does this number divide 96, and if so, is the result a valid date? For example, does 3 divide 96? You can answer this by dividing. However, if you know the divisibility rule for 3, you immediately know that 3 does indeed divide 96.

Before we consider some of the common divisibility rules, let me ask you a question: What rules do you remember from grade school? If you don't remember any of them, take a few minutes to see whether you can *discover* any of the rules. In other words, just by looking at a large number, such as 96 or 142, can you tell whether it is divisible by 2 without dividing by 2, whether it is divisible by 3 without dividing by 3, and so on? Play around for a while and see how many divisibility rules you can remember and/or construct before reading on. . . . 

Because one of the themes of this book is the connectedness of mathematics, you may not be surprised to hear that many of the divisibility rules are related to one another. We will discuss them from this perspective. Several of the divisibility rules are quite simple. Let us examine them first.

Divisibility by 2 When will a number be divisible by 2? Another way of asking this question is: If you think of (the set of) all numbers that are divisible by 2, what do they have in common?

What they have in common is that they are all even numbers. Thus, the divisibility rule for 2 is:

A number is divisible by 2 iff it is an even number.

An equivalent statement is:

A number is divisible by 2 iff its ones digit is a 0, 2, 4, 6, or 8.

Note that this statement holds for base 10. Divisibility rules in other bases will be left as exercises.

Divisibility by 5 This rule is simple in base 10:

A number is divisible by 5 iff its ones digit is a 0 or a 5.

Do you see why this rule is so easy in base 10? Think and then read on. . . . 

Think back to base 10 manipulatives. When we count by 5s, we are counting by half longs.

Divisibility by 10 A number is divisible by 10 iff the ones digit is 0.

These three divisibility rules are generally the easiest to remember. Do you think it is a coincidence that 2 and 5 both divide 10?

Divisibility by 3 Make up several numbers and determine whether they are divisible by 3. Look at the numbers that are divisible by 3. Can you see any patterns that would enable you to determine that a number is or is not divisible by 3 without having to divide? If you don't, consider this statement: "Divisible by 3" is analogous to an exclusive club. That is, every number that is divisible by 3 has some characteristic that every number not divisible by 3 does not have. Look again at your two sets of numbers (those that are divisible by 3 and those that are not); make up more numbers and place them



CHILDREN

Most elementary teachers have the students do skip counting before multiplication. The set of multiples of 5 is the first such set that most children learn because it is the simplest set of multiples in base 10.

in the appropriate set, for this is a great chance to develop the part of the problem-solving toolbox dealing with looking for patterns. Then read on. . . .

INVESTIGATION 4.5 Understanding Why the Divisibility Rule for 3 Works

Formally, we say that

A number is divisible by 3 iff the sum of the digits of the number is divisible by 3.

For example, $3 \mid 567$ because $3 \mid (5 + 6 + 7)$. Let us use the divisibility relationships that we explored earlier to explain *why* the divisibility rule for 3 works. Recall that one of the main purposes of this course is to learn not just the whats of mathematics but also the whys.

Pick a few numbers that are divisible by 3 and see whether you can tell what the sum of the digits has to do with divisibility by 3. For example, 243 is divisible by 3. Why? Explore this for a while and then read on. . . .

DISCUSSION

Begin with manipulatives We will first work through a concrete representation and then move to the more general case. If we represent 243 with base 10 manipulatives, we have Figure 4.6.

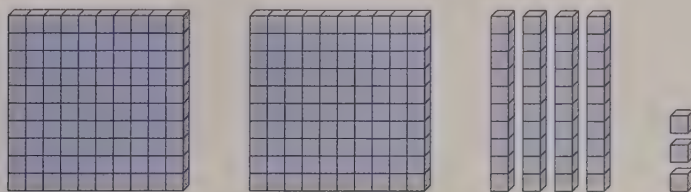


FIGURE 4.6

In one sense, we can look at this representation as three distinct groups: flats, longs, and units. The key to understanding why this rule works lies in the fact that if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$. We will use an unproved extension of that theorem: If $a \mid b$, $a \mid c$, and $a \mid d$, then $a \mid (b + c + d)$ for all natural numbers a , b , c , and d . Do you see what this theorem has to do with showing why 243 is divisible by 3? Think and then read on. . . .

If we can break down our number into various components so that each component is divisible by 3, then the natural number will be divisible by 3. What can we do to the flats and the longs to make them divisible by 3? Think and then read on. . . .

If we cut one unit from each flat and each long, we still have the same amount—that is, 243 (see Figure 4.7). But now the 99 blocks and the 9 blocks are divisible by 3.

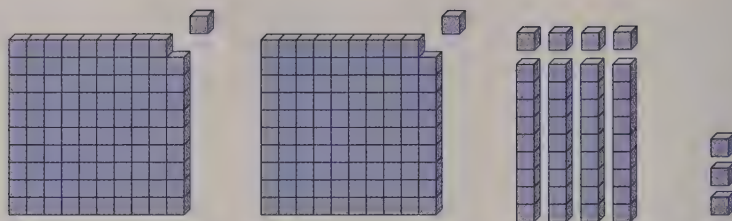


FIGURE 4.7

We simply move the “extra” units created in cutting the flats and longs and put them with the original 3 units (see Figure 4.8). Now we have 9 ones, and 9 is divisible by 3.

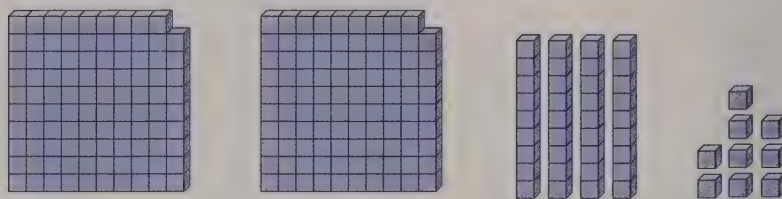


FIGURE 4.8

Now use expanded notation We can use expanded notation now to show what we have done and to understand better why the divisibility rule for 3 works.

Statement	Reason
$243 = 200 + 40 + 3$	We have rewritten the amount using expanded notation.
$= (100 + 100) + (10 + 10 + 10 + 10) + 3$	We have substituted equivalent amounts (for example, $100 + 100$ for 200).
$= (99 + 1 + 99 + 1) + (9 + 1 + 9 + 1 + 9 + 1 + 9 + 1) + 3$	100 and 10 are not divisible by 3, but if we take 1 away from each 100 and each 10, what remains is divisible by 3.
$= 2(99) + 4(9) + (2 + 4 + 3)$	We have used the commutative and associative properties to rearrange what we had.

Finally, use the theorem We can now use the theorem: If $a \mid b$, $a \mid c$, and $a \mid d$, then $a \mid (b + c + d)$. That is, if we can show that 3 divides each of the three expressions above [$2(99)$, $4(9)$, and $(2 + 4 + 3)$], then 3 must divide the sum 243; that is, 243 is divisible by 3! Let us examine each of the three expressions in turn.

We know that $3 \mid 2(99)$ because $3 \mid 99$. This knowledge comes from another (unproved) theorem that states that if $a \mid b$, then $a \mid bc$ for all natural numbers a , b , and c . This can be stated in English as follows: If a number divides another number, then it divides any multiple of that number.

Similarly, $3 \mid 4(9)$ because $3 \mid 9$.

Therefore, demonstrating that $3 \mid 243$ rests upon knowing that $3 \mid (2 + 4 + 3)$. However, $2 + 4 + 3$ is simply the sum of the digits of 243!

Generalizing the justification This does not constitute a proof of the divisibility rule for 3 but is, rather, an examination into the structure of the rule. You

may want to do the above kind of analysis with a few other numbers to better understand why it works. If we had represented 243 as abc , we would have a proof of the divisibility rule for 3 for any three-digit number. That is,

$$abc = 100a + 10b + c = 99a + a + 9b + b + c = 99a + 9b + (a + b + c)$$

Divisibility by 9 As you discovered when you worked with the multiplication table in Exploration 3.7, the 3 family and the 9 family are related. So too are their divisibility rules. Before reading the rule below, see whether you can guess what it is. You can approach this task in two different ways: Choose several numbers and multiply them by 9 or make up a bunch of numbers and divide them by 9. The first approach creates one set; the second approach creates two sets, divisible by 9 and not divisible by 9. Here is another chance to apply your problem-solving skills (looking for patterns) and content knowledge (looking to see whether you can adapt the divisibility rule for 3). In either case, see what you can come up with before reading on. . . . 

A number is divisible by 9 iff the sum of its digits is divisible by 9.

The justification of this rule is left as an exercise.

INVESTIGATION 4.6 Divisibility by 4 and 8

 Just as the divisibility rules for 3 and 9 are similar, so too are the divisibility rules for 2, 4, and 8.

- A. As before, choose several numbers and determine which are divisible by 4. Then find patterns that will lead you to be able to predict whether a number is divisible by 4 without dividing. Then read on. . . . 
- B. Choose several numbers and determine which are divisible by 8. Find patterns to help you predict whether a number is divisible by 8. Then read on. . . . 

DISCUSSION

A. The divisibility rule for 4 is not terribly obvious. Let me offer a hint before presenting the rule.

Consider the number 532. The test for divisibility by 4 centers on whether $4 \mid 32$. Similarly, with the number 123,456, the test for divisibility by 4 centers on whether $4 \mid 56$.

Now try to express this rule in English as though you were explaining it to someone on the phone. Then read on. . . . 

There are many valid ways to express the rule. One looks like this:

A number is divisible by 4 iff the number represented by the last two digits is divisible by 4.

By the last two digits, I mean the ones digit and the tens digit.

Before we examine why this rule works, you may want to try it with a few numbers.

The key to understanding how this rule works is to realize that no matter how many hundreds we have, *that* part of the number will always be divisible

by 4. This is true because $4 \mid 100$, and therefore any multiple of 100 will also be divisible by 4.

Proof for three-digit numbers:

Statement	Reason
$xyz = 100x + 10y + z$	Rewrite using expanded form.
$4 \mid 100x$	Because $4 \mid 100$, $4 \mid 100a$.
If $4 \mid (10y + z)$, then $4 \mid xyz$	Because of the theorem, if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

B. Before stating the rule for divisibility by 8, let me give you a hint and see whether you can figure it out. The rule for 8 is similar to the rule for 4. Think and then read on . . . 

To help you understand the rule rather than just memorizing it, consider the number 2328 and whether it is divisible by 8. This is another case in which expanded notation can be helpful:

$$2328 = 2000 + 300 + 20 + 8$$

Will any of the parts be divisible by 8? Think and read on. . . 

Because $8 \mid 1000$, any multiple of 1000 will be divisible by 8, so $8 \mid 2000$. However, 8 does not divide 100, so we need to look at the last *three* digits to determine divisibility by 8. That is, if $8 \mid 328$, then the original number, 2328, will be divisible by 8.

The divisibility rule for 8 uses language similar to the rule for 4:

A number is divisible by 8 iff the number represented by the last three digits is divisible by 8.

Some students have noted that although 8 does not divide 100, 8 does divide any even multiple of 100. In other words, $8 \mid 200$, $8 \mid 400$, and so forth. Thus, we can say that if the digit representing the hundreds place is even, we have to look at only the last two digits. This led me to believe that if the digit representing the hundreds place is odd, there might be a pattern that might enable us to determine divisibility by 8 by looking at only the last two digits. What do you think? This question will be left as an exercise.

Divisibility by 6 The divisibility rule for 6 has a different flavor from the other rules. How might you determine whether a large number was divisible by 6 without actually dividing? Think and then read on. . . 

A number n is divisible by 6 iff n is divisible by 2 and by 3.

This can also be stated as

$$6 \mid n \text{ iff } 2 \mid n \text{ and } 3 \mid n$$

Why do you think this rule must be true?

Let us now investigate how we might apply this divisibility rule to a larger number.

INVESTIGATION 4.7 Creating a Divisibility Rule for 12

What about a divisibility rule for 12? Can we use what we know, or do we just have to divide (longhand or with a calculator)? Think and then read on. . . . 

DISCUSSION

When I have asked my classes this, I generally get different hypotheses. Let us examine two of them. Some students say that if a number is divisible by 2 and by 6, then it must be divisible by 12. Other students say that if a number is divisible by 3 and by 4, then it must be divisible by 12. Is this one of those cases in which both answers are correct, or is only one correct? Think and then read on. . . . 

In this case, the first hypothesis is not always true. Can you see why? For example, $2 \mid 18$ and $6 \mid 18$ but $12 \nmid 18$. If you thought this first hypothesis was correct, can you now see why it isn't? If you knew that it wasn't correct, can you explain why? In either case, take a few minutes to think about how you would explain to someone why this hypothesis is not true. Then read on. . . . 

This hypothesis doesn't work because "divisible by 2" is a redundant condition. If a number is divisible by 6, then it *must* be divisible by 2. Thus, saying that a number is divisible by 2 and divisible by 6 is mathematically equivalent to saying simply that it is divisible by 6.

Revisiting Venn diagrams The preceding sentence makes great sense to some students and absolutely no sense to others. If you are in the latter set, the old adage that a picture is worth a thousand words may be relevant. Let us examine some Venn diagrams and see how they can contribute to our understanding of divisibility rules.

To say that if a number is divisible by 6, it must be divisible by 2 means that the set of numbers divisible by 6 is a subset of the set of numbers divisible by 2, as shown in the figure at the left in Figure 4.9. When we look at divisibility by 3 and by 4, however, we see that they are not disjoint sets (like the sets of even and odd numbers), but rather overlapping sets. Thus, the diagram at the right in Figure 4.9 illustrates the relationship between the numbers divisible by 3 and the numbers divisible by 4. The intersection of these two sets is the set of numbers divisible by 12.

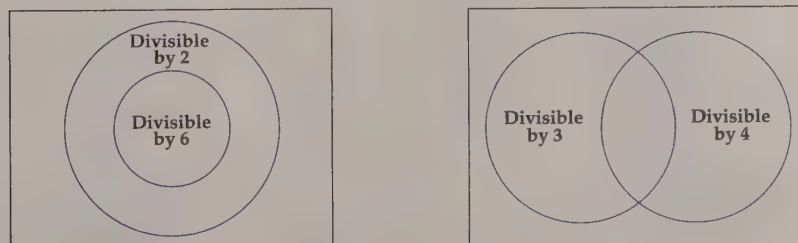


FIGURE 4.9

We can say that a number is divisible by 12 iff it is divisible by 3 and by 4.

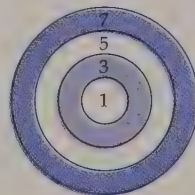
Summary

The ideas of divisibility, divides, factor, and multiple are highly interconnected. As we examined patterns in dates and patterns in even and odd numbers, we saw how the concepts of factor, multiple, and divisibility enabled us to understand better the problems we were investigating. Which term we use often depends on our perspective. For example, we might say that 6 divides 24, that 24 is divisible by 6, that 6 is a factor of 24, that 24 is a multiple of 6, or that 6 is a divisor of 24. This notion of divisibility is useful because the relationship between two numbers is often important in explaining patterns and making predictions. We also used divisibility ideas to develop and explain the various divisibility rules. In Exploration 4.4, for example, when the number of points in the stars is divisible by the number denoting how to connect the dots, a polygon is formed. We also found that in order to understand why divisibility rules work, it was helpful to understand some basic principles and terminology of deductive reasoning: the inverse, converse, and contrapositive of a statement.

The notion of divisibility is highly connected to another important idea that you have seen several times in this book already: decomposition. Just as an understanding of computation algorithms in Chapter 3 depended on decomposing the numbers, so too does an understanding of divisibility require that one be comfortable with taking numbers apart and putting them together—that is, decomposing and composing them. As we saw in Chapter 3, we can decompose numbers additively or multiplicatively; for example, we can see 6 as $3 + 3$ or as $2 \cdot 3$. In this section, we found that divisibility relates to multiplicative decompositions.

EXERCISES 4.1

- Write at least two different definitions of *odd number*. When would it be preferable to use each definition?
- Earlier in this section, we examined the relationships between adding odd numbers and adding even numbers. This question asks you to examine the relationships between multiplying odd numbers and multiplying even numbers.
 - Explain why the product of an odd number and an odd number is an odd number. *Hint:* Look at the discussion of why the sum of an even number of odd numbers is an even number.
 - How would you describe the rule for determining whether the product of many natural numbers is even or odd? For example, is $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$ an odd number or an even number?
- Find four different odd numbers whose sum is 20.
 - How many combinations of four different odd numbers whose sum is 30 can you find? Describe your strategies.
- What can you say about two numbers if their sum is even and their product is odd? Justify your answer.
- The sum of three numbers is even. Must the product of the three numbers be even or odd, or are both possible? Justify your answer.
- The game of darts is mathematically rich with respect to number theory.
 - If you throw five darts, in how many ways can you get a score of 21 if we assume that every dart hits the dartboard? Write your solution and summarize your strategy.
 - In how many different ways can you get a score of 22?
 - Make a list of all possible scores from throwing five darts. Describe your strategy for finding *all possible scores*. If you were stumped on part (b), does this exercise shed any light?
- In how many different ways can one make a score of 22 with 6 darts?
 - What if we take away the restriction that all darts must hit the board? Now how many possibilities for a score of 22 are there?



- c. Why do all combinations with missed darts include 2 misses?
8. There are some interesting connections between the set of odd and even numbers and the sets of figurate numbers you encountered in Exploration 2.5.
- a. The table below is generated by finding the sum of consecutive odd numbers. Can you explain why the sum of consecutive odd numbers generates the set S of square numbers, $S = \{1, 4, 9, 16, 25, 36, \dots\}$?
- $$1$$
- $$1 + 3 = 4$$
- $$1 + 3 + 5 = 9$$
- $$1 + 3 + 5 + 7 = 16$$
- b. Another way to generate the set of square numbers, is shown below. Can you explain how the following sequence leads to the set of square numbers?
- $$1$$
- $$1 + 2 + 1$$
- $$1 + 2 + 3 + 2 + 1$$
- $$1 + 2 + 3 + 4 + 3 + 2 + 1$$
- $$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$$
9. Look at the following three examples:
- $$2 \cdot 4 = 3^2 - 1^2$$
- $$4 \cdot 6 = 5^2 - 1^2$$
- $$6 \cdot 8 = 7^2 - 1^2$$
- a. Describe what you see.
- b. Predict what the next row will look like.
- c. Describe the n th row and justify your description.
10. a. Describe the following sequence:
- $$1 \cdot 8 + 1 = 9$$
- $$12 \cdot 8 + 2 = 98$$
- $$123 \cdot 8 + 3 = 987$$
- $$1234 \cdot 8 + 4 = 9876$$
- b. What will be the ninth row in this sequence?
- c. Does the pattern break after that, does it continue, or is it modified?
11. I'm thinking of two numbers. Determine the numbers if:
- a. Their sum is 45 and their difference is 15.
- b. Their sum is 48 and their quotient is 3.
- c. Their sum is 61, and if the digits in each of the numbers are reversed, their sum is 115.
12. Begin a sequence by selecting any two numbers between 1 and 9. To continue the sequence, add the two numbers and record the digit that appears in the ones place. For example, 5, 7, 2, 9, 1. You will note that eventually the pattern repeats. Try this with a variety of starting numbers: $5 + 7 = 12$, $7 + 2 = 9$, $2 + 9 = 11$
- a. What is the shortest sequence you can find?

- b. What is the longest sequence you can find?
- c. How many different number sequences can you generate?
- d. What other patterns do you notice?
13. a. Find the following products: $37 \cdot 3$ and then $37 \cdot 6$. What do you predict will be the product of $37 \cdot 9$? Describe, in words, the pattern that you see.
- b. Now predict how long you think the pattern will continue and explain your prediction.
- c. Some students believe that the pattern breaks when we get to $37 \cdot 30$, and others don't. What do you think? Why?
14. In Investigation 4.1, we explored one set of dates.
- a. Consider the following dates: 2/3/45 and 6/7/89. First, write a general description for all dates that will be members of this set. Second, determine how many instances of this pattern will occur in this century.
- b. Create your own set of dates that have a number theory connection. First, write a general description for all dates that will be members of this set. Second, determine how many instances of this pattern will occur in this century.
15. This problem has been around for many years. Can you determine the ages of Alfie's children from analyzing the following conversation?
- Alfie: Can you guess the ages of my three children?
The product of their ages is 36.
- Bernice: That's not enough information. Give me another hint.
- Alfie: The sum of their ages is the same as our street address.
- Bernice: That's still not enough information. Give me another hint.
- Alfie: My oldest child is a girl.
- Bernice: Now I can figure it out.
- Note: Problems 16 and 17 can consist of a rote practice of the divisibility rules or they can be a wonderful opportunity for thinking. Before checking each number for the specific divisibility rule, try to predict the answer. For example, does 123,456 "feel" like the sum of the digits will be divisible by 3? Or in Exercise 17(b), before checking to see whether $4 \mid 98$, does it "feel" like it will or it won't? If your intuition is often right, can you explain why? In many cases, what is behind a good intuition is a sense of patterns.
16. Use divisibility rules to answer the following. Briefly explain how you arrived at your conclusion. Simply dividing the larger number by the smaller is not acceptable.
- a. Is the number 123,456 divisible by 3? Why or why not?
- b. Is the number 123,456 divisible by 6? Why or why not?

- c. Is the number 123,456 divisible by 8? Why or why not?
 - d. Is the number 2,345,678 divisible by 4? Why or why not?
 - e. Is the number 2,345,678 divisible by 6? Why or why not?
 - f. Is the number 2,345,678 divisible by 9? Why or why not?
17. Test the numbers below for divisibility by 2, 3, 4, 5, 6, 8, 9, and 12.
 - a. 222,444 b. 213,498 c. 987,987,987
 18. Write a definition for *divides* in your own words. Then have someone who is not in this course read your definition and explain to you, on the basis of your definition, what *divides* means.
 19. In the following number there are two missing digits: 467, __2. You are told that this number is divisible by 144. What is the number? Explain how you found it.
 20. One aspect of parenting that has nearly driven me crazy is that kids lose things. My children are still young and love to play card games. However, they lose cards, so we have several decks of cards, none of which is complete. This situation reminded me of a problem that I have seen in many guises. I offer it to you now in the context of card games. We have assembled quite a pile of cards and are playing a game in which all the cards are dealt out before playing. We have made the following amazing discovery:

If two people are playing, one card will be left over.

If three people are playing, two cards will be left over.

If four people are playing, three cards will be left over.

If five people are playing, four cards will be left over.

If six people are playing, five cards will be left over.

If seven people are playing, no cards will be left over.

How many cards are in this deck?
 21. This is a variation on the card game problem. If two, three, four, five, or six people are playing, one card will be left over (in each case). If seven people are playing, no cards will be left over. How many cards are in the deck?
 22. Joe wrote the divisibility rule for 4 in the following way: A number is divisible by 4 if the last two digits are divisible by 4. Is his expression accurate? Is it clear? If you like it, explain why. If you think it is not accurate or not clear, explain what is inaccurate or vague.
 23. We learned earlier that if the digit representing the hundreds place is even, then a number is divisible by 8 iff the number represented by the last two digits is divisible by 8. Finish and justify the second part of this modification of the divisibility rule for 8: If the digit representing the hundreds place is an odd number, then a number is divisible by 8 iff _____.
 24. Prove the divisibility rule for 9 for any three-digit number.
 25. Some textbooks present two other divisibility rules. Examine each of the rules below to make sure you can use them. Then explain why they work.
 - a. A number is divisible by 7 if the number obtained by subtracting double the ones digit from the number represented by the remaining digits is divisible by 7.
 - b. A whole number is divisible by 11 if the sum of the digits representing even powers of 10 minus the sum of the digits representing odd powers of ten is also divisible by 11.
 26. Devise and justify a divisibility rule for
 - a. 18
 - b. 24
 - c. 25
 - d. 2 in base 5
 - e. 4 in base 5
 - f. 3 in base 12
 27. Classify each of the following as true or false, assuming that a , b , and c are integers. If the statement is true, briefly explain why it is true. If the statement is false, give a counterexample, and then rewrite it so that it is true, and then justify the new statement.
 - a. If $a \mid b$ and $a \nmid c$, then $a \mid bc$.
 - b. If a number is divisible by 3, then every digit of that number is divisible by 3.
 - c. If $c \mid ab$, then $c \mid a$ or $c \mid b$.
 - d. If $4 \mid a$ and $6 \mid a$, then $24 \mid a$.
 - e. If $12 \mid a$, then $6 \mid a$.
 - f. If $d \mid a$, then $d \mid a^2$.
 28. Each of the conjectures below is either true or false. For each, if you think it is true, try to prove that it is true or justify why you think it is true. If you think it is false, provide a counterexample.
 - a. The sum of 3 consecutive numbers is divisible by 3.
 - b. The product of any 3 consecutive numbers is divisible by 6.
 - c. The product of any 4 consecutive numbers is divisible by 24.
 - d. The square of any odd number can be represented in the form $8n + 1$, where $n \in \mathbb{W}$.
 29.
 - a. What can you say about the divisibility of the sum of three consecutive numbers? In other words, are there any numbers that will always divide this sum, regardless of the three numbers?
 - b. What about the sum of any four consecutive numbers?
 - c. What about the sum of any five consecutive numbers?
 30. For 6 consecutive years, Joan's age was divisible by her granddaughter's age. What were their ages during that period?

31. a. Let's say you have a bunch of 6¢ and 5¢ stamps and a package for which the postage is \$1.93. Can you make \$1.93 with only 5¢ and 6¢ stamps?
 b. What if you have only 5¢ stamps and 6¢ stamps and the postage is \$1.80? How many different ways can you make the postage?
32. a. What is the units digit of 3^{900} ? *Hint:* Make a table and look for a pattern.
 b. What is the remainder when 3^{123} is divided by 5?
 c. What is the remainder when 3^{900} is divided by 2?
33. Look at the following:
 $17 \mid 10^2 - 7^2$
 $35 \mid 30^2 - 5^2$
 a. Create another example that fits this pattern.
 b. Can you generalize this pattern? That is, can you express the relationship in such a way that someone else could use your directions to create more examples?
 c. Can you prove your generalization?
34. At the end of this section, we used Venn diagrams to represent some divisibility relationships.
- a. Represent with a Venn diagram the relationship between the set of numbers divisible by 4 and the set of numbers divisible by 8.
 b. Represent with a Venn diagram the relationship between the set of numbers divisible by 3 and the set of numbers divisible by 9.
 c. Represent with a Venn diagram the relationship between the set of numbers divisible by 5 and the set of numbers divisible by 10.
 d. Represent with a Venn diagram the relationship between the set of numbers divisible by 5 and the set of numbers divisible by 25.
 e. Represent with a Venn diagram the relationship between the set of numbers divisible by 2, the set of numbers divisible by 4, the set of numbers divisible by 8, and the set of numbers divisible by 16.
35. Can you tell whether the following sum will be an odd or an even number without actually adding the numbers? Play with some other examples and then summarize and justify whatever hypotheses/rules you come up with.

$$21 + 56 + 37 + 23 + 49 + 20 + 73 + 34$$

SECTION

4.2

PRIME AND COMPOSITE NUMBERS

WHAT DO YOU THINK?

- Why is 1 neither a prime nor a composite number?
- How are prime numbers the building blocks for the set of natural numbers?

Over the centuries, many famous mathematicians have been fascinated by prime numbers. The Greeks are believed to have discovered prime numbers and were also fascinated by them. The prime numbers were considered to be the building blocks of all natural numbers greater than 1. Children are often intrigued by prime numbers. Recall the statement from Chapter 3 that “(p)robably the major conceptual achievement of the early school years is the interpretation of numbers in terms of part and whole relationships.” As in the previous section, we will again be focusing on multiplicative compositions and decompositions of natural numbers.

Let us begin our examination of prime and composite numbers by first revisiting Investigation 4.1. Are there any dates in the year 1997 that fit that pattern? Work on this and then read on. . . 

It turns out that there are no such dates, because 97 is a prime number. That is, it has only two factors: 1 and itself.

A natural number is a **prime** number iff it has exactly two factors: 1 and itself.

A natural number that has more than two factors is called a **composite** number. Note the relationship between the words *composite* and *composition*, a term we have encountered several times already in this course.

The number 1 doesn't fit into either of the sets above, and so we say that 1 is neither a prime number nor a composite number. Do you see why?

Over the years, I have found that my strongest students are the ones who not only know facts and procedures but also have good number sense. One of the ways to develop this number sense in elementary children is through investigations that deal with prime numbers.

Determining Whether a Number Is Prime or Composite

A question that has fascinated both ancient and modern mathematicians concerns being able to determine whether a large number is a prime number or a composite number. We don't need a large number for this question to be difficult. For example, is the number 103 prime or composite? Work on this question for a minute or so and then read on. . . . 

Immediately, we can see that no even number will divide 103. However, without sophisticated techniques, in order to determine if 103 is prime, we must check to see whether it is divisible by each odd number up to 51. Thus, we would have to check: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51. Do you see why we can stop at 51?

There is a very elegant shortcut that saves us from having to test all these numbers. This shortcut was discovered over 2000 years ago and emerges from the following investigation.

INVESTIGATION 4.8 The Sieve of Eratosthenes

The sieve of Eratosthenes was developed by the Greek mathematician Eratosthenes, who lived about 230 B.C., as a tool for determining all prime numbers less than a given number. Our demonstrating the sieve here has several purposes. First, it will uncover a more efficient way to determine whether a large number is prime. Second, it will further develop your ability to see and to extend patterns. As usual, you will get far more from the following activity if you do it with pencil and paper rather than just read.

TABLE 4.3

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150

In Table 4.3, cross out 1, which is neither composite nor prime.

Circle 2, which is the first prime number. Now cross out all multiples of 2.

Circle the next unmarked number (3), which must be prime (Do you see why?), and cross out all multiples of 3. Did you make use of any patterns to help you cross out multiples of 3?

Circle the next unmarked number (5), and then cross out all multiples of 5.

Circle the next unmarked number (7) and stop. On the basis of the crossing out that you have already done, what do you predict will be the first multiple of 7 that you will have to cross out? Think and then cross out all multiples of 7 and then read on. . . . 

You found that the first multiple of 7 that wasn't already crossed out was 49; that is, $7 \cdot 7$. The other multiples of 7 that you had to cross out were $7 \cdot 11$, $7 \cdot 13$, and $7 \cdot 17$.

Now circle the next prime number (11), and stop. Before you cross out all multiples of 11, what do you predict will be the first multiple of 11 that you will have to cross out? Think and then cross out all multiples of 11 and then read on. . . . 

You found that the first multiple of 11 that wasn't already crossed out was 121; that is $11 \cdot 11$. Now circle all the numbers in the table up to 121 that have not yet been crossed out. What do you think these numbers have in common? Think and then read on. . . . 

HISTORY

Many famous mathematicians have looked for functions that would produce only prime numbers. For example, consider all numbers of the form $M = 2^p - 1$, where p is a prime number. Numbers of this form that are prime are called Mersenne primes, after the monk Marin Mersenne (1588–1648). Can you find three Mersenne primes?

Now it turns out that sometimes this function generates prime numbers and sometimes not.

Pierre Fermat (1601–1665), arguably the king of number theory, played with the following function: $F(n) = 2^{(2^n)} + 1$. He was hopeful that this function would generate only prime numbers. For example, if $n = 2$, then $2^{(2^2)} + 1 = 2^{(4)} + 1 = 17$.

This conjecture looked good until Leonard Euler (1707–1783) showed in 1732 that $F(5) = 4,294,967,297$ can be factored into $641 \cdot 6,700,417$. This was before any kind of calculating machine existed! No one has yet created a function that produces only prime numbers.

DISCUSSION

If your intuition said that these are all prime numbers, you are right. Can you explain *why* all these numbers *must* be prime numbers? For example, can you explain why 149, for example, cannot be a multiple of 7, 11, 13, 17, 19, or any other prime number less than 149? Work on this question and then read on. . . . 

One way of demonstrating why all the remaining numbers must be prime involves **indirect reasoning**, which is a method of logic in which we prove that something is false by assuming that it is true and then showing that this assumption leads to a contradiction.

Let us assume that one of the circled numbers in the sieve, for example 149, is not prime.

- 149 is clearly not a multiple of 2, because it's an odd number.
- 149 cannot be a multiple of 3, because it would have been crossed out with the multiples of 3.
- 149 cannot be a multiple of 5, because it would have been crossed out with the multiples of 5.
- 149 cannot be a multiple of 7, because it would have been crossed out with the multiples of 7.
- 149 cannot be a multiple of 11, because it would have been crossed out with the multiples of 11.
- 149 cannot be a multiple of 13, because we know from our investigation with the sieve that the first possible multiple of 13 that hasn't already been crossed out would be $13 \cdot 13$, which is 169.

In similar fashion, we can show that 149 cannot be a multiple of any other number. This disproves the initial assumption that 149 was not prime. Therefore, it is impossible that 149 is a composite number, and therefore, it must be a prime number.

Actually, once we know that 13 is not a factor of 149, we are finished. Do you see why?

If we begin to cross out multiples of 13, we find that all the multiples of 13 from $1 \cdot 13$ to $12 \cdot 13$ had already been crossed out. Therefore, the first multiple of 13 that we needed to cross out is $13 \cdot 13$. But $13 \cdot 13$ is 169, which is greater than 149, and so we know that 13 cannot be a factor of 149. Similarly, all numbers greater than 13 (which must have squares that are greater than 169) cannot be factors of 149. We can turn this observation into a generalization: If the square of a number is greater than the number we are testing for, we can stop.

This very powerful generalization, in turn, can be stated as a rule:

Test for determining whether a number is prime:

To determine whether a natural number n is a prime number, list all the prime numbers p that satisfy the equation $p \leq \sqrt{n}$. If none of those prime numbers divides n , then n is a prime number.

Prime Factorization

Recall the question about possible dates in 1996 in which the product of the day and month will be 96. We mentioned that another strategy would be to find all the factors of 96. One way to find all the factors of a number is to determine the **prime factorization** of that number—that is, to represent that number as the product of numbers, each of which is prime.

The prime factorization of 96 is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$. Do you see how this representation can help us to determine all the factors of 96? Think and then read on. . . . 

We simply check to see whether our number can be constructed from the prime factorization. Clearly, 2 and 3 are factors of 96.

- 4 is a factor because $4 = 2 \cdot 2$.
- 5 is not a factor of 96 because 5 is not in this set.
- 6 is a factor because $6 = 2 \cdot 3$.
- 7 is not a factor of 96 because 7 is not in this set.
- 8 is a factor because $8 = 2 \cdot 2 \cdot 2$.
- 9 is not a factor of 96 because $3 \cdot 3$ is not in this set.
- 10 is not a factor of 96 because $2 \cdot 5$ is not in this set.
- 11 is not a factor of 96 because 11 is not in this set.
- 12 is a factor because $12 = 2 \cdot 2 \cdot 3$.

Because $8 \cdot 12 = 96$, we don't have to look any further. Do you see why?

Recall the first investigation, in which we saw that many dates came in pairs. In this case, each factor up to 12 is part of a pair: $1 \cdot 96$, $2 \cdot 48$, $3 \cdot 32$, $4 \cdot 24$, $6 \cdot 16$, and $8 \cdot 12$.

Thus, the factors of 96 are 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, and 96.



HISTORY

The number 1 was a puzzle to the early Greeks. They did not consider 1 to be a number at all because it was the "indivisible unit from which all other numbers arose."³ However, it was not just the early Greeks who found 1 to be a "different kind of number." The German Kobel wrote in 1537 that "Wherefrom thou understandest that 1 is no number, but it is a generatrix, beginning, and foundation for all other numbers."⁴

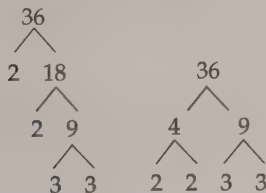
³David Wells, *The Penguin Dictionary of Curious and Interesting Numbers* (New York: Penguin Books, 1986), p. 30.

⁴Ibid., p. 30.

Now it is simply a matter of determining which possibilities are valid dates; for example, 1/96/96 is not a valid date, nor is 2/48/96.

Determining the prime factorization of a number is not an easy task for many students, and there are many different methods. Take a few minutes to determine the prime factorization of 36 and 84. Then read the following discussion. 

Tree diagrams and prime factorization One technique for determining the prime factorization of a number that appeals to many students is to use a tree diagram. The top of the tree is the original number. Each number is equal to the product of the two numbers immediately below it. Below are two tree diagrams that produce the prime factorization for 36—that is, $2 \cdot 2 \cdot 3 \cdot 3$.



Division and prime factorization Other students prefer a technique that looks like long division that goes up instead of down. The division is done mentally. Some students like to do it “in order”; that is, they divide by 2 until they can do it no more and then go to bigger and bigger prime numbers. When they do this, their final answer is already in order from smallest to largest.

$$\begin{array}{r}
 7 \\
 3 \overline{)21} \\
 2 \overline{)42} \\
 2 \overline{)84}
 \end{array}
 \quad 84 = 2 \cdot 2 \cdot 3 \cdot 7$$

The Fundamental Theorem of Arithmetic

If you compare the two tree diagrams above, you can see that we got the same prime factorization for 36 regardless of how we started. It turns out that this is true in all cases. This seemingly obvious statement actually has quite a formal name, the **Fundamental Theorem of Arithmetic**. It states that every natural number greater than 1 has one unique way of being represented as a product of its prime factors.

Let us use the number 60 to illustrate this theorem. As you can see, it doesn't matter which path we take. When we get to the end, all the factors are prime, and we find that $60 = 2 \cdot 2 \cdot 3 \cdot 5$.

$$\begin{aligned}
 60 &= 2 \cdot 30 = 2 \cdot (6 \cdot 5) = 2 \cdot [(3 \cdot 2) \cdot 5] = 2 \cdot 3 \cdot 2 \cdot 5 = 2 \cdot 2 \cdot 3 \cdot 5 \\
 60 &= 10 \cdot 6 = (2 \cdot 5) \cdot (2 \cdot 3) = 2 \cdot 5 \cdot 2 \cdot 3 = 2 \cdot 2 \cdot 3 \cdot 5
 \end{aligned}$$

A Largest Prime Number?

“And the boy in Winnetka, Illinois, who wanted to know: Is there a train so long you can't count the cars? Is there a blackboard so long it will hold all the numbers?” (Carl Sandburg).

This quote brings to mind a related question: Is there a largest prime number? Do you think there is one or not? What arguments could be made either way? 

The Fundamental Theorem of Arithmetic enables us to answer this question. It turns out that there is no largest prime number; there are infinitely many. The first proof of this conjecture occurs in Euclid's *Elements*. The proof uses indirect reasoning, in which we begin by assuming that there is a largest prime number, which we will call P .

Let us now create a number X , in the following way:

$$X = (2 \cdot 3 \cdot 4 \cdots P) + 1$$

We have assumed that P is the largest *prime* number, so X must be a *composite* number; that is, it must have at least one factor other than 1 and itself.

Let F represent the smallest *prime factor* of X . Because F is a prime number and P is the largest prime number, F must be a prime number less than P ; that is, it must be one of the numbers 2, 3, 4, . . . up to P .

However, it cannot be one of these numbers. Do you see why? Think before reading on. . . . 

We constructed our number X in such a way that it is not divisible by 2, 3, 4, or any number up to P . That is equivalent to saying that 2 is not a factor of X , 3 is not a factor, and so forth. Thus, the smallest possible prime factor of X is greater than P , which is supposed to be the largest prime number. This contradiction shows that the assumption that there is a largest prime number cannot be true.



BEYOND THE CLASSROOM

The invention of radio earlier in this century had many ramifications for people's lives. During World War II, the ability to communicate by radio enabled commanders to be more effective both before and during battles. However, the other side was able to receive the messages too, as long as they knew the radio frequency on which the messages were transmitted. Therefore, important messages were sent in code. Functions were created to encode messages. One very simple coding function is to replace each letter with the next letter in the alphabet. Using this code, what is the following message: I B Q Q Z C J S U I E B Z? The answer is at the end

of the next paragraph. As time went on, each side became better at cracking the other side's codes. A team of British analysts developed a machine to crack German codes, which became known as the German Enigma cipher machine.

More recently, several M.I.T. scientists have developed what is called a safe coding system. The process involves selecting two very large prime numbers. The coding key is the product of these two numbers. Because it is very difficult to factor extremely large numbers, it is virtually impossible to crack such codes. Answer to coded message: happy birthday!

INVESTIGATION 4.9 Numbers with Personalities: Perfect and Other Numbers

Some of the names that have been given to certain kinds of numbers convey human characteristics: for example, *perfect*, *amicable*, *sociable*, *weird*, *narcissistic*, and *irrational*. We will examine some of these numbers, because they provide rich ground for developing mathematical thinking; they are also good "rainy-day activities" when you teach.

- A.** A number n is a **perfect** number iff the sum of its *proper divisors* is equal to n . A **proper divisor** of a number is a divisor that is less than the number. Do you see why 6 is the first perfect number? 

DISCUSSION

Six is the first perfect number because 1, 2, and 3 are the proper divisors of 6 and their sum is 6.

-  **B.** Perfect numbers lead to two other kinds of numbers, which are called abundant and deficient. Before looking at the definitions of these terms, try to define them yourself. . . . 

A number n is an **abundant** number iff the sum of its proper divisors is greater than n .

A number n is a **deficient** number iff the sum of its proper divisors is less than n .

Make a table listing the first 30 natural numbers. Put an A next to ones you think will be abundant, a D next to ones you think will be deficient, and a ? next to ones you feel could go either way. Try to articulate the reasoning behind your predictions. Then read on. . . . 

DISCUSSION

All prime numbers are deficient by definition. Do you see why? Because the only proper factor of a prime number is 1, the sum of the proper factors of a prime number is 1. Of the nonprime numbers less than 30, the deficient numbers are 4, 8, 9, 10, 14, 15, 16, 21, 22, 25, 26, 27.

Does this list shed light on any other relationships? What does it show?

Notice that the subset of the square numbers in the above set {4, 9, 16, 25} has only deficient numbers. Also, except for 6, any doubles of members of the set of prime numbers less than 30 {2, 3, 5, 7, 11, 13, 17, 19, 23, 29} will also be deficient. Why is this?

The following numbers are abundant: 12, 18, 20, 24, and 30. What do you notice about this set?

One pattern that students often observe that leads to a hypothesis is that all of these numbers are even. Is this just coincidence, or is it true that all abundant numbers are even numbers? What do you think? This question will be left unanswered for you to solve on your own.

The active reader will have noticed that we overlooked 28 in our lists of deficient and abundant numbers. Why?

Summary

In this section, we have examined another way of classifying natural numbers: prime and composite (and, of course, 1, which is neither prime nor composite). The Sieve of Eratosthenes not only gave us a tool to determine whether a number is prime but also helped us to develop and refine other problem-solving tools. We have also found that each number has a unique prime factorization, which is another kind of decomposition (funny how this idea keeps coming up!). It is critically important to note that the decomposition of composite numbers into their prime factors is a multiplicative decomposition; this is one of many places where seeing a number in additive or multiplicative terms is not a matter of preference but a matter of structure.

We have explored several strategies that enable us to determine the prime factorization of a natural number. When you teach children, you will find that the Fundamental Theorem of Arithmetic, like the commutative property, is not immediately obvious to children, and in both cases, an understanding of these ideas makes work with mathematics much easier. Determining the prime

factorization is like looking through a microscope to see the atomic structure of a molecule. In one sense, prime numbers are the building blocks for whole numbers. We have also seen that the concept of prime and composite numbers has applications outside the classroom.

The notions of prime and composite enable us to understand better Explorations 4.1 and 4.2. Think back to the ones that you did. How does this section add to your understanding of those explorations? Then read on. . . 

In Exploration 4.1 (Taxman), we realize that the first move is to select the largest prime number. We realize that numbers with many factors (like 24) are better left for the end of the game. Do you see why? In Exploration 4.2 (Factors), we found that although all prime numbers have only two factors, the set of composite numbers is not so simple. Although most composite numbers have an even number of factors, not all do, and some composite numbers have many factors. Some have so many factors that we call them abundant numbers!

EXERCISES 4.2

- How many factors does a prime number have?
- Write the first 20 prime numbers.
- Find the prime factorization of the numbers below:

a. 48	b. 75	c. 92
d. 144	e. 196	f. 504
g. 756	h. 14,586	
- In each case below, determine whether the number is prime or composite. You will probably want to use a combination of divisibility tests, common sense, and concepts from our work with prime numbers.

a. 61	b. 65	c. 71
d. 89	e. 223	f. 347
g. 437	h. 961	i. 3,437
- Which of the numbers between 511 and 525 are prime? Briefly explain your reason for denoting each number as either composite or prime.
- Consider the numbers representing the second half of the twentieth century. Which of those numbers are prime?
- To determine whether 617 is a prime number, it suffices to check that no prime number less than ____ is a factor of it. Explain your answer.
- A **superprime** is defined in the following way: It is a prime number such that each time a digit is removed (one at a time), starting at the right side, the remaining number is still prime. For example, 2391 is a superprime because 239 is a prime and 23 is a prime and 2 is a prime. Find three three-digit superprime numbers. Describe strategies that helped in your search.
- There is an all-star team of abundant numbers called **superabundant** numbers. Here are the membership criteria for inclusion in this exclusive club:

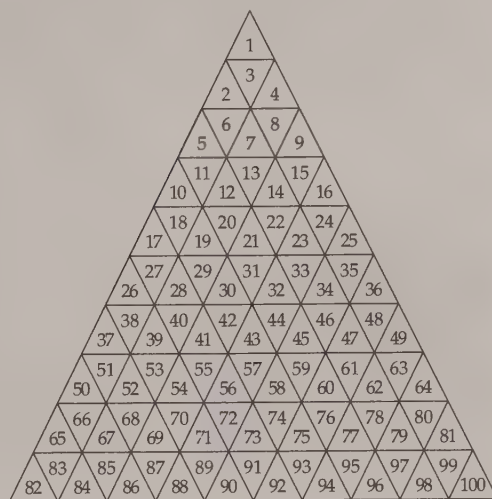
Take the sum of all of a number's divisors, including itself. Divide by this sum by the number itself.

If this quotient is greater than the quotient for any natural number less than this number, then the number is called superabundant.

 - Design and run a spreadsheet to determine which of the first 50 natural numbers are superabundant, and then write directions for your program so that others may use it.
 - Describe any new patterns or observations about characteristics of superabundant numbers that you found in making and analyzing this table.
 - Make a line graph of the first 50 natural numbers in which the horizontal axis consists of successive natural numbers and the vertical axis represents the value of the quotient used to determine whether the number is superabundant. What new observations come from the graph?
 - Predict the next superabundant number and explain the reasoning behind your prediction.
- Construct a number that has exactly five divisors.
- Find the smallest natural number that is divisible by the first ten counting numbers.
 - How else could this question have been worded?
- What is the smallest four-digit number that has exactly three factors? Justify your answer and explain how you arrived at your solution.

- b. What is the largest three-digit number that has exactly four factors? Justify your answer and explain how you arrived at your solution.
13. What is the smallest composite number that is divisible by none of the primes 2, 3, 5, and 7? Briefly explain your answer.
14. Here are partial directions for creating five consecutive composite numbers. First, begin with $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$. Then use the divisibility theorem from the previous section; that is, if $a|b$ and $a|c$, then $a|(b+c)$. Can you use these hints to create five consecutive composite numbers?
15. An organization with 100 people will be divided into smaller committees.
- What are the possible committee sizes if every committee must be the same size?
 - What are the possible committee sizes if the organization wants to have some large committees (each the same size) and some smaller committees (each the same size), and the larger committees must be at least twice the size of the smaller ones?
 - Create at least one more restriction on part (b) that would lead to fewer possible answers.
16. a. If 21 divides m , what else must divide m ?
b. If 36 divides m , what else must divide m ?
17. If you examine all the prime numbers less than 100, some of them are "next-door neighbors"—for example, 11 and 13, 17 and 19, 71 and 73. Mathematicians call these twin primes or prime twins. What about prime triplets? Only one set of prime triplets exists: 3, 5, 7. Prove that there cannot be another set of prime triplets.
18. This is a famous problem in mathematics education. Suppose you lined up 1000 dolls in a row, all facing up. Then suppose you walked down the line and turned over every other doll. Now, one-half of the dolls would be facing up (those representing odd numbers) and one-half of the dolls would be facing down. Now suppose you walked down the line again, and this time turned over every third doll. That is, if it was facing up, you would turn it so that it was facing down; if it was facing down, you would turn it so that it was facing up. Continue in this manner 1000 times. At this point, which dolls will be facing up and which dolls will be facing down?
- The following suggestions may help you with the problem: Some students find it helpful to make the problem smaller—for example, 50 dolls or 25 dolls. Some students find it helpful to model the problem. What other objects could you use to represent the dolls? Most students find it helpful to make a table to represent their findings.
19. a. Describe the patterns you find in the following figure.
b. Explain how to make the figure as though you were talking on the phone to someone.

- c. Find a shortcut for finding the sum of any hexagon.
d. Find a shortcut for finding the sum of any row.



20. The following questions have come from students over the years. Explore these questions with a partner. After your exploration, summarize and justify your findings.
- Of the first 100 numbers, 25 are prime. How many of the second hundred numbers will be prime? Will the frequency (density) of prime numbers increase, decrease, or stay at about 25 percent as the numbers get larger and larger?
 - Can you predict when twin primes will occur? How many are there in the first 100 numbers? Will there be the same number of twin primes in the next 100 numbers?
 - The largest difference between two prime numbers less than 100 is 8; it is between the 89 and 97. What is the largest difference you can find between any two prime numbers?
 - Are there more primes with a 1, a 3, a 7, or a 9 in the ones place?
21. These are some famous conjectures, some of which have yet to be proved or disproved! Explore these conjectures with a partner. Then explain why you think the conjecture is true.
- Any prime greater than 3 is within 1 unit of a multiple of 6.
 - Every even number can be expressed as the difference of two primes.
 - This conjecture comes from Fermat: Any prime number of the form $p = 4n + 1$, $n \in \mathbb{N}$, can be expressed as the sum of two squares in only one way.
 - This conjecture is called Goldbach's conjecture after Christian Goldbach (1690–1764): Any even number greater than 2 can be expressed as the sum of two

primes. To this day, no one has found a counterexample or proved this conjecture.

- e. Another conjecture made by Goldbach: Any odd number greater than 5 can be expressed as the sum of three prime numbers.
- f. Every triangular number can be represented as the sum of a square number and two triangular numbers.

SECTION

4.3

GREATEST COMMON FACTOR AND
LEAST COMMON MULTIPLE

WHAT DO YOU THINK?

- What does it mean to say that the GCF and the LCM let us decompose composite numbers?
- How are the GCF and the LCM related to each other?

One of the many decisions I had to make when writing this book was where to place the section on the greatest common factor (GCF) and least common multiple (LCM). In elementary school, these concepts are used primarily when adding and simplifying fractions, and so this section could be placed in Chapter 5. However, these concepts have applications beyond fractions problems, and one of the goals of this book is to get students to see that mathematics is more than just procedures and that many mathematical ideas are connected to other mathematical ideas. Therefore, we will first examine these concepts in this chapter and then apply them to fraction ideas in Chapter 5.

INVESTIGATION 4.10 Cutting Squares Using Number Theory Concepts

The following problem is similar to Investigation 1.3, in which the teacher wanted to cut out squares for her students to use in making paper dinosaurs. Let's say that a teacher has a rectangular sheet of cardboard 420 centimeters long and 378 centimeters wide and that he wants to cut that sheet into many squares, all of the same size. What are the dimensions of the largest possible square (whose length is a whole number) that will create no waste? Think about this problem and then read on. . . . 

DISCUSSION

STRATEGY 1: Use guess-check-revise

Looking at the numbers, we can see that they are both divisible by 2, so we could make 2 by 2 squares. Using the divisibility rule for 3, we can see that they are also divisible by 3, and so we can make 3 by 3 squares. We can proceed in this manner, checking for divisibility by 4, 5, 6, and so on, until we find that we can go no further.

STRATEGY 2: Use prime factorization

What if we found the prime factorization of both numbers? How might that help us? Think about this before reading on. . . . 

$$378 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7$$

$$420 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7$$

Because we are looking for a number that divides *both* 378 and 420, an equivalent statement is that we are looking for a number that is a factor of both. In fact, we are looking for the largest such common factor. By looking at the prime factorization of both numbers, we can see that they have in common a 2, a 3, and a 7. How does this help? Some students say, "Add the three

numbers, and that will be the largest square—that is, 12 by 12.” Some students say, “Multiply the three numbers, and that will be the largest square—that is, 42 by 42.” What do you think? Why? 

When we examine factors, we are looking at numbers from a *multiplicative* perspective, and thus adding them is not connected to this question. In fact, 12 does not divide 378; do you see why? However, 42 does divide both 378 and 420. Do you see why?

In fact, 42 is the *greatest* number that divides both 378 and 420. If we express this conclusion using the term *factor* instead of *divides*, we can say that 42 is the greatest factor that 378 and 420 have in common. A more concise way to say this is that 42 is the *greatest common factor* of 378 and 420.

The Greatest Common Factor

Whenever we examine two natural numbers, we can create a set of numbers called their common factors. In some cases, the only common factor is 1; for example, examine 7 and 10. However, in many cases, the numbers have several common factors. The greatest of these common factors is called the **greatest common factor** (GCF). We use the notation $\text{GCF}(a, b)$ to express the GCF of two natural numbers a and b .

Note: In a similar fashion, we can speak of the GCF of three or more numbers. This will be explored in the exercises.

Exploring the greatest common factor concept Before we examine techniques for finding the GCF of two or more numbers, let us take some time to explore the concept. This is analogous to the work we did in Chapter 3 when we explored the concepts of addition and subtraction to understand the algorithms better, and then explored the concepts of multiplication and division to understand those algorithms better. What can we say about the size of the GCF of *any* two numbers a and b ? For example, will it always be smaller than the two numbers? Can it be in between two numbers? What do you think? One student compared this question to thinking of the qualifications for a job; for example, what qualities would you expect in an excellent elementary teacher: likes children, has eyes in the back of the head, doesn't require much sleep? Take a few moments to think about this before reading on. . . . 

The GCF of two numbers is generally smaller than either of them, because it has to be a factor of each. However, there are some cases in which the GCF of two numbers is *not* smaller than either of them. Can you think of such a case? Think and then read on. . . . 

Although the GCF can never be larger than either of the numbers, it can be equal to the smaller of the two numbers. For example, the GCF of 4 and 8 is 4.

In a manner similar to a detective story, we can make predictions about the GCF if we know just a little bit about the two numbers. For example, if you know that two numbers are prime, what can you say about their GCF? Think and then read on. . . . 

Prime numbers have only two factors, and one of them is 1. Because the other factor will be the number itself, we can conclude that if the two numbers are prime, their GCF is 1.

What can you say about the GCF of two even numbers? Think and then read on. . . . 

Their GCF will be at least 2. Why is this?

What about the GCF of two odd numbers? Think and then read on. . . . 

This clue alone is not enough for us to say much, except that the GCF will be an odd number also. Why is this?

INVESTIGATION 4.11 Methods for Finding the GCF

Let us now investigate how we might determine the GCF of two numbers. Rather than give an efficient procedure right away, we will take some time to build the foundation of this procedure, much like taking care while constructing the foundation of a house.

Using only the definition of GCF, how would you determine $\text{GCF}(45, 60)$? Work on this any way you want to. My only recommendation is that you think about the meaning of whatever you do, as opposed to “groping-and-hope.” Think and then read on. . . . 

DISCUSSION

STRATEGY 1: Use factorization

We could, as we just saw, determine all the factors of each number and then find the largest of the common factors:

$$\text{Factors of } 45 = \{1, 3, 5, 9, 15, 45\}$$

$$\text{Factors of } 60 = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

$$\text{Common factors} = \{1, 3, 5, 15\}$$

We see from this list that 15 is the GCF of 45 and 60.

STRATEGY 2: Use intuition or number sense

A student who is highly intuitive and has good number sense might just know that 15 divides both these numbers. The fact that 15 divides both numbers simply means that 15 is a common factor. How might you reason that, in fact, 15 is the GCF? Think before reading on. . . . 

Let us represent the results of dividing each number by 5 (which we know is not the GCF) and by 15. What do you notice?

$$45 = 5 \cdot 9 \qquad 45 = 15 \cdot 3$$

$$60 = 5 \cdot 12 \qquad 60 = 15 \cdot 4$$

When we divide 45 and 60 by 5, we are left with 9 and 12. When we divide 45 and 60 by 15, we are left with 3 and 4. One difference between 9 and 12 and 3 and 4 is that 3 and 4 have no common factors.

When two numbers have no factors in common other than 1, they are said to be **relatively prime**. One of my students came up with an interesting paraphrase of this concept: If two numbers are relatively prime, they are prime to each other.

Because 3 and 4 are relatively prime, 15 is the GCF of 45 and 60. Do you see why?

STRATEGY 3: Repeatedly divide by prime numbers

What about people who did not have the intuition and number sense to see this? Is their only recourse the long way we saw above? Another procedure in-

volves an adaptation of the long-division algorithm. The following problem illustrates a systematic application in that we begin with the smallest prime divisor and then move up. That is, we first divide both numbers by 3. At this point, we move up to 5 because 15 and 20 are both divisible by 5. The resulting quotients, 3 and 4, have no factors in common. The GCF of 45 and 60 is the product of their common factors: $3 \cdot 5 = 15$.

$$\begin{array}{r} 3, 4 \\ 5 \overline{)15, 20} \\ 3 \overline{)45, 60} \end{array}$$

STRATEGY 4: Use prime factorization

This strategy uses the Fundamental Theorem of Arithmetic, which we explored in Section 4.2. We first determine the prime factorization of each number and then look for common factors. If we look at the prime factorizations of 45 and 60 and circle the factors that the two numbers have in common, we have the following:

$$\begin{aligned} 45 &= 3 \cdot \textcircled{3} \cdot 5 \\ 60 &= 2 \cdot 2 \cdot \textcircled{3} \cdot 5 \end{aligned}$$

We can further refine this procedure by using exponents:

$$\begin{aligned} 45 &= 3^2 \cdot 5^1 \\ 60 &= 2^2 \cdot 3^1 \cdot 5^1 \end{aligned}$$

The GCF is determined by examining those factors that both numbers have in common and then taking the *smallest* exponent in each case. The common factors of 45 and 60 are 3 and 5. The smallest exponent of 3 is 1, and the smallest exponent of 5 is 1. Thus $3 \cdot 5$ is the GCF.



CHILDREN

Using Cuisenaire rods, how many different ways can you make a 12 train and an 8 train using the same colors? (See Figure 4.10.)

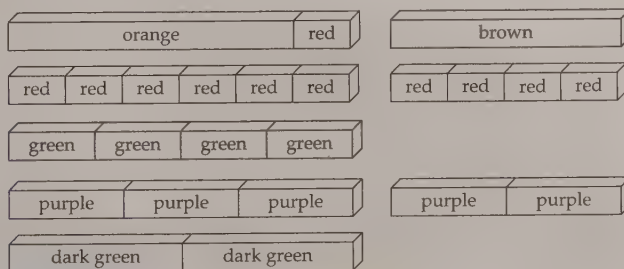


FIGURE 4.10

Do you see that the answer to this question is the GCF of 8 and 12?

Least Common Multiple

Let us begin our investigation of the least common multiple (LCM) by having you think about the following question before reading on: What do you think “least common multiple” means? Write a definition in your own language.

Let us examine this concept word by word, using a specific example. Suppose we wanted to find the least common multiple of 8 and 12. At the most basic level, we can generate the set of multiples of 8 and 12, and then find the smallest number that the two sets have in common.

Multiples of 8 = {8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, . . .}

Multiples of 12 = {12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, . . .}

What multiples do the two numbers have in common?

These two numbers have many common multiples: {24, 48, 72, 96, . . .}. Because the least of the common multiples is 24, we say that $\text{LCM}(8, 12) = 24$.

Whenever we examine two natural numbers, we can create a set of numbers called their common multiples. The least of these common multiples is called the **least common multiple** (LCM). We use the notation $\text{LCM}(a, b)$ to express the LCM of two natural numbers, a and b .

Note: In a similar fashion, we can speak of the LCM of three or more numbers. This will be explored in the exercises.

Exploring the least common multiple concept The active reader might think at this point, “Yes, this will work, but what if the numbers are larger? This approach could get tedious pretty fast!” This is true. However, as we did with the greatest common factor, let us extend that definition to learn more about the general attributes of the LCM. What can we say about the size of the LCM of *any* two numbers a and b ? For example, will it always be greater than the two numbers? Can it be in between the two numbers? What do you think? Take a few moments to think about this before reading on. . . . 

The LCM of two numbers is generally greater than either of them, because it has to be a multiple of each. However, there are some cases in which the LCM of two numbers is *not* greater than either of them. Can you think of such a case? Think and then read on. . . . 

When one number is a multiple of the other, the LCM will be equal to the larger number; for example, $\text{LCM}(5, 10) = 10$. Why is this?

Now let me give you two more numbers. Without doing any computation, what can you tell me about the LCM of 18 and 40? Think and read on. . . . 

Several possible responses include

- It is greater than 40.
- It is less than or equal to $18 \cdot 40$, or 720.
- It will be a multiple of both numbers.
- It ends in a zero, because all multiples of 40 end in 0.

Strategies for Finding the LCM

Now let us examine how we might find the actual LCM of 18 and 40. Again, there are many ways to determine the LCM of two numbers. We will examine several ways.

One way of determining the LCM of 18 and 40 is to construct the LCM by beginning with one of the numbers and applying our understanding of LCM. This process is illustrated in the following discussion:

Reasoning

The LCM *must* contain all the factors of 18.

Now, in order to *also* be a multiple of 40, the LCM will have to contain all the factors in the prime factorization of 40.

Looking now at the factors of 18, what factors of 40 are we missing? Think and then read on. . . . 

We need to put two more 2s and one 5 into our prime factorization of LCM(18, 40).

The work

$$18 = 2 \cdot 3 \cdot 3,$$

$$\text{LCM}(18, 40) \text{ must contain: } 2 \cdot 3 \cdot 3$$

$$40 = 2 \cdot 2 \cdot 2 \cdot 5$$

$$\begin{aligned} \text{LCM}(18, 40) \text{ must contain: } & 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \\ \text{That is, } \text{LCM}(18, 40) = & 360. \end{aligned}$$

Reasoning

Another way of illustrating this process is to note the prime factorizations of each number and realize that the least common multiple must contain all the factors in either number with no redundancies. That is, the LCM needs to contain three 2s, two 3s, and one 5.

The work

$$\begin{array}{rcl} & 2 \cdot 3 \cdot 3 & \\ 2 \cdot 2 \cdot 2 \cdot & & \cdot 5 \end{array}$$

Using Prime Factorization and Exponents to Find the LCM

There is a more formal way to find the LCM, and this is connected to one of the ways in which we found the GCF. This method comes from representing the prime factorization of each number in exponential form:

$$18 = 2 \cdot 3 \cdot 3 = 2^1 \cdot 3^2$$

$$40 = 2 \cdot 2 \cdot 2 \cdot 5 = 2^3 \cdot 5^1$$

When finding the GCF, we took the smaller exponent of all common factors. What do you think we will do when finding the LCM? Think and read on. . . . 

In order for a number to be the LCM, it must contain *all* the factors in either number. For example, because the prime factorization of 18 contains a 2, the LCM must contain a 2. However, because the prime factorization of 40 contains three 2s, the LCM must contain three 2s. Thus, when we examine the prime factorization of each number, whenever there is a common factor, we must take the greater exponent.

Using this method, we find that $\text{LCM}(18, 40) = 2^3 \cdot 3^2 \cdot 5^1$. Do you see why? 

The only factor that 18 and 40 have in common is 2, and the greatest exponent above 2 is 3 (meaning that $2 \cdot 2 \cdot 2$ is a factor of 40). Therefore, the prime factorization of the LCM must contain 2^3 . The noncommon factors are 3 and 5, and so 3^2 and 5^1 are also placed in the prime factorization of the LCM.

Note: This procedure with exponents makes sense to some students and not to others. The point is not that everyone should use this procedure because it is the *best* procedure. Rather, the point is that this is a very efficient procedure *if* you feel comfortable with it. If you don't feel comfortable with this procedure, then you can work with it until it makes more sense or you can use another procedure.

The concepts of GCF and LCM are two essential number theory concepts. Investigation 4.10 (Cutting Squares) demonstrates one real-life application. The most direct application of these concepts in the elementary school curriculum has to do with fractions. We need to find the GCF when we are simplifying fractions, and we need to find the LCM when we are adding fractions with unlike denominators. In the remainder of this section, we will develop and explore relationships between GCF and LCM.



CHILDREN

Below are two different ways in which the concept of LCM can emerge in elementary school. Teachers may give their students a 100 chart and ask them to cross out multiples of 8 with one color and cross out multiples of 12 with another color, or to draw a circle around multiples of 8 and a square around multiples of 12, as in Table 4.4. When students discover that some

numbers have both a circle and a square around them, the teacher can ask, "How would you describe those numbers?" In this manner, the students have constructed the concept of LCM in their heads. Then, when the more formal definition is presented, they already have experience that connects to this idea.

TABLE 4.4

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Cuisenaire rods can also be used to introduce this concept to children. The teacher might ask the students to select a 4 rod and a 6 rod and then ask, "How many different ways can you

make a 4 train and a 6 train that have the same length?" (See Figure 4.11.) Do you see how the answer to this question connects to the LCM concept?

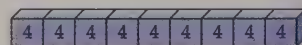
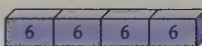
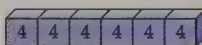
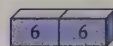
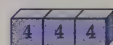


FIGURE 4.11

INVESTIGATION 4.12 Relationships Between the GCF and the LCM

It is important to note that although the GCF and LCM are *different* concepts, they are also closely related. One of the goals of this section is for you to understand each of these concepts and also to understand the many ways in which they are related. The purpose of this investigation is to help you see one of the ways in which they are connected.

Complete Table 4.5 and note what patterns you observe and what hypotheses you make. Think and then read on. . .



TABLE 4.5

a	b	$\text{GCF}(a, b)$	$\text{LCM}(a, b)$
4	6	2	12
6	10		
8	12		
9	12		
10	15		

DISCUSSION

If you take any two numbers, it just so happens that the product of those numbers is always identical to the product of their GCF and LCM; that is, for any natural numbers a and b ,

$$a \cdot b = \text{GCF}(a, b) \cdot \text{LCM}(a, b)$$

Using Venn diagrams The proof of this is beyond the scope of this book, but Venn diagrams can help us to understand *why* this relationship is true. However, using these diagrams here requires a slightly unorthodox use of the concepts of set, intersection, and union. This connection among GCF, LCM, and sets came from a tutorial session in my office with several students; one student was trying to explain the concepts to another student and suddenly “saw” the connection among GCF, LCM, and intersection and union. One of the outcomes of our discourse was a deeper understanding of the connection between the two concepts.

Let us examine this connection now. Consider the prime factorization of 18 and 40 as sets; that is,

Prime factorization of 18 = {2, 3, 3}

Prime factorization of 40 = {2, 2, 2, 5}

If we place these elements in a Venn diagram, we have Figure 4.12. How does this Venn diagram connect to the GCF of 18 and 40? How does it connect to the LCM of 18 and 40?

From this unorthodox representation, the GCF of 18 and 40 can be seen as the intersection of the two sets; that is, $\text{GCF}(18, 40) = 2$. The LCM of 18 and 40 can be seen as the union of the two sets; that is, $\text{LCM}(18, 40) = 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = 360$.

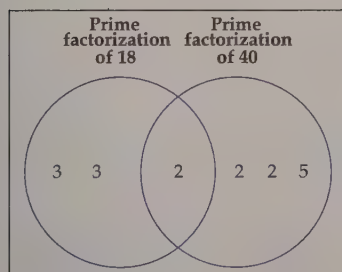


FIGURE 4.12

In this case, we see that the factors fall within three distinct regions of the Venn diagram. The first region contains those numbers that are factors of 18 but not of 40 ($3 \cdot 3$), the second region contains common factors (2), and the third region contains those numbers that are factors of 40 but not of 18 ($2 \cdot 2 \cdot 5$). Now examine the equation connecting the two concepts in this light: $18 \cdot 40 = \text{GCF}(18, 40) \cdot \text{LCM}(18, 40)$. We can see that 18 and 40 both consist of numbers from two regions, the GCF consists of numbers from the common region, and the LCM consists of numbers from all three regions.



LEARNING

For some mathematicians, the unorthodox use of Venn diagrams is heresy. However, for many mathematics educators, there is much value in having students invent (sometimes unorthodox) procedures and algorithms. This is alluded to in several places in the NCTM standards. They speak of increasing attention to “creating algorithms and procedures” (*Curriculum Standards*, p. 70). In the

professional standards, they speak of “invented and conventional terms and symbols” among Tools for Enhancing Discourse (*Professional Standards*, p. 52). One of the cornerstone assumptions of the NCTM standards is the belief in the value of “conjecturing, inventing, and problem solving—away from an emphasis on mechanistic answer-finding” (*Professional Standards*, p. 3).

Yet Another Way to Find the LCM

This relationship between the GCF and the LCM has a very practical application. It yields another way to find the LCM of two numbers. Do you see how? Think and then read on. . . . 

Consider, for example, finding $\text{LCM}(40, 72)$. The equation in Investigation 4.12 is an algebraic equation, and thus we can use the rules of algebra to solve the equation for $\text{LCM}(a, b)$. Doing so, we have

$$\text{LCM}(a, b) = \frac{a \cdot b}{\text{GCF}(a, b)}$$

As you may have already discovered, finding the GCF of two numbers is generally much easier than finding the LCM. Thus, to find $\text{LCM}(40, 72)$, we have only to find $\text{GCF}(40, 72)$ and then solve the equation. Because $\text{GCF}(40, 72) = 8$, we have

$$\text{LCM}(40, 72) = \frac{40 \cdot 72}{8}$$

If you pause for a few seconds, you should be able to see how you can determine this computation in your head. Do so and then read on. . . . 

If you first divide the 72 by 8, you then have the relatively simple multiplication problem $40 \cdot 9$, which is 360.

INVESTIGATION 4.13 Going Deeper into the GCF and the LCM

The following problem provides an opportunity to develop your problem-solving toolbox while applying your understanding of GCF, LCM, and their relationship.

If the GCF of 45 and x is 9, and the LCM of 45 and x is 135, find x .

Take a few minutes to work on this problem on your own and then read on. . . . 

DISCUSSION

There are several different ways to solve this problem. I want to take this time to emphasize the value of thinking *before* jumping into a problem. Think about the characteristics of x . What do we know about x from the given information? First, let's look at the other two numbers.

$$45 = 3 \cdot 3 \cdot 5$$

$$135 = 3 \cdot 3 \cdot 3 \cdot 5$$

If 9 is the GCF of 45 and x , that means x is a multiple of 9. Do you see why? If the LCM of 45 and x is 135, that means that the prime factorization of x has to have at least three 3s. Do you see why?

Using this line of reasoning, our first candidate for the answer is a multiple of 9 that has three 3s: $3 \cdot 3 \cdot 3 = 27$

It also turns out, not surprisingly, that 27 is the correct answer. The value of doing a bit of qualitative thinking at the beginning of the problem recalls the old proverb "A stitch in time saves nine!"

Relationships of Operations on the Set of Natural Numbers

One of the more important goals of elementary school mathematics is for students not only to be able to compute but also to understand the relatedness of various mathematical concepts and procedures. There is much evidence, from the National Assessment of Educational Progress and elsewhere, that this is not happening for many students. This represents an ironic twist. As mentioned at the beginning of the chapter, the Pythagoreans were fascinated by the properties of and relationships among natural numbers. They called the study of natural numbers *arithmetic* to distinguish it from *logistic*, by which they meant the practical art of computing with numbers. Thus, what they called logistics, and felt was beneath arithmetic, is what we have come to call arithmetic! The Pythagoreans are also credited with developing the word *mathematics*, by which they meant the following four subjects: geometry, arithmetic, music, and astronomy.

One of the important "big ideas" that the Pythagoreans meant by arithmetic is the part-whole aspect of numbers—what some people refer to as understanding how numbers are decomposed and composed. In the lower elementary grades, the primary focus is on additive decompositions. For example, a student who understands the various ways in which 10 can be decomposed— $9 + 1$, $8 + 2$, $7 + 3$, $6 + 4$, $5 + 5$, and so on—knows more than just "number facts." This decomposition is a critical part of knowing how to add and subtract when regrouping and is a critical part of good mental arithmetic.

and estimating. In the upper elementary grades, understanding multiplicative decompositions is developed. For example, 12 can be decomposed multiplicatively in several ways— $12 \cdot 1$, $6 \cdot 2$, and $4 \cdot 3$, as well as its prime decomposition, $2 \cdot 2 \cdot 3$.

Additive and multiplicative decomposition are “big ideas” that occur repeatedly throughout mathematics. Understanding additive and multiplicative relationships enables us to compare amounts in two different ways. For example, if an item costs \$10 in one store and \$15 in another store, we can compare the two prices in additive terms—it costs \$5 more in the second store—or we can compare the two prices in multiplicative terms—it costs $1\frac{1}{2}$ times as much or it costs 50 percent more in the second store. These comparisons have to do with how we choose to decompose the amounts: 15 as $10 + 5$ or 15 as $3 \cdot 5$ and 10 as $2 \cdot 5$. These two different ways of comparing amounts multiplicatively will be developed in Chapter 5 and Chapter 6.

Being aware of the relationships among these mathematical ideas (divisibility, decomposition, and additive versus multiplicative comparisons) is a key to developing what the NCTM has called mathematical power. Therefore, it is worth repeating. Because people learn and perceive ideas in different ways, I also offer the diagram in Figure 4.13, which illustrates the relationships among these three ideas. The figure shows that when we examine the way two numbers are related, we can decompose them additively or multiplicatively. In the former case, relating two numbers often leads to statements that use the words *more than* or *less than*. In the latter case, relating two numbers often leads to statements involving concepts and terms such as *divisibility*, *ratio*, *percent more*, and *percent increase*. This is only a partial map of how these three concepts are related, but it addresses the oft-asked question “Am I ever going to need this stuff?”

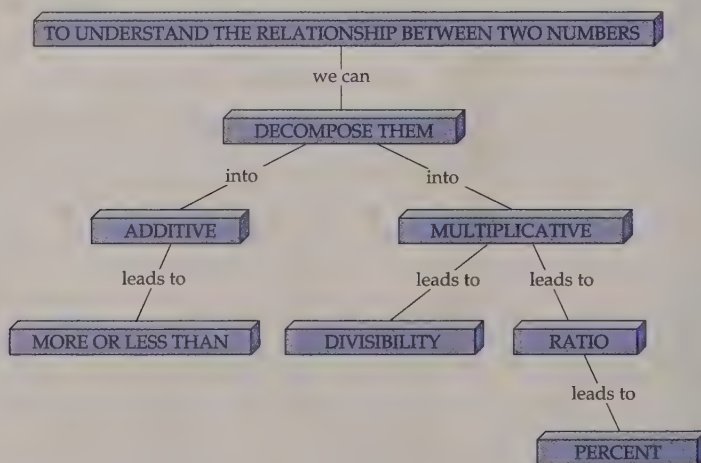


FIGURE 4.13

Summary

In this section, we have investigated the concepts of the greatest common factor and the least common multiple. In developing more efficient strategies for determining the GCF or the LCM of two or more numbers, we used concepts from the first two sections of the chapter (divisibility, prime, and prime factorization), and we again used the idea of decomposition. In this case, the additive

decompositions were not relevant; rather, the multiplicative decompositions enabled us to determine the GCF and LCM and to understand better how these two ideas are related.

These ideas of GCF, LCM, and relatively prime shed new light on our understanding of the subtler star patterns in Exploration 4.4. Some star patterns are polygons (triangles, squares, pentagons, and so on), and some star patterns look like stars. If we consider the relationship between x and n in star (x, n) and if x is a multiple of n (but not double n), then a polygon is created. Otherwise, a star is created. If x and n are relatively prime, then the star will have one orbit.

EXERCISES 4.3

- Find the following:
 - GCF(30, 75)
 - GCF(45, 54)
 - GCF(12, 35)
 - GCF(27, 189)
 - GCF(75, 144)
 - GCF(105, 132)
 - GCF(156, 910)
 - GCF(630, 1848)
- Find the following:
 - GCF(12, 30, 75)
 - GCF(12, 333, 8415)
- Find the following:
 - LCM(12, 20)
 - LCM(30, 75)
 - LCM(44, 66)
 - LCM(462, 630)
- Find the following:
 - LCM(12, 30, 75)
 - LCM(20, 30, 40)
- Find two numbers such that their GCF = 2.
 - Find two numbers, both greater than 100, such that their GCF = 2.
 - Find two numbers such that their GCF = 6.
 - Find two numbers, both greater than 100, such that their GCF = 6.
- Find the GCF and the LCM of the following pairs of numbers:
 - 25, 40
 - 72, 108
 - 252, 420
 - 3780, 3960
- The GCF of two numbers is 18. Their LCM is 630. What are the two numbers?
- The GCF of 66 and x is 11; the LCM of 66 and x is 858. Find x .
- The GCF of two numbers m and n is 12, their LCM is 600, and both m and n are less than 500. Find m and n .
- A child has a large supply of dominoes, each of which measures 32 millimeters by 56 millimeters. She wants to lay them out to form a solid square, and she wants them all to be laid out horizontally. What will be the dimensions of the smallest square that she can form, and how many dominoes will it require?
- George vacuums the rugs every 18 days, mows the grass every 12 days, and pays the bills every 15 days. Today he did all three. How long will it be before he has another day like today?
- In each of the following, explain your answer.
 - What is the relationship between a and b if $\text{GCF}(a, b) = a$?
 - $\text{GCF}(a, a) = ?$
 - If $\text{GCF}(a, b) = b$, then $\text{LCM}(a, b) = ?$
- Classify each of the following as true or false, assuming that a , b , and c are integers. If a statement is true, justify it. If it is false, give a counterexample, and then revise the statement to make it a true statement.
 - If a and b are different and prime, then $\text{GCF}(a, b) = 1$.
 - If a and b are relatively prime, then $\text{LCM}(a, b) = ab$.
 - If a and b are different and are both even, then $\text{GCF}(a, b) = 2$.
 - If $\text{GCF}(a, b) = 2$, then a and b are both even.
 - $\text{LCM}(a, b) \mid ab$.
- Ann and Bob are cycling on a track. Ann completes one lap every 12 seconds, and Bob completes one lap every 15 seconds. When will Ann lap Bob, assuming that they started together?
- A manufacturer sells widgets, each of which is packaged in a box whose dimensions are 4 centimeters by 6 centimeters by 10 centimeters. Design the dimensions of a box to ship widgets. Your box must hold at least 1000 widgets.
- I am thinking of a 2-digit number in which the sum of the digits is 9 and the number is even. These clues alone do not create one answer. Add one more clue so that there is only one answer.
- What is the largest square that can be used to fill a 6 by 10 rectangle? For example, you would need sixty 1 by 1 squares to make a 6 by 10 rectangle. However, if you started with a 3 by 3 square, you would not be able to fill the rectangle using only 3 by 3 squares. The most concise way to ask this question is "What is the largest square that can be used to tile a 6 by 10 rectangle?" We will study tiling in Chapter 9.

- b. What is the largest square that can be used to fill a 12 by 18 rectangle?
- c. Now comes the more general question: How would you go about finding the largest square that can be used to fill an x by y rectangle?
18. Obtain a set of Cuisenaire rods from your instructor or from the library on your campus.
- a. What is the longest train that can be measured by both the purple rod and the dark green rod? Note: a longer train can be measured by a shorter train if you can make the longer train by using only the shorter train. Thus, a 9 rod cannot be measured by a 2 rod, but it can be measured by a 3 rod.
- b. What is the shortest train that can be measured by both the dark green rod and the brown rod?
- c. Now comes the more general question: How would you go about finding the shortest train that can be “measured” by both a rod that is x units long and a rod that is y units long?
19. a. What is the longest train that will measure the brown rod and a train that is 12 units long?
- b. What is the longest train that will measure a train that is 15 units long and a train that is 20 units long?
- c. Now comes the more general question: How would you go about finding the longest train that will measure a train that is x units long and a train that is y units long?
20. We have talked about the GCF of two or more numbers and the LCM of two or more numbers. Why don't we talk about the LCF (least common factor) or the GCM (greatest common multiple) of two or more numbers?

CHAPTER SUMMARY

1. Prime numbers are the building blocks for the set of natural numbers.
2. Divisibility, GCF, and LCM enable us to decompose composite numbers. If elementary students understand these concepts, they can more easily compose and decompose mathematical expressions.
3. These decompositions are multiplicative rather than additive.
4. These mathematical concepts and a good problem-solving toolbox are essential to being able to see patterns that lead to hypotheses, which are then tested and either confirmed or revised.
5. Number theory connects with other mathematical concepts and fields.
6. Exploring numbers and number patterns can be fascinating and enjoyable.

Basic Concepts

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relatively prime 216

Extending the Number System

- ◆ 5.1 Integers
- ◆ 5.2 Fractions and Rational Numbers
- ◆ 5.3 Understanding Operations with Fractions
- ◆ 5.4 Beyond Integers and Fractions: Decimals, Exponents, and Real Numbers

For many thousands of years, whole numbers were adequate for most people's needs. However, the limitations of whole numbers became more and more problematic as time went on. In this chapter, we will examine four extensions of the set of whole numbers: negative numbers, fractions, decimals, and irrational numbers.¹ Most people do not realize how recent the invention of these kinds of numbers is and that these sets of numbers represent significant and important leaps in the development of mathematics. In this chapter, it is important that you come to appreciate the significance of each of these sets of numbers and that you understand how they are related to one another. In Chapter 3, we examined the various meanings of the four basic operations and how various algorithms enable us to compute quickly. In this chapter, we will examine what each of these new sets of numbers means. Knowing what they mean and understanding the four operations will then enable you to make sense of the computation algorithms that we use for integers, fractions, and decimals. As before, knowing why as well as how increases one's mathematical power:

▶ Teachers [of all grades] should be able to extend the number systems from the whole numbers to fractions and integers, then rationals and real numbers, including a discussion of the extension of the operations, properties, and ordering. Notions of fractions, decimals, percents, ratio, and proportion should be developed through problems with an applied flavor.

(Professional Standards, p. 136)

¹ Although there are still numbers beyond irrational numbers, we will limit ourselves to the sets of numbers mentioned here because these are the sets that we focus on in K–8 mathematics.

SECTION

5.1

INTEGERS

WHAT DO YOU THINK?

- Why is it that the sum of two negative numbers is a negative number, but the product of two negative numbers is a positive number?
- How do operations with integers connect with whole-number operations? How do they differ?
- Why is -8 less than -7 ?

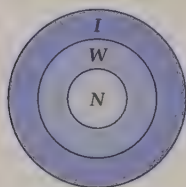


FIGURE 5.1

LANGUAGE

The term 'integer' often causes some confusion for many students because the terms 'integer' and 'whole number' do not have identical meanings in mathematics and in everyday English. In mathematics, 'integers' refers to the following set of numbers: $\{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$. In everyday English, the description of this set is not identical to the description in the first sentence of this section. That is, in everyday English, one might say that the integers can be broken down into three sets—negative whole numbers, zero, and positive whole numbers. However, 'whole numbers,' in mathematics, refers to the set $\{0, 1, 2, 3, \dots\}$. Thus, 'negative whole numbers' is like saying "he goed to the store yesterday." My resolution to this dilemma is to use the terms 'positive integers' and 'negative integers' but not the terms 'positive whole numbers' and 'negative whole numbers.'

Our first extension of the set of whole numbers is the set of **integers**, which is simply the union of the set of positive integers, the set of **negative integers**, and zero. Although most of people's everyday use of mathematics involves positive numbers, we encounter negative numbers in various ways. One of the major goals of this section is for you to understand how the procedures for computing with positive numbers are related to computing when one or more of the numbers are negative.

Integer Connections

Before we examine operations with integers, let us take a little time to see how integers connect to the set of whole numbers we have been working with up to now.

- We began our study of numbers with the set of natural numbers, N .

$$N: \{1, 2, 3, 4, \dots\}$$

- With the invention of zero, we have the set of whole numbers, W .

$$W: \{0, 1, 2, 3, 4, \dots\}$$

- With the invention of negative numbers, we have the set of integers, I .

$$I: \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

Figure 5.1 shows a Venn diagram illustrating the notion of extending our set of numbers as our ancestors invented new kinds of numbers. Each set of numbers contains the previous set. We will extend this diagram as we discuss fractions, decimals, and irrational numbers.

HISTORY

Like our base 10 numeration system, the set of integers was developed relatively recently. Negative numbers were either ignored or dismissed as absurd by most of the ancients. We find the earliest mention of negative numbers and how we might compute with them in the works of Brahmagupta (A.D. 628) in India. Other early work with the concept of negative numbers is found in the writings of al-Khwarizmi (A.D. 825) in Persia and Chu Shi-Ku

(A.D. 1300) in China. The first modern mathematician to treat negative numbers seriously was Cardan in 1545. He accepted negative numbers as solutions of equations and, like the three authors above, stated the rules for computing with them. However, he referred to them as "false" numbers. Descartes, who developed the coordinate system, still referred to negative numbers as false numbers.

Integers in our world When do we use negative numbers? Stop to reflect on where you have encountered negative numbers before reading on. . . 

People use negative numbers both on and off the job. For instance:

- Businesses use negative numbers to indicate a business deficit, or "negative profit."
- We often use negative numbers when describing change. For example, graphs often have negative numbers.

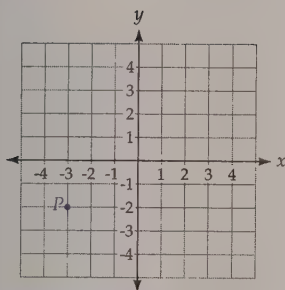


FIGURE 5.2

- We use negative numbers to indicate temperatures below zero.
- Locations below sea level are often represented with negative numbers.
- A golfer uses negative numbers to indicate a score below par.
- Physicists use negative numbers to indicate negatively charged particles.
- We use a coordinate system to indicate the position of an object in space. For example, the location of point P in Figure 5.2 is $(-3, -2)$.

Representing Integers

There are many models for representing integers. We will focus on the number line model because it is the model to which most real-life applications connect.

Number lines can be represented horizontally or vertically, as shown in Figure 5.3.

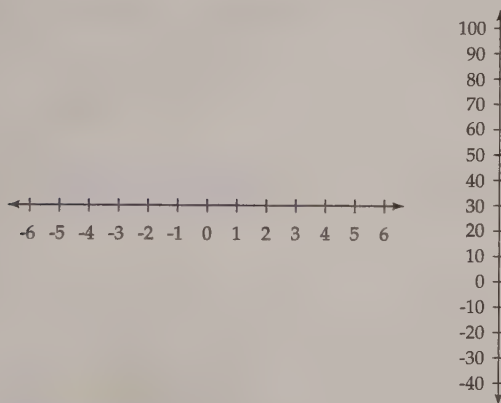


FIGURE 5.3

Some other models commonly used to represent integers and their applications include black and red chips, positively and negatively charged particles, and money (assets and liabilities). If you did Exploration 5.1, you used black and white dots to represent integers. The discrete model is a representation of integers that generally makes more sense to young children than the number line model.

Operations with Integers

In this section, we will examine the four operations with negative numbers. As you recall from Chapter 3, we developed models for each of the operations:

- addition: combine, increase
- subtraction: take away, comparison, missing addend
- multiplication: repeated addition, rectangular array, Cartesian product
- division: partitioning, repeated subtraction, missing factor

As we strive to develop ways to operate *meaningfully* with negative numbers, we will also examine which of these models for the operations work and which do not.

Many textbooks present the students with various algorithms (that is, rules) for adding, subtracting, multiplying, and dividing with negative numbers.

There are several problems with this approach. First, students remember the rules only as long as they use them. I call this rented or Teflon knowledge; after a few months, the rules are gone. If the students do not have a strong understanding of where the rules came from, then they have nothing to show for all that work. Second, in many real-life situations, the most difficult part of the problem is figuring out how best to approach it. Most real-life problems lend themselves to different solution strategies, not one “right” way that everyone should use. Thus, rather than present you with rules, we will investigate how we obtain sums, differences, products, and quotients with negative numbers. If you first solve, or at least attempt to solve, each problem *on your own*, you will find that not only will you be able to compute confidently with negative numbers, but also your problem-solving toolbox will be fuller. Furthermore, even if you forget the rules a year from now, you will be able to use your knowledge to re-create them.

Understanding Addition with Integers

We will use a problem-solving tool from higher mathematics to develop algorithms for adding integers: We will examine all possible combinations (cases) and then look for patterns. The four cases are represented below:

	<i>a</i>	<i>b</i>	Example
Case 1: Both numbers are positive.	+	+	$3 + 4$
Case 2: One number is positive, one number is negative, and the magnitude of the negative number is greater than the magnitude of the positive number.	+	−	$3 + ^{-}4$
Case 3: One number is positive, one number is negative, and the magnitude of the positive number is greater than the magnitude of the negative number.	−	+	$^{-}3 + 4$
Case 4: Both numbers are negative.	−	−	$^{-}3 + ^{-}4$

Language You may have noticed that sometimes the + and − signs have been raised (printed as superscripts) and sometimes not. This convention allows us to avoid meaningless sentences. For example, we will read the problem $^{-}6 - ^{-}8$ as “negative 6 minus negative 8” instead of “minus 6 minus 8.”

The words *plus* and *minus* will be used to refer to the *operations* of addition and subtraction. The words *positive* and *negative* will be used to refer to the *value* of the number.

For practice, translate each of the equations below into English. Check your translations with the sentences at the right.

$$5 - ^{-}4 = 9 \qquad 5 \text{ minus negative } 4 \text{ is equal to positive } 9.$$

$$32 + ^{-}48 = ^{-}16 \qquad 32 \text{ plus negative } 48 \text{ is equal to negative } 16.$$

Let us now examine the first two cases.

Case 1 Both numbers are positive. We explored this case in Chapter 3.

Case 2 One number is positive, one number is negative, and the magnitude of the negative number is greater than the magnitude of the positive number.

Do several other examples and then try to express a rule that would apply to any addition problem in which the magnitude of the negative number is greater than the magnitude of the positive number. Then read on. . . 

We will examine a specific problem in detail and then look at generalizations. How would you represent the problem $3 + -4$?

Number line model From our work with number lines in Chapter 3, we learned that the problem $3 + 4$ can be represented as shown at the left in Figure 5.4. That is, we begin at zero and move 3 units to the right, then we move 4 more units to the right; thus, we find that $3 + 4 = 7$. If a positive number is represented by an arrow pointing to the right (the positive direction), then a negative number is represented by an arrow pointing to the left (the negative direction). Thus, the problem $3 + -4$ can be represented as shown at the right in Figure 5.4. That is, we begin at zero and move 3 units to the right and then move 4 units to the left; thus, we find that $3 + -4 = -1$. As you may have already found, sometimes the sum of a positive number and a negative number is positive (for example, $12 + -5 = +7$) and sometimes it is negative, as in Figure 5.4.

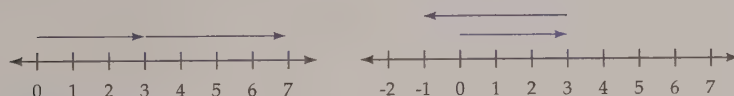


FIGURE 5.4

If you do a number of problems, you realize that the procedure for adding a positive number and a negative number feels a lot like subtraction. In fact, one description of a general rule is: Disregard the signs and subtract the *smaller* number from the *larger* number; the sign of the sum is the same as the sign of the *larger* number.

Absolute Value

There is a mathematical way to say, “Disregard the sign in front of the number,” and that is to use the term *absolute value*. Let us first define this concept and then use it to state more precisely the rule for adding a positive number and a negative number.

One way to illustrate the concept of absolute value is to consider how far the number is from zero (the origin). The **absolute value** tells us the distance of the number from zero. For example, the numbers $+5$ and -5 are both the same distance from zero, so they both have the same absolute value, which is 5 (see Figure 5.5).

With notation, we say, $|-5| = 5$. Similarly, $|+5| = 5$.

In English, we say that the absolute value of negative 5 is 5, and the absolute value of positive 5 is also 5.

We can also use a balance scale (a common manipulative used in elementary schools) to illustrate the concept of absolute value. If we place a weight under -5 and an equal weight under $+5$, the scale will balance (see Figure 5.6).

There is also language for referring to pairs of numbers whose absolute values are equal: We say they are opposites or negatives of each other. Thus, the **opposite** of $+6$ is -6 and the opposite of -6 is $+6$. Using another meaning of **negative**, we say that the negative of $+6$ is -6 and the negative of -6 is $+6$.

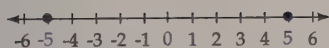


FIGURE 5.5

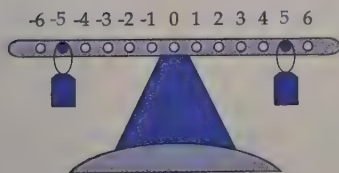


FIGURE 5.6

As you can readily see, when we add any integer and its opposite, the result is zero; that is,

$$a + -a = 0$$

Thus, we can say that every integer has an additive inverse.

In this text, we will use the term **additive inverse** instead of *negative* for two reasons. First, the term *negative* often creates a false impression. For example, if $x = -6$, then $-x = +6$. In other words, when we are working with variables, the value of $-x$ is often positive. Second, when working with fractions, we will develop a similar concept with a similar term: the *multiplicative inverse*.

The concept of absolute value lets us now state more precisely the rule for adding a positive number and a negative number: We first find the difference of the absolute values of the two numbers; the sign of the sum is the sign of the number with the larger absolute value.



MATHEMATICS

With one exception, all the properties of addition, subtraction, multiplication, and division that we developed in Chapter 3 hold for *all numbers*. That exception is the closure property. Whether or not a set is closed under an operation depends on the nature of the set. This concept will be pursued in an exercise.

The Other Two Cases of Integer Addition

Case 3 One number is positive, one number is negative, and the magnitude of the positive number is greater than the magnitude of the negative number (an example is $4 + -3$).

When we look carefully at all integer addition problems that fall in this category, we find that the generalization stated in the preceding paragraph applies to this case too. In other words, the similarity between Case 2 problems ($3 + -4$, $2 + -8$, $-7 + 3$, $-9 + 4$) and Case 3 problems ($4 + -3$, $7 + -2$, $-6 + 9$, $-1 + 5$) is that one number is positive and one number is negative. Regardless of which number has the larger magnitude, we can use the same procedure to determine the answer.

This discussion connects to NCTM Standard 7 (Reasoning and Proof), to sets language and to NCTM Standard 9 (Connections). When we first examine all the possibilities when adding integers, we have four distinct subsets. However, when we look closely at two of these subsets, we find that their similarity (one positive, one negative) outweighs their difference (which number has the greater absolute value).

Case 4 In Case 4, both numbers are negative. Let us examine a specific problem in detail and then look for generalizations. Figure 5.7 shows a representation of $-3 + -4$ on a number line. Do several other examples and then try to express a rule that would apply to any addition problem in which both numbers are negative. Many students find that they can do this more easily if they have a context. For example, Jeremy borrowed 3 dollars from Annie and then borrowed 4 more dollars from her. How much does he owe her? Then read on. . . .

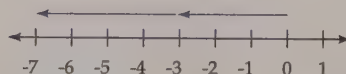


FIGURE 5.7

With the concept of absolute value, we can state the rule for adding two negative numbers precisely: To find the sum of two negative numbers, we first find the sum of the absolute values of the two numbers and then place a negative sign in front of this sum.

Understanding Subtraction with Integers

In Exploration 5.2, you examined integer subtraction with the take-away model and the comparison model. In the text, we focus on connecting whole-number subtraction to integer subtraction.

INVESTIGATION 5.1 Subtraction with Integers

Do the following subtraction problems yourself before reading on. As you work, check to make sure that you are using your understanding of integers rather than just guessing. 

1. $14 - -25 =$ 2. $-5 - 17 =$ 3. $-6 - -8 =$ 4. $-12 - 5 =$

DISCUSSION

If you found that you did better with adding negative numbers than with subtracting negative numbers, I have good news: We define subtraction of negative numbers in terms of addition, which most people understand more easily.

To make the connection stronger, let us examine how we might use what we know about subtraction to determine the answer to $4 - 6$. In one sense, we cannot “take away” 6. However, if we think in terms of a checking account, if we take away 6 from 4, we will have a deficit of 2; that is, we have -2 . We could apply our work with subtraction and number lines from Chapter 3: When we subtract one number from another, we move to the left. When we begin at the point 4 and move 6 units to the left, we end up at the point negative 2, as shown in Figure 5.8.

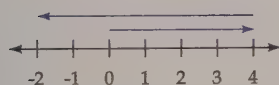


FIGURE 5.8

Thus, we can conclude that $4 - 6 = -2$.

However, this is similar to the addition problem $4 + -6 = -2$.

We use this knowledge to define subtraction of integers formally in terms of addition:

$$a - b = a + -b$$

That is, subtracting is equivalent to adding the additive inverse.

Connecting Whole-Number Subtraction to Integer Subtraction

In Chapter 3, we examined different models for the four operations and different algorithms. In order for your knowledge of mathematics to be as connected as possible, it is important that you see how we apply these models to each new set of numbers.

Go back to Chapter 3 and examine how we defined subtraction there. In what ways are the two definitions of subtraction the same (that is, equivalent)? In what ways are they different? Why did we define subtraction differently there from the way we define it here? Think and then read on. . . . 

Let us compare the two definitions:

Chapter 3: $a - b = c$ iff there is a number c such that $c + b = a$

Chapter 5: $a - b = a + -b$

If you examine the actual problems involved in subtracting a positive number from a positive number, you find that we are not adding the opposite as

much as we are taking away a positive amount or comparing the size of two sets (each with positive values). Therefore, although the definition given in this chapter seems simpler and more practical (to most of my students), it doesn't connect as well to subtraction with two positive numbers. Therefore, I chose to defer this definition until this chapter so that the work in Chapter 3 was more focused on the context in which subtraction occurs when all three of the numbers (minuend, subtrahend, and difference) are positive numbers.

Connecting Subtraction Contexts to Algorithms

One essential component of what it means to understand an operation is recognizing conditions in real-world situations that indicate that the operation would be useful in those situations.

(Curriculum Standards, p. 41)

We have now developed efficient procedures for integer addition and subtraction. However, as you have discovered, in many real-life settings, translating words into a mathematical sentence is not always simple. Therefore, we will examine a few such settings to apply our knowledge.

Problem 1 Denine realized that she had overdrawn her checking account by \$60, and she was fined \$15 for a returned check. What is her present balance? Work on this problem and then read on. . . 

If we translate this problem to mathematical language, we find that we need to take away \$15 from negative \$60. Thus, the problem is $-60 - 15$. Using our understanding of subtraction, we translate this subtraction problem to the following addition problem: $-60 + -15$. Applying our understanding of integer addition, we have an answer of -75 ; that is, her present balance is negative \$75 (or she is \$75 in the red).

Problem 2 On one day the high temperature in Nome, Alaska, was -6 degrees. On that same day, the high temperature at the North Pole was -64 degrees. How much warmer was Nome than the North Pole? Work on this problem and then read on. . . 

If we translate this problem to mathematical language, we find that we are using the comparison model of subtraction. Thus, the problem is $-6 - -64$, which we can translate as $-6 + +64$, and the answer is 58; that is, it was 58 degrees warmer in Nome.

We could also have interpreted this as a missing addend problem: $-64 + x = -6$.

Using algebra, $x = -6 - -64 = 58$.

Understanding Multiplication with Integers

There are very few simple real-world problems in which we multiply and divide integers. Most cases of integer multiplication and division occur in solving equations. Because the ability to work confidently with algebraic language is critical in high school and college, understanding the procedures is important. "The study of mathematics should emphasize reasoning so that students can believe that mathematics makes sense" (Curriculum Standards, p. 29). If you did Exploration 5.3, you realized that although the rules for integer multiplication are simple, justifying them is more difficult.

HISTORY

Our present notation, putting a minus sign in front of a negative number, was a long time in the making. The early Hindus designated negative numbers by circling them—for example, ⑤—whereas the later Hindus and Arabs represented them by placing a dot over the number—for example, 5̣—and the Chinese used red to represent positive and black to represent negative (just the opposite of what we use today). Cardan used the symbol m to represent a negative number—for example, $m5$.

INVESTIGATION 5.2 The Product of a Positive and a Negative Number

C Consider the following problem: $5 \cdot -3$. Can we apply our understanding of multiplication with positive numbers to determine the product? Think and read on. . . . 

DISCUSSION

Applying the repeated addition model of multiplication, $5 \cdot -3$ literally means to add -3 five times, that is, $-3 + -3 + -3 + -3 + -3$. We know from integer addition that the sum must be -15 . That is, we can deduce that $5 \cdot -3 = -15$.

What about $-3 \cdot 5$? Think and read on. . . . 

Trying to apply repeated addition here is problematical; we must add 5 *negative 3 times*. However, the commutative property makes the task easier: $-3 \cdot 5 = 5 \cdot -3 = -15$.

From this discussion, we can recall the rule that you may have memorized in school: The product of a positive number and a negative number is a negative number.

Two negative numbers What about $-3 \cdot -5$? None of the models for positive whole-number multiplication adapt nicely to this problem, and the commutative property does us no good here. However, we can use our knowledge that a positive times a negative is a negative and make use of patterns:

We know that $4 \cdot -3 = -12$

Thus, $3 \cdot -3 = -9$

Similarly, $2 \cdot -3 = -6$

And so, $1 \cdot -3 = -3$

$0 \cdot -3 = 0$

Recall the zero property of multiplication from Chapter 3.

$-1 \cdot -3 = ?$

What does $-1 \cdot -3$ *have to* equal if the pattern is to continue? Think and then read on. . . . 

As the number line in Figure 5.9 illustrates, in this case, each product is 3 more than the previous product, and 3 more than 0 is $+3$.

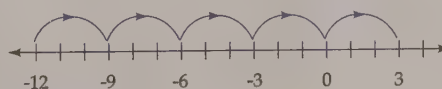


FIGURE 5.9

From this discussion, we can recall the rule that you may have memorized in school: The product of two negative numbers is a positive number.

Understanding Division with Integers

Just as we interpreted integer subtraction by applying our knowledge of the relationship between subtraction and addition, we can now understand integer division by applying our knowledge of the relationship between division and



MATHEMATICS

Leonard Euler (1770) produced one of the earliest proofs that $-1 \cdot -1 = 1$. However, many great mathematicians thought that the idea of a negative number times a negative number was absurd. Many eighteenth-century algebra textbooks rejected multiplication of negative numbers because this was not a problem that they saw as having any real-life relevance.

multiplication. From an intuitive perspective, you may sense that the multiplication rules translate quite directly to division:

- The quotient of a positive and a negative number is a negative number.
- The quotient of two negative numbers is a positive number.

We can apply the missing factor model of division ($y \div n = x$ iff $x \cdot n = y$) to verify these rules.

For example, consider the problem $-12 \div 4$. There is little doubt that the quotient is either -3 or $+3$. Many students simply guess, but we can apply this model to help us find the correct answer. Applying our definition of division, we can say that

$$-12 \div 4 = x \quad \text{iff} \quad x \cdot 4 = -12$$

What number times 4 is equal to negative 12? We know from multiplication that this number must be -3 . Therefore, $-12 \div 4$ must equal -3 .

Similarly, consider $-15 \div -3$. Applying our definition of division, we can say that

$$-15 \div -3 = x \quad \text{iff} \quad x \cdot -3 = -15$$

We know from multiplication that this number must be 5. Therefore, $-15 \div -3$ must equal 5.

Summary

In this section, we have learned that the set of integers includes both the set of positive and negative numbers and zero. We have discovered that there are many ways to represent integers, and we have explored two models for representing integers: number lines in the text and dots in Explorations 5.1–5.3. We have adapted our understanding of the four basic operations, learned in Chapter 3, so that the algorithms for adding, subtracting, multiplying, and dividing integers make sense. In the course of exploring these operations, it became important to understand several new terms: *absolute value*, *opposite*, *negative*, and *additive inverse*.

In seeking to make sense of integer operations, we find that we cannot rely solely on common sense and manipulatives, as we did with operations with the set of natural numbers. For example, $-20 \div -4$ does not readily lend itself to a concrete, real-life problem or to a division-as-partitioning representation. At some point in the study of mathematics, the students must metaphorically “leap from the nest” and be able to use more analytic methods of reasoning to construct new understandings on the foundations of old ones.

Although operations with integers are not generally introduced until middle school, this section is important for several reasons. First, teachers need to have a sense of what lies beyond the concepts their students are learning at the present time. Even if you teach younger students, it is important that you know how the more sophisticated sets of numbers connect to the set of natural numbers and that the four basic operations, which students first learn in the context of natural numbers, will need to make sense with these more sophisticated kinds of numbers. Second, many students have trouble with algebra because there are too many rules to memorize. However, as you have seen in this section, we can apply our understanding of the meaning of the four operations and the properties of the four operations to understand the procedures for operating with integers. Third, you are likely to encounter gifted students. I happened to be visiting a second-grade teacher one day when one of her students invented the procedure for integer subtraction on the spot. The

teacher was explaining how to subtract with regrouping with the problem $72 - 58$. She said, "You can't take 8 from 2." He replied, "Yes, you can; you get minus 6; then you take 50 from 70, which is 20, and 20 minus 6 is 14!"

EXERCISES 5.1

1. Perform the computations:

a. $-356 - 138$	b. $\frac{-36 + 48}{-6}$
c. $16 - (-3)$	d. $-217 + 139$
e. $-2(3 - 10)$	f. $-3 \cdot 5 + (-6 \div 2)$
g. $16 - (-3)^2$	h. $-6 + (24 \div -3)$
i. $-124 - -345$	j. $(47 + -11)/(-6)$

2. Find the missing number:

a. $-65 + \square = 173$	b. $36 - \square = 83$
c. $331 + \square = -86$	d. $-812 + \square = -223$
e. $-342 + \square = -129$	f. $-12,348 - \square = -348$

3. Show two different ways to determine the following sum:

$$-19 + -6 + -22 + 8 + -4 + 7 + 1$$

4. The origin of our symbols for $+$ and $-$ can be traced back to the practice of merchants in the Middle Ages, who used the signs p for *piu* (more) and m for *meno* (less) to indicate how much above or below a standard weight each sack was. Thus, if the standard weight was 100 pounds, a sack weighing 96 pounds would have $m4$ written on it. How much overweight or underweight is the following shipment? Describe at least two different ways you could have solved this problem.



5. The formula for converting from Fahrenheit to Celsius is given by $C = \frac{5(F - 32)}{9}$. If the temperature outside is 23° Fahrenheit, what would the temperature be in Celsius?
6. At what temperature would the Fahrenheit and Celsius temperatures be equal?
7. Here are the low temperatures (Fahrenheit) for one week in Minneapolis, Minnesota: $-19, -6, -22, 8, -4, 7, 1$. What was the mean (average) low temperature for the week?
8. The Crabby Apple restaurant lost \$2500 in January. If its net worth at the end of the month was $-\$400$, what was its net worth at the beginning of the month?
9. Jacob opens a savings account on January 1 with a deposit of \$250. He has "direct deposit," in which \$25 is deposited every other week. The bank also charges a \$3 monthly processing fee. How much money will he have at the end of the year?

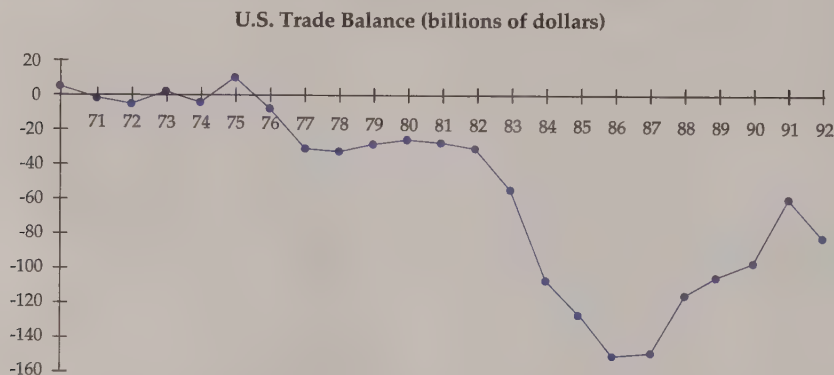
10. Have you ever been on an airplane and heard the pilot say that the plane would be a little late because it would be flying into a strong headwind or that even though the plane was taking off a bit late, you would be making up time because you would be flying with a tailwind? This problem asks you to analyze such a situation. You have the following data: A plane flying at its maximum speed can go 200 miles per hour with a tailwind or 160 miles per hour into a headwind.

- What is the wind speed?
 - What would be the maximum speed of the plane if there were no wind?
11. If you are flying to a city 800 miles away, what will be the difference in flight time between flying into a 40 mph headwind and flying with a 40 mph tailwind if the plane's maximum speed with no wind is 160 mph?
12. Let's say the countdown for a space shuttle launch has begun. At "T minus 27 hours" (that is, twenty-seven hours before launch), a problem occurs. If the technicians have not fixed the problem by T minus 8 hours, the launch will have to be scratched. How much time do the technicians have to correct the problem?
13. John had \$123 in the bank but wrote a check for \$56 and another check for \$86. What is his current balance?
14. Many Americans do not realize that our calendar is not universal. For example, the copyright date for this textbook is 2000. However, this year would be reported as 5761 in the Jewish calendar. According to the standard calendar, when was the world created?
15. In Chapter 1, we investigated magic squares, in which all rows, columns, and diagonals had the same sum. The magic square below is a subtraction magic square. Why is this?

15	7	8
5	4	1
10	3	7

- Make another subtraction magic square.
- Make a subtraction magic square in which all the numbers are negative numbers.
- Make a subtraction magic square in which four of the numbers are positive, four of the numbers are negative, and one number is zero.
- Write instructions explaining to someone how to make a subtraction magic square that works for any number.

16. The graph below shows the United States trade balance (the difference between exports and imports) from 1970 to 1992.²
- When was the lowest balance? Approximately what was it?
 - What was the net change from 1980 to 1990?
 - What was the year of the biggest decline? How much was the decline?



² *The American Almanac 1993–1994: Statistical Abstract of the United States* (Austin, TX: The Reference Press), No. 1344, p. 808.

17. Below are the low and high temperatures on a winter day in several cities. Make up and answer two questions from the chart.

	Low	High
Anchorage	-28	-7
Minneapolis	-12	5
Honolulu	68	73
Miami Beach	68	87
Portland	26	46

18. Examine some daily newspapers or weekly magazines to find articles that require some knowledge of negative numbers. Either make up a story problem that might come out of the article or explain how a knowledge of negative numbers is needed in order to understand the article. For example, recall Exercise 16 about the U.S. trade balance.

19. Consider the positive and negative values of the integers from 1 to 9:

$\pm 1 \quad \pm 2 \quad \pm 3 \quad \pm 4 \quad \pm 5 \quad \pm 6 \quad \pm 7 \quad \pm 8 \quad \pm 9$

The sum of the nine positive integers is +45, and the sum of the nine negative integers is -45. How many numbers between -45 and +45 can be made by adding and using each number or its opposite exactly once? For example, here is one combination whose sum is -3.

$$1 + 2 + 3 + 4 + 5 + 6 + -7 + -8 + -9 = -3$$

20. Let x and y represent any positive integers, $x \neq y$. For each of the operations below, tell whether the result will be always positive, will be always negative, or might be one or the other. Explain your reasoning.

- $|x - y|$
- $x^2 - y^2$
- $x^2 - xy + y^2$
- $x^2 + 2xy - y^2$

21. Let x and y represent any integers, $x \neq y$. For each of the operations below, tell whether the result will be always positive, will be always negative, or might be one or the other. Explain your reasoning.

- $|x - y|$
- $x^2 - y^2$
- $x^2 - xy + y^2$
- $x^2 + 2xy - y^2$

22. Sam was told by the doctor that he had to lose about 40 pounds. Six weeks ago he weighed 185 pounds. Below is a weekly record of his progress. How much does he weigh now? Do this problem at least two different ways.

Week	1	2	3	4	5	6
Change	-3	-2	+1	-6	+3	-2

23. The set of integers is closed under which of the four fundamental operations?

24. The term *wind chill factor* is an important concept for people living in or visiting a cold climate in winter. Most body heat is lost through the skin, and the wind can accelerate this cooling process. For example, if the air temperature is 20°F and there is a 30 mph wind, then the wind chill temperature is -2°F . That is, you are losing body heat at the same rate as though it were 2 degrees below zero on a calm day! Examine the wind chill table below.
- If the air temperature is -20°F and there is a 15 mph wind, what is the wind chill temperature?
 - Describe two observations from the chart.

Wind Chill Temperature ($^{\circ}\text{F}$)

Air temperature (°F)	Wind speed (mph)									
	0	5	10	15	20	25	30	35	40	
32	34	32	27	24	21	17	14	12	10	
28	30	28	23	19	15	12	9	6	4	
24	26	24	19	14	10	7	3	0	-3	
20	23	20	14	9	5	1	-2	-6	-9	
16	19	16	10	5	0	-4	-8	-12	-17	
12	15	12	6	0	-5	-10	-15	-19	-24	
8	11	8	1	-5	-11	-16	-21	-26	-32	
4	7	4	-3	-10	-16	-22	-28	-34	-40	
0	3	0	-7	-15	-22	-28	-35	-42	-49	
-4	-1	-4	-12	-20	-28	-35	-42	-50	-58	
-8	-4	-8	-16	-25	-33	-41	-50	-59	-67	
-12	-8	-12	-21	-30	-39	-48	-58	-68		
-16	-12	-16	-26	-36	-45	-55	-66			
-20	-16	-20	-30	-41	-52	-63				
-24	-20	-24	-35	-47	-58					
-28	-24	-28	-39	-52	-65					
-32	-27	-32	-44	-58						
-36	-31	-36	-49	-64						
-40	-35	-40	-54	-69						

Source: R.G. Steadman, "Indices of Windchill of Clothed Persons." *Journal of Applied Meteorology*, vol. 10, August 1971. Reprinted with permission from the American Meteorological Society.

SECTION

5.2

FRACTIONS AND RATIONAL NUMBERS

WHAT DO YOU THINK?

- What does a fraction represent?
- Why is $1/5$ less than $1/3$?
- What is the difference between the whole and the unit?

The concept of rational numbers is one of the "big ideas" of elementary mathematics, and it is also one of the more abstract and difficult. As you will soon discover, fractions are more than just numbers, and the numerator and denominator are more than just the top and bottom numbers.

In this chapter, we will build on the part-whole relationships developed in Chapter 2, and we will continue to use composition and decomposition (first seen with expanded form in Chapter 2) to probe more deeply into the whys of elementary mathematics.

In this section, we will introduce the four interpretations of fractions. We will explore the "fraction as a measure" interpretation in different settings and with different models so that it makes sense. We will also come to understand what it means to say that two fractions are equivalent and what finding the simplest form, commonly called reducing fractions, means.

Although the concept of fractions may be almost as old as mathematical thinking, operations with fractions are a relatively recent part of the history of mathematics. In Section 5.1, we discovered that the set of integers simply involves an extension of the number system. Fractions, on the other hand, are different from counting numbers and integers in a very significant way: *Two* numbers are needed to represent one amount! From another perspective, when we work with rational numbers, we change the question from *how many?* to *how much?* I hope that by the end of this section, you will see what a difference this makes!



CHILDREN

Children encounter the notion of rational numbers and fractions even before they enter school through the notion of sharing between two people. That is, when one whole is divided into two equal parts, each person has one-half.

Two Different Sets

Before we begin, it is necessary to make a distinction between *fraction* and *rational number*.

A **rational number** is a number whose value can be expressed as the quotient or ratio of two *integers* a and b , represented as $\frac{a}{b}$, where $b \neq 0$.

A **fraction** is a number whose value can be expressed as the quotient or ratio of *any two numbers* a and b , represented as $\frac{a}{b}$, where $b \neq 0$. For example, $\sqrt{2}/3$ is a fraction but not a rational number.

In either case, a is called the **numerator** and b is called the **denominator**.

Technically, the set of rational numbers is a subset of the set of fractions, which can include amounts like $\sqrt{2}/2$ and $\pi/6$. In elementary school, children work with fractions that are rational numbers, and these will be our primary focus in this chapter. Therefore, we will generally use the term *fraction*.

Fractions in History

The notion of $3/4$ may make sense to you now, but it is not easy for many children, and it is such an abstraction that it is relatively recent in human history.

The Egyptians expressed all fractions (with the exception of $2/3$) as unit fractions—that is, fractions whose numerator is 1. They used the symbol $\bigcirc$, which they placed above a numeral to indicate a fraction. Thus, $1/12$ was written as $\bigcirc$ III. The Egyptians' decision to represent fractional amounts using only unit fractions was a consequence of their difficulty with using two numbers to represent a single amount.

As we saw earlier, the idea of representing *all* amounts with whole numbers was very appealing to the ancient Greeks, and so they did not even consider the idea of creating numbers that were not whole numbers. Rather, they worked with ratios. For example, instead of saying that $2/5$ of the students at a college are male, they would say that the ratio of males to females is 2 to 3. (We will examine ratios more closely in Chapter 6.)

The Romans also avoided fractions. We live with the effects of one of their ways of avoiding fractions: Rather than dealing with parts of a unit, they created smaller units. Their word for twelfth was *unica*, which is where our words *ounce* and *inch* come from.

Our present method of writing fractions (for example, $\frac{2}{3}$) was probably invented by the Hindus. Brahmagupta (A.D. 628) wrote $\frac{2}{3}$. The bar seems to have been introduced by the Arabs, but just as the base 10 system took many years to be accepted, it was some time before most writers used the bar when expressing fractions. Part of the problem seems to have been caused by printing difficulties, and even today few word processors are able to express fractions vertically; instead they write $2/3$.



LANGUAGE

The origin of the word *fraction* is also interesting; it is derived from the Latin word *fractio*, which comes from the Latin word *frangere*, meaning “to break.” In early American arithmetic books, the term *broken numbers* was often used instead of the word *fraction*. The first known mention of the actual word *fraction* was by Chaucer in 1321.³

INVESTIGATION 5.3 Rational Number Contexts: What Does $3/4$ Mean?

This investigation uncovers the different meanings of fractions. Thus, although all of the following situations involve the fraction $3/4$, they represent $3/4$ in different contexts. One of the goals of our work with rational numbers is for you to understand the similarities and differences among these contexts. Thus, think carefully as you work.

³ Louis Karpinski, *The History of Arithmetic* (New York: Russell & Russell, 1965), p. 127.

Represent each of the following situations with a diagram. Then read on. . . . 

- A. Joey grew $\frac{3}{4}$ of an inch last month.
- B. Four children want to share 3 pies equally. How much pie does each person get?
- C. At a recent meeting, $\frac{3}{4}$ of the participants were women. If there were 12 people at the meeting, how many were women?
- D. At a college, $\frac{3}{4}$ of the students are women.

DISCUSSION

Do you see how the pictures in Figure 5.10 connect to the verbal descriptions?



FIGURE 5.10

Each of the situations embodies one of the four major contexts in which we use rational numbers.

A. The first situation illustrates **rational number as a measure**. To measure the appropriate location of $\frac{3}{4}$, we must divide (partition) the unit length (1 inch) into 4 equal lengths. The length of 3 of those equal lengths shows how much Joey grew.

When we are dealing with the fraction a/b in the context of measuring, we have some amount or object that has been divided into b equal amounts, and we are considering a of those pieces. In this sense, $1/b$ is a unit of measure.

B. The second situation illustrates **rational number as a quotient**. The initial problem situation is 3 (pies) divided among 4 (people). The answer of $\frac{3}{4}$ pie per person represents the value of each person's share; that is, the denominator once again is in relation to the unit (1 pie). If we divide each pie into 4 pieces, each person's share is 3 of those pieces.

When we are dealing with a fraction in the context of a quotient, an amount a needs to be shared or divided equally into b groups: In this context, a/b means that the amount a is to be distributed (or put) into b parts.

C. The third situation embodies **rational number as an operator**. In this case, $\frac{3}{4}$ acts as a function machine (recall "What's my rule?" from Chapter 2) by operating on the whole (in this case, 12 people). In this example, the input is 12 people, and $\frac{3}{4}$ is the rule that tells us what to do—that is, take 3 of every 4 people in the whole. The output is thus 9 people.

The operator context can be thought of as a stretching or shrinking process; a/b is a function machine that tells us the extent to which the given object or amount is to be stretched or shrunk. This context has also been called "the most algebraic of the basic ideas."⁴

⁴Thomas E. Kieran, quoted in Frances R. Curcio and Nadine S. Bezuk, *Understanding Rational Numbers and Proportions*, Reston (VA.: NCTM, 1994), p. 3.

D. The fourth situation illustrates **rational number as a ratio**. That is, if $3/4$ of the students are women, and if we count all the men as 1 group, then the number of women would be equal to 3 of those groups. In this case, we can represent this statement in fraction language— $3/4$ of the students are women—or we can represent the statement in ratio language—the ratio of women to men is 3 to 1—3:1.

A ratio is a relationship between two quantities. Ratios can involve comparisons between parts and parts (for example, women:men = 3:1), comparisons between parts and wholes (women:total = 3:4), or comparisons between two different wholes (when we say that a car averages 30 miles per gallon, we are comparing the number of miles traveled to the number of gallons consumed).

We will examine the measure context in this section, the quotient and operator contexts in Section 5.3, and the ratio context in Chapter 6. However, we shall soon find that they are highly interconnected and that many problems can be interpreted in more than one context. See Exploration 5.11 (Meanings of Operations with Fractions).

What do these contexts have in common? There are certain important ideas that abide in all four contexts:

- Each context can be interpreted in part-whole relationships.
- Something is to be partitioned into parts of equal size (value).
The something can have a value of 1, in which case the unit = the whole.
The something can have a value $\neq 1$, in which case the unit $\neq$ the whole.
- The numerator and denominator are like codes that tell us about the relative sizes of the parts and the unit, and the code is multiplicative in nature. For example, when we say $1/2$, it is not the difference between the two numbers that contains the key to the value; rather, it is the fact that the value of the denominator is twice the value of the numerator that contains the key. Thus, $1/2$ has the same value as $4/8$, not $7/8$.

It is also important to note that the diagrams in Figure 5.10 also illustrate three basic ways in which we can represent rational numbers: **length models** (e.g., number lines and Cuisenaire rods), **area models** (e.g., squares, circles, Geoboards, Pattern Blocks™), and **set models** (e.g., discrete circles and egg cartons). These are not the only representations (for example, we can speak of volume models), but these three have been found useful in helping children to understand fraction concepts and useful in terms of representing problems visually.

The unit and the whole are not always the same! Equating the unit with the whole is one of the most common misconceptions that people have about working with fractions. Let us revisit the four problems posed in connection with Figure 5.10. In each case, what is the whole? What is the unit? What do we mean by whole and by unit? Think and then read on. . . 

In the first case, the whole is $3/4$ inch—that is how much Joey grew; the unit is 1 inch. In the second case, the whole is 3 pies; the unit is 1 pie. In the third case, the whole is 12 people; the unit is 1 person. In the fourth case, the whole is the number of students at the college; the unit is 1 person.

The whole is the given object or amount. The unit is that amount to which we give a value of 1—1 inch, 1 pie, 1 person. In some cases, the whole and the unit are coincident (the same). For example, if Lisa gets $1/2$ of a pizza and Liam gets $1/3$ of the pizza, the whole is 1 pizza and the unit is also 1 pizza. However, if Ramon buys $2\frac{1}{2}$ gallons of gas and his brother takes $1/2$ gallon, the whole is

$2\frac{1}{2}$ gallons and the unit is 1 gallon. The need to think about wholes and units comes up in several explorations (5.7, 5.10, 5.11, 5.12 and 5.13) and throughout this chapter.

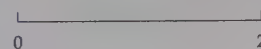
Let us now investigate the context of rational number as measure.

INVESTIGATION 5.4 Fractions as Measure

Do each of the following before reading on. If possible, compare answers and explanations with a partner.

A. If  = 1, show $\frac{4}{5}$.

B. Place $\frac{5}{6}$ on this number line.



C. If  = 1, show $\frac{5}{6}$.

D. If  = $\frac{4}{3}$, show 1.

DISCUSSION

A. This question is relatively straightforward: Divide the rectangle into 5 equal regions, and shade in 4 of them [Figure 5.11(a)].

B. This question requires you to grapple with the difference between the unit and the whole.

In this case, the whole and the unit are not the same. This is important, because the meaning of the denominator is *in relation to the unit*. That is, to find the location of $5/6$, we do not take the whole line and divide it into 6 equal lengths. Rather, we first must determine the unit length and then divide that length into 6 equal lengths. Recall the wire problem (Investigation 1.15). In that problem, the unit is 1 inch but the whole is 50 inches. This difference between the whole and the unit cannot be overemphasized [Figure 5.11(b)].

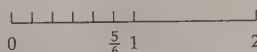
C. To make sense of this question, we must partition the dots into 6 equal-size groups. Recall the partitioning model of whole-number division. We then take 5 of those groups to show $5/6$ [Figure 5.11(c)].

D. For many students, this question is the most difficult. There are different ways to make sense of this situation and answer the question. For example, we can focus on the numerator, which indicates that we have 4 equal parts. When we then focus on the denominator, we find that 3 of these equal parts represent the unit (that is, 3 of those parts have a value of 1).

On the other hand, we can interpret this statement from a ratio context. For example, if 8 dots have a value of $4/3$, then 4 dots will have a value of $2/3$, and so 2 dots will have a value of $1/3$. Many people feel more confident once they get to a unit fraction; they reason that if 2 dots have a value of $1/3$, then 6 dots will have a value of $3/3$ —that is, 1 [Figure 5.11(d)].



(a)



(b)



(c)



(d)

FIGURE 5.11

Let us work with another problem to develop these fraction ideas further.

INVESTIGATION 5.5 Determining an Appropriate Representation

Jose paid \$6 for a box of chocolates that weighed $\frac{3}{4}$ pound (see Figure 5.12). What is the price of 1 pound (at this rate)? Work on this and then read on. . . .



FIGURE 5.12

DISCUSSION

An area model is an appropriate representation, because boxes of chocolates are often rectangular in shape. If we look at the box in terms of weight, we have $\frac{3}{4}$ of a pound. If we look at the box in terms of money, it costs \$6. In one sense, we are saying that $\frac{3}{4}$ of a pound is equivalent to \$6. That is, there is a functional relationship between $\frac{3}{4}$ (box) and 6 (dollars). There are many strategies for transforming this information into an answer.

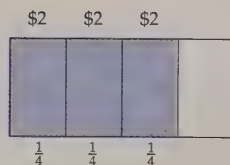


FIGURE 5.13

STRATEGY 1: "See" the multiplication in the question

Thinking of the problem as "we have 3 parts, which are equivalent to \$6" invites the question. "What number do we add three times to get 6?" In other words, "3 times what is 6?"

From this perspective, we realize that each part (that is, each quarter-pound) has a value of \$2 (see Figure 5.13), and so one pound will cost \$8.

STRATEGY 2: Apply the function concept

Input	Output	Reasoning
$\frac{3}{4}$ pound	\$6	Essentially, we now have the rule (recall "What's my rule?").
$1\frac{1}{2}$ pounds	\$12	We are using the idea of proportional reasoning, which we will explore in more detail in Chapter 6. If we double the amount, we must double the price.
3 pounds	\$24	We double the amount again. Why do we do this?
1 pound	\$8	Most people work more confidently with whole numbers and can now confidently reason that if 3 pounds cost \$24, then 1 pound costs \$8.

It is important to note that there is no one "right" way to use the function concept. We could also have reasoned that if $\frac{3}{4}$ pound costs \$6, then $\frac{1}{4}$ pound must cost \$2, and therefore 1 pound must cost \$8.

Let us continue our investigation of the meaning of fractions with a story problem.

INVESTIGATION 5.6 How Many Blue Balloons?

Josh goes through a large bag of balloons containing red and blue balloons. He finds that $\frac{1}{8}$ of them are blue. He also notes that there are 120 more red balloons than blue balloons. How many blue balloons are in the bag? Work on this and then read on. . . .

DISCUSSION

Balloons are discrete amounts, and so we could represent this problem with a discrete model. However, this would be tedious; drawing 120 more red balloons would take some time! In this case, even though the actual problem involves discrete objects, a more appropriate model to represent the problem is an area model.

This idea of constructing appropriate representations for a problem situation is one that has been neglected in school mathematics and one that the NCTM and other educational bodies (for example, the National Science Teachers Association) are encouraging teachers to emphasize more. As mentioned in Chapter 1, Representation was not a separate standard in the 1989 NCTM Curriculum and Evaluation Standards. Partly for the reason described above, a decision was made to make Representation one of the basic standards in school mathematics. Do you understand Figure 5.14? If you were unable to solve this problem, before reading on, try to solve the problem using Figure 5.14 and then read on. . . .

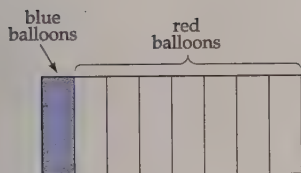


FIGURE 5.14

STRATEGY 1: Use reasoning

If we now try to connect the problem to Figure 5.14, one way (of many) to verbalize this would be to say that the blue balloons are represented by 1 box, and the red balloons are represented by 7 boxes. We unlock the secret to the solution when we connect the figure to the other piece of given information: There are 120 more red balloons than blue balloons. This means that 6 boxes must represent 120 balloons. Consider why this is and then read on. . . .

From this, we can conclude that 1 box represents 20 balloons. Thus, there are 20 blue balloons and $7 \cdot 20 = 140$ red balloons.

Check this solution with the original problem. Does this answer work?

If there are 20 blue balloons and 140 red balloons, then there are 160 balloons in all. Because 20 is $\frac{1}{8}$ of 160 and 140 is 120 more than 20, our solution fits the given information.

STRATEGY 2: Use algebra

Those students who are fluent algebraically may have wondered why we didn't use algebra. In this case, if you are comfortable with algebra, it certainly is quicker. However, there are several ways to do the algebra. Try to solve this problem algebraically and then read on. . . .

One algebraic solution path:

- We can let $x =$ the number of blue balloons.
- We can then represent the number of red balloons as $x + 120$.
- This means that $x + (x + 120)$ —that is, $2x + 120$ —represents the total number of balloons.

- Because $1/8$ of them are blue, we can say that $x = \frac{1}{8}(2x + 120)$.
- It is important to note that we can connect language nicely to the equation; that is, the number of blue balloons is $1/8$ the total number of balloons.
- When we solve this equation for x , we find that $x = 20$.

Another algebraic solution path:

- If $1/8$ of the balloons are blue, that means $7/8$ of them are red.
- If we let $x = \text{the number of balloons in all}$, we have the following equation:

$$\frac{7}{8}x - \frac{1}{8}x = 120$$

- As before, we can connect language nicely to the equation; that is, the difference between the number of red and blue balloons is 120.
- Solving this equation for x , we get $x = 160$, and $1/8$ of 160 is 20.

Not only are there different ways to solve this problem algebraically, but it is also important to note that x did not represent the same thing in the two cases we examined. This is another reason for taking the time on your paper to make sure that you (and the reader) know what the x refers to.

Equivalent Fractions

Let us now examine the concept of equivalent fractions.

Two fractions are **equivalent fractions** if they have the same value.

When you made your own manipulatives in Exploration 5.5, you could *literally* see the equivalence of certain fractions as you developed strategies to cut the unit squares or circles into pieces. However, it is not always obvious that two fractions do indeed have the same value.

Almost all of the essential fraction ideas surface in the concept of equivalent fractions, and this concept permeates many of the fraction problems that we encounter. One of the biggest challenges in approaching this section is that you already know the procedures for finding equivalent fractions. For example, you know that $3/4$ and $6/8$ are equivalent fractions because you can multiply $3/4$ by $2/2$.

$$\frac{3}{4} = \frac{3}{4} \cdot \frac{2}{2} = \frac{6}{8}$$

As we have stressed repeatedly in this book, mathematical power and number sense come not just from knowing procedures but also from knowing why these procedures work and how they connect to other mathematical ideas. Therefore, we will do some work to make sense of the procedures for determining equivalent fractions. To the greatest extent possible, try to assume that you don't know the procedure when answering the questions.

Equivalence Please address these two questions before reading on. . . . 

- What does “equivalent” mean?
- Where else have you encountered the notion of equivalence in mathematics and in life outside school?

One way of looking at “equivalent” comes from taking apart the actual word: *equi-valent*, or equal value. We needed the notion of equivalence when we added and subtracted whole numbers with regrouping; for example, 1 ten and 2 ones is equivalent to 12 ones. We use equivalence with money every day; for example, one quarter is equivalent to 25 pennies.

Using models Similarly, two fractions are equivalent if they have the same value. Using this notion, can you illustrate the equivalence of $\frac{3}{4}$ and $\frac{6}{8}$ using one or more of the fraction models we have discussed: area, length, or set? Do this before reading on. . . . 

We can use the set model to illustrate equivalence. For example, consider a set of 8 dots [Figure 5.15(a)]. If we take 6 of them, we literally have $\frac{6}{8}$ [Figure 5.15(b)]. However, we can also partition this set of 8 dots into 4 equal groups of 2 dots; if we then take 3 of these 4 equal groups, we have, by definition, taken $\frac{3}{4}$ of the set [Figure 5.15(c)].



(a)



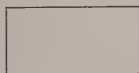
(b)



(c)

FIGURE 5.15

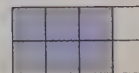
However, we could also have used an area model. For example, consider a rectangle representing 1 [Figure 5.16(a)]. We can partition the rectangle into 4 equal regions and shade in 3 of them to represent $\frac{3}{4}$ [Figure 5.16(b)]. However, if we draw a horizontal line through the middle of the rectangle, the rectangle has now been partitioned into 8 equal regions. Because 6 of those regions have been shaded in, we have, by definition, $\frac{6}{8}$ of the rectangle [Figure 5.16(c)].



(a)



(b)



(c)

FIGURE 5.16

Patterns and Order: Approaches to Procedures

Patterns in a sequence As stated before in this book, mathematics has been referred to as “a science of pattern and order.” Let us examine equivalent fractions from the perspective of pattern and order. Look at the sequence of equivalent fractions below. What “patterns” and “order” do you notice? What is the next fraction in the sequence? Why? 

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} \dots$$

Some of the many aspects of pattern and order include the following:

1. As we move from one fraction to another, the numerator increases by 3 and the denominator increases by 4. In this sense, the $\frac{3}{4}$ acts as a “function machine.”
2. The numerator of each of the fractions is a multiple of 3, and the denominator of each of the fractions is the same multiple of 4.

CHILDREN

Many teachers use paper folding to introduce the idea of equivalent fractions. For example, take a piece of paper and fold it in half. Then fold it in half again. [Note: There are several ways to do this.] Shade 3 of the 4 regions to show $\frac{3}{4}$. Now fold the paper in half again. Now we see that the paper has been folded into 8 regions and that 6 are shaded. Thus $\frac{3}{4} = \frac{6}{8}$; that is, these two fractions represent the same *part* of the whole.

3. All the denominators are even numbers, and every other numerator is an even number.

We can translate the first statement into notation:

Specific example: $\frac{3}{4} = \frac{3+3}{4+4} = \frac{6}{8}$

More general case: $\frac{a}{b} = \frac{\overbrace{a+a+\cdots+a}^{n \text{ times}}}{\underbrace{b+b+\cdots+b}_{n \text{ times}}} = \frac{an}{bn}$

We can also translate the second statement into notation:

Specific example: $\frac{3}{4} = \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8}$

More general case: $\frac{a}{b} = \frac{a \cdot n}{b \cdot n} = \frac{an}{bn}$

Another pattern Another aspect of equivalent fractions is used extensively in algebra. Look at the following pairs of equivalent fractions. What can you say about each pair?

$$\frac{3}{4} = \frac{6}{8} \quad \frac{5}{6} = \frac{10}{12} \quad \frac{3}{8} = \frac{9}{24}$$

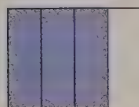
In each case, the product of the first numerator and the second denominator is equal to the product of the second numerator and the first denominator. This relationship (frequently referred to as **cross multiplication**) is used (often incorrectly) as a shortcut in solving equations. It is expressed most abstractly as

$$\frac{a}{b} = \frac{c}{d} \quad \text{iff} \quad ad = bc$$

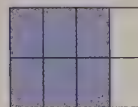
Addressing a Common Difficulty in Language

When trying to explain why $3/4$ and $6/8$ are equivalent, many students use the word *divide*—for example, “We divided the rectangle in half, and now there are 8 equal regions compared to 4 before.” This choice of words is interesting and points to a problem that many children have in trying to understand equivalent fractions at a conceptual level. When we look at why fractions are equivalent, we commonly encounter this word *divide*. However, in the procedure for creating equivalent fractions, we *multiply* the top and bottom by the same number! If we physically divide, why do we mathematically multiply? Think about this and then read on. . . . 

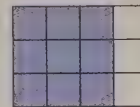
There is no simple answer to this question. It has to do with the reciprocal relationship between multiplication and division. Figure 5.17(a) shows $3/4$. If we divide each of the regions by 2, we are also multiplying the total number of regions by 2 [Figure 5.17(b)]. We now have 6 out of 8 regions shaded—that is, $6/8$. Starting with $3/4$, we could have divided each region into 3 smaller pieces. By dividing each region into 3 smaller pieces, we are multiplying the number of pieces by 3, and we can name this shaded amount $9/12$ [Figure 5.17(c)].



(a)



(b)



(c)

FIGURE 5.17

Simplest Form

A fraction is in **simplest form** if the numerator and the denominator are relatively prime numbers. The notion of simplifying fractions is clearly connected to the concept of equivalent fractions, and we will discuss it here. One important connection is that when we are simplifying fractions, we are essentially finding an equivalent fraction in which the numerator and denominator are smaller (and thus simpler) numbers. For example,

$$\frac{15}{20} = \frac{15 \div 5}{20 \div 5} = \frac{3}{4}$$

As has been true for other procedures, there are many strategies for simplifying fractions. Before we discuss them, simplify the following fractions yourself: $24/40$, $42/60$, and $63/105$. Did you use the same procedure in each case? Did you encounter any difficulties? How did you overcome those difficulties? 

One strategy is to use guess–test–revise. For example, you might divide both numerator and denominator by 4, which produces $6/10$. A quick glance reveals that this can be simplified further to $3/5$.

$$\frac{24}{40} = \frac{24 \div 4}{40 \div 4} = \frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$$

Another strategy is to determine the prime factorization of each number and then either cross out the common factors or select the “uncommon” factors.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

$$40 = 2 \cdot 2 \cdot 2 \cdot 5$$

Let us examine more closely what happens when we are able to simplify the fraction in one step, as we did above to simplify $15/20$. What is the relationship between the divisor and the original numerator and denominator in each of the three fractions given above ($24/40$, $42/60$, and $63/105$)?

$$\begin{array}{ccc} \frac{24 \div 8}{40 \div 8} & \frac{42 \div 6}{60 \div 6} & \frac{63 \div 21}{105 \div 21} \end{array}$$

In each case, to simplify the fraction in one step, we divide both the numerator and the denominator by their GCF. By doing so, we ensure that the resulting numerator and denominator will have no factors in common. In other words, a fraction is in simplest terms when the numerator and the denominator are relatively prime.

Developing fraction sense One goal of this work and Exploration 5.8 is to develop “fraction sense.” Number sense simply cannot be directly taught; rather, it emerges from being aware of the connectedness of and subtle relationships among various concepts and procedures. We see this awareness all the time

LANGUAGE

Just as we recommend not using the terms *carry* and *borrow* when regrouping during addition and subtraction of whole numbers, many mathematics educators recommend not using the term *reduce*, but rather using the term *simplify* for this process. Why might they make this recommendation?

when we watch experts at work: the dancer who can “feel” the subtle rhythms in a new piece of music, the electrician who rewired my house who could tell so much about the wiring problems from so little evidence, the mechanic who can look at a nut and know that he needs a 5/16-inch wrench.

As you are coming to realize, the set of fractions is in many ways very different from the set of whole numbers. Another important difference between fractions and whole numbers is that the whole numbers are evenly spaced on the number line. This is not true for fractions.

INVESTIGATION 5.7 Ordering Rational Numbers

Arrange these fractions from smallest to largest without converting to equivalent fractions or converting to decimals—that is, by focusing on fraction concepts and reasoning tools. Think and then read on. . . . 

$$\frac{3}{4} \quad \frac{2}{5} \quad \frac{5}{6}$$

DISCUSSION

One of the reasons for assigning Exploration 5.5 is to increase the chance that all students will develop a sense of the relative size of fractions through hands-on work with fractions with different denominators. There are a variety of ways in which students can validly answer the question. We will explore several of these ways with the intention of refining or expanding your “fraction sense” toolbox.

Some people start by looking for the largest or the smallest, whereas others start by picking two whose order they know. In this example, let us start that way. We know that $\frac{2}{5} < \frac{3}{4}$. An area diagram would readily show this.

Since one goal of this course is to develop mathematical reasoning (Standard 7), let us examine tools that go beyond visual evidence (i.e. “it looks bigger”) which has limitations, e.g. which is bigger: 17/20 or 5/7?

We can use another tool, which we will use in many other situations, to conclude that $\frac{2}{5} < \frac{3}{4}$. This tool uses the fraction 1/2 as a reference point or benchmark.

We know that 3/4 is greater than 1/2 because 3 is more than 1/2 of 4; similarly, we know that 2/5 is less than 1/2 because 2 is less than 1/2 of 5. Thus, we can conclude that $\frac{2}{5} < \frac{3}{4}$.

The next debate concerns 5/6 and 3/4. How would you explain which is larger? Think and then read on. . . . 

Looking at the two fractions, we see that they are both one piece away from 1. Because sixths are smaller than fourths, we can reason that 5/6 must be greater than 3/4 because the distance between 5/6 and 1 is 1/6, whereas the distance between 3/4 and 1 is 1/4 (Figure 5.18).

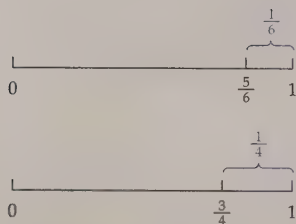


FIGURE 5.18

The Denseness of the Set of Rational Numbers

This kind of activity brings up a question that children sometimes ask: Can we name *any* point on the number line with a rational number? What do you think? . . . 

For example, name a rational number between $1/2$ and $3/4$. If we did this with a whole class, we would get a number of correct responses, although $2/3$ might be the most common.

Can you name another rational number between $1/2$ and $3/4$? How many can you name?

In fact, we can say that between any two rational numbers, there are an infinite number of rational numbers. Mathematicians refer to this property by saying that rational numbers are **dense**. Think of naming any point on a number line, knowing that no matter how close two rational numbers are, we can find an infinite number of rational numbers between those two rational numbers.

At the same time, there are points on the number line that cannot be named by rational numbers! Can you think of any such points? Think and then read on. . . . 

Such points (for example, $\sqrt{2}$ and π) are called **irrational numbers**; these numbers cannot be expressed as a ratio between two integers. We will explore such numbers in Section 5.4.

Summary

This section and Explorations 5.5 to 5.8 lay the groundwork for understanding operations with fractions. In this section, we have seen that there are four major interpretations of fractions—as measures, as quotients, as operators, and as ratios. We have seen that we need to pay attention when we work with fractions because the fraction concept is not at all a simple one.

You have learned that a fraction is not simply a number; rather, a fraction expresses a relationship between two quantities. The numerator and denominator can be seen as a code that tells us the relative size of the fraction.

In order to work effectively with fractions as measures, we need a strong understanding of several basic ideas:

- We are dealing with part-whole relationships.
- Something is to be partitioned into parts.
- All of the parts must have the same value.
- We need to take care not to confuse the unit and the whole, which are the same in some situations but not in others.
- We can use various models to represent fractions and fraction situations. In this book, we emphasize number line, area, and discrete models.

We then applied these ideas in order to understand more fully the notions of equivalent fraction and simplifying fractions. A deeper understanding of rational numbers enables us to sense their relative size; this fraction sense is useful in estimating and problem-solving.

If you look back over the investigations in this section and Explorations 5.5 to 5.8, you will find that to the extent that you could apply these ideas, you were able to develop strategies that would enable you to solve the problems confidently. These ideas will continue to be important as we explore other fraction ideas and algorithms in the next section.

EXERCISES 5.2

- Let's say that Amy made 15 out of 21 free throws and Jane made 8 out of 12. Which student had a better rate of success? Explain your answer.
- Two elementary school principals are comparing attendance figures for their schools.

	Total enrollment	Absent
Wheelock	264	32
Fuller	402	58

Which school had the larger fraction of students absent? Justify your answer.

- In each of the questions below, justify your answer.

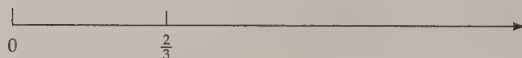
- If  has a value of $4/5$, draw and shade in the amount that would have a value of 1. Justify your answer.

- If  = $1\frac{1}{3}$, then shade in the amount equal to 1. Justify your answer.

- If  = $3/4$, show 1.

- If  = $5/4$, show 1.

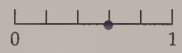
- On the number line below, mark the approximate location of 1. Explain your reasoning.



- On the number line below, approximate the value of x . Explain your reasoning.



- Name a fraction between $2/3$ and $3/4$.
 - Name a fraction between $1/4$ and $1/5$.
 - Name a fraction closer to 1 than to $7/8$.
 - Name a fraction between $1/4$ and $1/2$ that has a denominator of 11.
- Let's say you are making rulers for your students to help them learn fractions. You have decided to have a ruler marking every $1/6$ inch and every $1/8$ inch. How many different marks will there be per inch?
 - Draw three diagrams to represent each of the following fractions. Justify your diagrams.

	Length	Area	Discrete
$\frac{3}{5}$			
$1\frac{2}{3}$			
$\frac{9}{4}$			

- Look at the sequence below and look at the multiplication table in Chapter 3. Do you see a connection? Can you explain the connection, or is it just coincidence?

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \dots$$

- Simplify the following fractions.

- $\frac{25}{40}$
- $\frac{32}{48}$
- $\frac{54}{60}$
- $\frac{26}{65}$
- $\frac{168}{216}$
- $\frac{84}{132}$
- $\frac{493}{510}$
- $\frac{101010}{505050}$

- Arrange the following fractions in order from least in value to greatest in value without converting to decimals or finding the LCM. Explain your reasoning.

- $\frac{3}{4}, \frac{4}{5}, \frac{5}{15}$
- $\frac{6}{7}, \frac{5}{11}, \frac{2}{3}$
- $\frac{1}{3}, \frac{2}{5}, \frac{5}{8}, \frac{3}{4}, \frac{3}{50}$
- $\frac{2}{5}, \frac{5}{6}, \frac{4}{7}, \frac{7}{8}, \frac{3}{10}$

- Is $10/13$ closer to $1/2$ or to 1? Justify your choice.

- Which of the following fractions is the farthest from 1? Explain why.

$$\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$$

- In the following exercise, $0 < a < b < c$. For the following pairs, tell which is larger and explain why, or explain why you cannot be sure which is larger. Try to answer the questions using only reasoning rather than plugging in numbers. You may then substitute actual numbers to check your reasoning.

$$\text{a. } \frac{a}{b} \text{ or } \frac{a}{c} \quad \text{b. } \frac{a}{b} \text{ or } \frac{b}{c} \quad \text{c. } \frac{a}{c} \text{ or } \frac{b}{c}$$

12. The fraction of Fortune 1000 companies that employed undercover security agents in 1974 was $\frac{1}{10}$. In 1985 it was $\frac{1}{2}$. Compare the two numbers.
13. Let's say you decide to give blood at a blood drive. The body of an average adult contains about 5 quarts of blood. If you donate 1 pint of blood, what fraction of your blood do you give?
14. You are driving to Memphis, 115 miles away. You have 24 miles to go. Approximately, what fraction of the trip is left? Justify your choice of fraction.
15. You have from 10 to 11:30 P.M. to do a project.
 - a. At 11, what fraction of the time remains?
 - b. At 11:20, what fraction of the time remains?
16. In 1950 the annual per capita consumption of eggs was 395; that is, the average person ate 395 eggs per year. In 1990 the per capita consumption had dropped to 236. Fill in the blank with a fraction: In 1990 the average person eats ____ as many eggs as the average person did in 1950.
17. Below are the median prices of homes sold in 1991, by region of the country:

South	\$88,200
Northeast	\$136,700
Midwest	\$76,800
West	\$147,600

The median selling price of a home in the South was approximately what fraction of the median selling price of a home in the West?

18. PhonesRUs did a survey to determine telephone usage at a college. On the day the survey was done, a total of 5243 calls were made.
 - a. If 1371 calls were made to directory assistance, approximately what fraction of the total calls were made to directory assistance? Justify your choice.
 - b. The three pizza restaurants that delivered pizza received 737 calls. Approximately what fraction of the total calls were for pizza? Justify your choice.
19. In 1997 the average salary for beginning public school teachers was \$25,012, and the average salary for all public school teachers was \$38,436. The average salary for beginning public school teachers is approximately what fraction of the average salary for all public school teachers?
20. The data below show the area (in square miles) of several states and some of the Great Lakes:

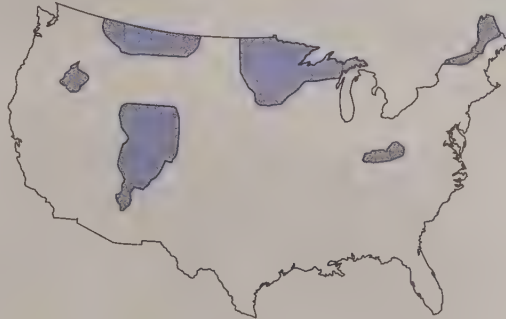
Lake Superior	31,800
Lake Huron	23,010
All five Great Lakes	94,710
New Hampshire	9,279
California	158,706
Tennessee	42,104
Rhode Island	1,212

For each of the questions below, using mental arithmetic and estimation, find a fraction with a small, "friendly" denominator (2, 3, 4, 5, 6, 8, 10, or 12) that most closely approximates the actual fraction. Justify your reasoning.

- a. Lake Huron is approximately what fraction of the size of Lake Superior?
 - b. New Hampshire is approximately what fraction of the size of Lake Superior?
 - c. Lake Superior is approximately what fraction of the size of Tennessee?
 - d. The area of all five Great Lakes is approximately what fraction of the size of California?
21. In 1996 there were approximately 4,800,000,000 people in the world. Using mental math and estimation, find a fraction to represent the fraction of the world's population that belonged to each of the following religions. Justify your reasoning.

Christianity	1,548,500,000
Islam	817,000,000
Hinduism	645,500,000
Buddhism	295,600,000
Judaism	17,800,000
Total	3,324,400,000

22. Each year the Weather Channel displays a map to indicate those portions of the country that will be celebrating a "white Christmas." If the shaded areas represent those regions that will have snow on the ground on Christmas Day, approximately what fraction of the contiguous United States will have a white Christmas? Explain your answer.



23. Give approximate values of this function for the given values of x :
- $$f(x) = \frac{2x - 1}{3x + 7}$$
- a. $x = 3$
 - b. $x = 10$
 - c. $x = 100$
 - d. What do you observe about the change in the value of the function as x gets larger and larger?

24. Write a fraction to represent the shaded portion. Justify your response.



25. If the numerator and the denominator of a fraction are increased by the same amount, is the new fraction greater than, equal to, or less than the original fraction? Justify your answer. Read this problem carefully; it is not a trick problem, but you need to make sure you interpret the wording correctly.
26. The music teacher would like to have the same number of girls and boys in the chorus. She finds that $\frac{5}{8}$ of the chorus are girls but that if she can get 12 more boys, the chorus will have the same number of boys and girls.
- How many students are in the chorus?
 - If there are 216 children in the school, which unit fraction would you select to represent the fraction of students in the present chorus?
27. Darcy loves tomatoes, but she lives in Minnesota, where the growing season is shorter than in most of the country. Therefore, she begins her tomato plants inside. However, she tells her friend that there were some casualties this spring. One-third of the plants were destroyed by her cat. Furthermore, one-fourth of the remaining plants were destroyed by a disease. What fraction survived?
28. I recall a teacher asking for fractions that were equivalent to $\frac{3}{4}$ and one student replying $1\frac{1}{2}/2$. How would you respond to this answer? Justify your response.
29. We have seen that there are two formulas for converting between the Celsius and Fahrenheit temperatures:

$$C = \frac{5(F - 32)}{9} \quad F = \frac{9C}{5} + 32$$

Fill in the blank in the following sentence: One Celsius degree is ____ of a Fahrenheit degree. Write a short paragraph to justify your choice.

30. Let us reinterpret the way gold is classified in terms of fractions. Pure gold is designated as 24-karat gold. Pure

gold is very expensive and very soft; therefore, gold is often mixed with other metals such as copper or zinc to make it stronger.

- Represent 14-karat gold in terms of a fraction.
 - If 1 ounce of pure gold costs \$750, approximately how much would 1 ounce of 14-karat gold cost? Estimate your answer first.
 - If you wanted to give a real-life answer to part (b), should you also know whether the other metal was copper or zinc? Why or why not?
 - Find out why the number 24 was picked to designate pure gold; for example, why wasn't 10 or 100 selected?
31. I have a 35 mm camera and am a very amateur photographer. However, I do know that I need to set the camera speed for $\frac{1}{60}$ of a second indoors and that $\frac{1}{250}$ of a second is a good outdoor speed. When I decide to take a picture, I see on my camera a dial like the one below. The numbers 1000, 500, 250, and so forth represent the fractions $\frac{1}{1000}$, $\frac{1}{500}$, and so forth. The numbers continue to decrease until they get to 1 and then they increase again.
- Using whole sentences, compare the shutter speeds of $\frac{1}{125}$ and $\frac{1}{500}$. Can you do this in more than one way?
 - Explain the meaning of the following part of the dial: 8, 4, 2, 1, 2, 4.
 - Find out the history of the shutter speeds. That is, why were these speeds chosen and not, for example, $\frac{1}{10}$ of a second or $\frac{1}{100}$ of a second. Why $\frac{1}{15}$ instead of $\frac{1}{16}$?



32. Justify the cross multiplication equation on page 248.

SECTION

5.3

UNDERSTANDING OPERATIONS WITH FRACTIONS

One of the sentences in the NCTM standards that has burned in my head is that students should “believe that mathematics makes sense.” My earliest memories of teaching (tutoring while in college) are of surprise that this subject, which was so much fun for me, made almost no sense to many other people. There are many algorithms that enable us to manipulate fractions quickly: algorithms for adding, subtracting, multiplying, and dividing, and algorithms for translating improper fractions into mixed numbers and vice versa. However, as I have stressed in this book so far, it is simply not enough for an elementary teacher to

WHAT DO YOU THINK?

- Why do we need a common denominator to add (or subtract) two fractions?
- Why don't we need a common denominator to multiply (or divide) two fractions?
- If whole-number multiplication "makes bigger," why does fraction multiplication sometimes "make smaller"?
- Why isn't $3\frac{1}{4} \cdot 2\frac{1}{3} = 6\frac{1}{12}$?

know how to compute. It is crucial that the teacher also know the *whys* behind the *hows*. If you know only the *hows*, your ability to apply that knowledge to nonroutine and multistep problems will be limited. After all, most real-life problems are nonroutine and multistep! Therefore, please take care in this section not just to skip over, saying "I know how to do that," but please ask yourself the questions that are posed. Can you really get into the *why*? I have seen many elementary teachers have wonderful discussions when students asked why and the teacher knew what to do with that question!

Addition of Rational Numbers

Addition of fractions is one of those situations that mathematics is famous for—the procedure goes against what most people's common sense tells them to do. When we add fractions, we do not add the numerators and the denominators; on the other hand, when we multiply fractions, we *do* multiply the numerators and the denominators.

$$\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5} \quad \text{but} \quad \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

Can you explain *why* we need to find a common denominator before we can add fractions with different denominators? Write (a first draft) in your notebook and then read on. . . . 

INVESTIGATION 5.8 Using Fraction Models to Understand Addition of Fractions

Using the area model, the length model, or the set model, determine the sum of $\frac{1}{2} + \frac{1}{3}$ and then try to explain *why* we need to find a common denominator in order to add fractions. This is your "second draft" for justifying the need for a common denominator. Then read on. . . . 

DISCUSSION

This is an excellent place to demonstrate the role of manipulatives and to show that different manipulatives do different things.

Although children bring to school a primitive understanding of fractions, schools often teach procedures at the expense of understanding. The NCTM standards encourage teachers to use "physical materials, diagrams, and real-world situations in conjunction with ongoing efforts to relate [the students'] learning experiences to oral language and symbols. . . . [Such efforts] will reduce the amount of time currently spent in the upper grades in correcting students' misconceptions and procedural difficulties."

(Curriculum Standards, p. 57)

Using pattern blocks In Exploration 5.5, you worked with two kinds of area manipulatives: squares and circles. Let us now consider pattern blocks (see Figure 5.19). How might you use pattern blocks to add $\frac{1}{2}$ and $\frac{1}{3}$?

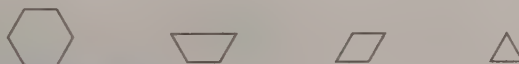


FIGURE 5.19

If we let the hexagon represent 1, then the trapezoid is $1/2$, the parallelogram is $1/3$, and the triangle is $1/6$. Using the notion of addition as

combining, we can combine the two and we have $1/2 + 1/3 = \frac{5}{6}$.

We could be humorous and say that parallelogram + trapezoid = baby carriage! The key question, though, is to determine the numerical value of this amount. The solution lies in realizing that to name an amount with a fraction, we must have equal-size parts. If you are familiar with pattern blocks, you realize that an equivalent representation is to cover this amount with 5 triangles, as in Figure 5.20.

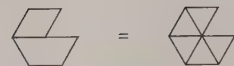


FIGURE 5.20

Because the value of each triangle is $1/6$, we can now say that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

Using Cuisenaire rods This is a more different question when using Cuisenaire rods, because there is no “natural” choice for a unit. Thus, you have to think of a color for which you can represent $1/2$ of that color *and* $1/3$ of that color. If students have worked with Cuisenaire rods in previous contexts (recall the discussion of GCF and LCM in Chapter 4), they will be familiar with the relationships among the various rods. One of many different solutions is to choose the dark green rod to have a value of 1. The red rod now has a value of $1/3$, and the green rod has a value of $1/2$. See Figure 5.21(a). As with pattern blocks, when we combine these two parts, in order to name the amount we have to find a way to represent this length with equal-size pieces, and 5 white rods have the same length. Because 6 white rods have the same length as the dark green rod, each of the white rods has a value of $1/6$, and therefore 5 of them have a value of $5/6$. See Figure 5.21(b).

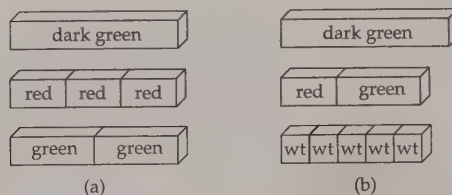


FIGURE 5.21

Looking back An important outcome of working with manipulatives is for the students to see that, for example, the yellow pattern block is not 6, but rather can be assigned any value, and the value of the other pieces is determined by their relationship to this block. So too with the Cuisenaire rods. Recall your work with the different models in Exploration 5.7.

Connecting Concepts to a Procedure for Adding Fractions

Let us now connect this work to the general procedure for adding fractions. Look at the addition problem below. Try to explain this procedure meaningfully—that is, to *justify* each step. Do this before reading on. . . .

$$\frac{2}{3} + \frac{1}{4} = \frac{8}{12} + \frac{3}{12} = \frac{11}{12}$$

In order to add two fractions that have different denominators, we determine the least common multiple of those two numbers, in this case 12. When working with fractions, we often refer to the LCM as the **lowest common**

denominator. We now determine an equivalent fraction (with denominator 12) for each of the addends. We do this because all of the parts have to be the same size. We know that $\frac{2}{3}$ is equivalent to $\frac{8}{12}$ and that $\frac{1}{4}$ is equivalent to $\frac{3}{12}$. Now that both fractions have the same denominator (that is, all pieces are the same size), we can combine them. We find that we have $8 + 3 = 11$ pieces, each of which has a value of $\frac{1}{12}$. Thus, the answer is $\frac{11}{12}$.

You may remember and use the algebraic shortcut that is presented below without proof:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

If this doesn't make sense, try it for the numbers above; that is, substitute 2 for a , 3 for b , and so forth.

Which method will be faster when the least common multiple is not the product of the two denominators? For example, $\frac{7}{8} + \frac{5}{6} = ?$ You might try it both ways to see which method you prefer.

Mixed Numbers and Improper Fractions

Up to this point, all of the addition problems have involved only **proper fractions**—that is, fractions whose value is between 0 and 1. Why do you think these fractions are called proper fractions? In many real-life situations, we encounter mixed numbers (for example, $5\frac{1}{4}$) and improper fractions (for example, $\frac{21}{4}$). Take a couple of minutes to write down your own definitions of these two terms and then read on. . . . 

An **improper fraction** is one in which the numerator is at least as large as the denominator.

A **mixed number** is a number that has a whole-number component and a fraction component.

The language we use to describe mixed numbers is important. How do you say $5\frac{1}{4}$ in English? Just as we found the value of expanded form helpful with whole numbers, there is an expanded form for mixed numbers. How do you think we would represent $5\frac{1}{4}$ in expanded form?

A recent National Assessment of Educational Progress contained the following item:

$5\frac{1}{4}$ is the same as:

- (a) $5 + 1/4$ (b) $5 - 1/4$ (c) $5 \times 1/4$ (d) $5 \div 1/4$ (e) I don't know

Only 47 percent of the seventh graders chose the correct response, (a). Even more startling, an even smaller percentage of eleventh graders chose the correct response—only 44 percent. This lack of connectedness between concepts and procedures is one of the reasons why students do so much worse on algebraic equations with fractions than on algebraic equations with whole numbers. In this case, the problem is not the algebra so much as it is the arithmetic!

INVESTIGATION 5.9 Connecting Improper Fractions and Mixed Numbers

Most of you remember the procedure for converting a mixed number into an improper fraction.

For example, to convert $3\frac{1}{4}$ into an improper fraction, we do the following:

$$3\frac{1}{4} = \frac{3 \cdot 4 + 1}{4}$$

However, did you ever stop to think about what this might feel like to a youngster? On the surface, this procedure is as far from common sense as almost any in mathematics: We multiply the *whole number* by the *denominator* and then we add the *numerator*, and then we put this number on top of the *original denominator*. This sounds a lot like “gazinta” and “bring down” from division.

How would you explain the why of this procedure to, let’s say, a fifth-grade student? Record your first attempt and then read on. . . 

DISCUSSION

As is often the case, concrete representations are most helpful to most students. Figure 5.22 shows two concrete representations of $3\frac{1}{4}$.

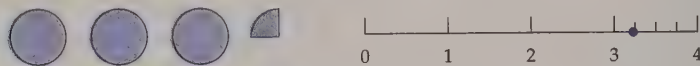


FIGURE 5.22

With these in mind, now make a second attempt to explain the why of the procedure. Then read on. . . 

In each of the representations, we see that the process is similar to the kinds of regrouping we did with whole numbers. In converting from the mixed number to the improper fraction, we need to convert all the units to fourths because in order for the improper fraction, $13/4$, to have meaning, all the pieces must be the same size—that is, they must be fourths. We can convert each 1 to four fourths and then add all the fourths to get $13/4$. However, because one context for multiplication is repeated addition, we see that we are adding three 4’s; hence $3 \cdot 4$. When we add these 12 fourths to the 1 fourth we already had, we have a total of 13 fourths.

$$\begin{aligned} 3\frac{1}{4} &= 1 + 1 + 1 + \frac{1}{4} \\ &= \frac{4}{4} + \frac{4}{4} + \frac{4}{4} + \frac{1}{4} \\ &= \frac{3 \cdot 4}{4} + \frac{1}{4} \\ &= \frac{3 \cdot 4 + 1}{4} \\ &= \frac{13}{4} \end{aligned}$$

Understanding Subtraction of Rational Numbers

For the most part, subtraction of fractions involves essentially the same processes as does addition. However, many students encounter difficulty in situations that involve regrouping. Therefore, it is important to examine this process also. Consider the following problem: $23\frac{2}{5} - 17\frac{4}{5}$. First do it on your own, writing down your justification of each step. Then read on. . . 

LEARNING

I have seen a number of students (in middle school, high school, and college) who do the problem below in the following way:

$$\begin{array}{r} 23\frac{2}{5} = \frac{117}{5} \\ -17\frac{4}{5} = \frac{89}{5} \\ \hline \frac{28}{5} = 5\frac{3}{5} \end{array}$$

I have even found some middle school teachers who confessed to teaching this procedure. Their rationale was that so many students had so much difficulty with fraction subtraction that they gave up and taught the students this “foolproof” way that works in all situations, even though it is very cumbersome in many cases. That is, the students were so weak in the *whys* that the teachers just gave up and focused exclusively on the *hows*. It is my deepest hope that even though many students will say, “Just show me how,” you will be one of those teachers who manages to keep awake or reawaken the curiosity (wanting to know why) that young children naturally possess but most people lose as they grow up.

$$\begin{array}{r}
 23\frac{2}{5} = 22\frac{7}{5} \\
 -17\frac{4}{5} = 17\frac{4}{5} \\
 \hline
 5\frac{3}{5}
 \end{array}$$

Let us discuss the transformation from $23\frac{2}{5}$ to $22\frac{7}{5}$, because this is difficult for many children. How would you explain this to someone who does not understand it?

We must do some renaming because we cannot subtract $\frac{4}{5}$ from $\frac{2}{5}$. With fractions, instead of converting 1 ten into 10 ones (as we did with whole numbers), we convert 1 into five fifths (or six sixths, or ten tenths, depending on the common denominator). Once again, expanded form will help to illustrate the process and justify the equivalence of $23\frac{2}{5}$ and $22\frac{7}{5}$.

$$23\frac{2}{5} = 23 + \frac{2}{5} = (22 + 1) + \frac{2}{5} = 22 + \left(\frac{5}{5} + \frac{2}{5}\right) = 22 + \frac{7}{5} = 22\frac{7}{5}$$

INVESTIGATION 5.10 Estimation and Mental Arithmetic: Sums and Differences of Fractions

As was the case with whole numbers, many everyday situations requiring computations with fractions do not require exact answers, but rather estimates. Think of circumstances in which you or other people might need to find the sum of fractions or the difference between fractions. Try to come up with realistic problem situations. In which of those situations do you need an exact answer? In which of those situations do you need an estimate, a ballpark answer?

When I do this exercise with students and teachers, we find that more often than not, an estimate rather than an exact answer is needed. Thus, it is important to examine strategies for estimating. As we have seen before with whole numbers, this also involves examining mental arithmetic. Because there are many strategies, we will examine them using a process similar to the one we used when developing estimation and mental math strategies with whole numbers.

Part 1: Mental addition and subtraction with fractions

In each of the following problems, try to find the sum or the difference entirely in your head.

Briefly describe your strategies. Then read the discussion below. 

A. $2\frac{1}{2}$	B. $3\frac{4}{5}$	C. $7\frac{3}{4}$	D. 7	E. $9\frac{1}{4}$
$+5\frac{3}{4}$	$+1\frac{2}{3}$	$5\frac{1}{8}$	$-2\frac{5}{8}$	$-3\frac{5}{8}$
		$+2\frac{1}{2}$		

DISCUSSION

Keep the following points in mind for each problem:

- Deciding which strategy is better is often not as helpful as knowing which strategy works best for you.
- All possible useful strategies have not been included here because too many strategies (and 1500-page books) can overwhelm students!

A. $2\frac{1}{2} + 5\frac{3}{4}$

STRATEGY 1: Represent the sum with expanded form and use the commutative and associative properties

$$2\frac{1}{2} + 5\frac{3}{4} = 2 + \frac{1}{2} + 5 + \frac{3}{4} = 2 + 5 + \frac{1}{2} + \frac{3}{4} = 7 + \frac{1}{2} + \frac{3}{4}$$

That is, we quickly transform the problem into $7 + \frac{1}{2} + \frac{3}{4}$.

Decomposing $\frac{3}{4}$ into $\frac{1}{2} + \frac{1}{4}$, we then have $7 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = 8\frac{1}{4}$.

STRATEGY 2: Add up and break into parts

$$\left(2\frac{1}{2} + 5\right) + \frac{3}{4} = 7\frac{1}{2} + \frac{3}{4} = 7\frac{1}{2} + \frac{1}{2} + \frac{1}{4} = 8 + \frac{1}{4} = 8\frac{1}{4}$$

B. $3\frac{4}{5} + 1\frac{2}{3}$

Use the shortcut

$$\frac{4}{5} + \frac{2}{3} = \frac{4 \cdot 3 + 5 \cdot 2}{15} = \frac{22}{15} = 1\frac{7}{15}$$

Add this to the sum obtained from the whole numbers and we have $4 + 1\frac{7}{15} = 5\frac{7}{15}$.

C. $7\frac{3}{4} + 5\frac{1}{8} + 2\frac{1}{2}$

STRATEGY 1: Look for compatible numbers

$\frac{3}{4}$ and $\frac{1}{2}$ can be added mentally to get $1\frac{1}{4}$; then add $1\frac{1}{4} + \frac{1}{8} = 1\frac{2}{8} + \frac{1}{8} = 1\frac{3}{8}$. Add this to the 14, and we have $15\frac{3}{8}$. *Note:* I find it easier to add $\frac{3}{4}$ and $\frac{1}{2}$ first rather than adding $\frac{3}{4}$ and $\frac{1}{8}$ and then $\frac{1}{2}$. Do you see why? Which would be easier for you?

STRATEGY 2: Convert to common denominator

Because the common denominator is relatively small and the conversions to eighths are relatively easy, some students find it easier to add all three at once, thinking something like this: “ $6 + 1 + 4 = 11$, that means 11 eighths, which is 8 eighths (1) + 3 eighths. Add this to the 14 and we have $15\frac{3}{8}$ as the answer.”

D. $7 - 2\frac{5}{8}$

STRATEGY 1: Break into parts and subtract

First, $7 - 2 = 5$; now take $\frac{5}{8}$ from 5 to get $4\frac{3}{8}$.

STRATEGY 2: Add up

$2\frac{5}{8} + 4 = 6\frac{5}{8}$. How much more do we need to get to 7? We need $\frac{3}{8}$ more. The answer is $4\frac{3}{8}$.

E $9\frac{1}{4} - 3\frac{5}{8}$

Add up:

$3\frac{5}{8}$ plus 5 equals $8\frac{5}{8}$ plus $\frac{3}{8}$ equals 9 plus $\frac{2}{8} = 9\frac{1}{4}$. What you have to remember to add now is $5 + \frac{3}{8} + \frac{2}{8} = 5\frac{5}{8}$.

A number line helps some students to better understand the process (see Figure 5.23).

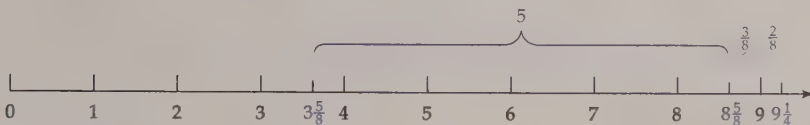


FIGURE 5.23

This is one place where many students find the explorations paying rich dividends. Making the manipulatives (Exploration 5.5), grappling with relative sizes of fractions (Exploration 5.6), and developing fraction sense (Exploration 5.8) all focus on understanding the relationships involved in fractions. Having a good understanding of these relationships enables the student to decompose and then recompose the various fractions more confidently. Again we find the composition and decomposition of numbers in this chapter.

We can now apply these strategies to problems with a story.

Part 2: Estimating sums and differences with fractions

- A.** In their retirement, the parents of a close friend of mine have created a dollhouse to represent their dream house. They are shopping, and they see some miniature furniture that might fit in one of the bedrooms. The bed is $\frac{7}{8}$ inch wide; the dresser is $1\frac{1}{2}$ inches long, and the desk is $1\frac{3}{4}$ inches long. If the three articles are placed as shown in Figure 5.24, will they fit into the dollhouse bedroom, which is 4 inches wide? Try to add the three lengths in your head, and then read on. . . .

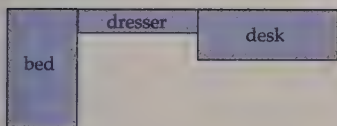


FIGURE 5.24

DISCUSSION

STRATEGY 1: Make a quick estimate

A quick estimation reveals that it will be close:

$$\frac{7}{8} + 1\frac{1}{2} + 1\frac{3}{4} \approx 1 + 1\frac{1}{2} + 1\frac{1}{2} = 4$$

Thus, we need a strategy that will be more precise.

STRATEGY 2: Add the two easier numbers

$$1\frac{1}{2} + 1\frac{3}{4} = 1\frac{1}{2} + 1\frac{1}{2} + \frac{1}{4} = 3\frac{1}{4}$$

We can use number sense here; that is, we know that $3\frac{1}{4} + \frac{3}{4} = 4$, and because $\frac{7}{8} > \frac{3}{4}$, we can conclude that the furniture is too big.

B. It rained almost every day this past week. Here are the amounts per day. How much rain fell during the week? First, make a rough estimate. Then make a refined estimate. Then read on. . . 

Monday	$1\frac{3}{4}$ inches	Friday	0 inches
Tuesday	$\frac{1}{2}$ inch	Saturday	$3\frac{3}{4}$ inches
Wednesday	$\frac{5}{8}$ inch	Sunday	$1\frac{5}{16}$ inches
Thursday	$\frac{7}{8}$ inch		

DISCUSSION

STRATEGY 1: Make a rough estimate

One way to get a rough estimate is to round each fraction to the nearest $\frac{1}{2}$ inch and keep a cumulative total:

$$1\frac{1}{2} + \frac{1}{2} = 2; 2 \text{ plus } \frac{1}{2} = 2\frac{1}{2}; 2\frac{1}{2} \text{ plus } 1 = 3\frac{1}{2}; 3\frac{1}{2} \text{ plus } 3\frac{1}{2} = 7; 7 \text{ plus } 1\frac{1}{2} = 8\frac{1}{2}$$

STRATEGY 2: Look for compatible numbers

One way to get a closer estimate is to look for compatible numbers—numbers whose sum is close to 1 or numbers that you can quickly add mentally.

$$\underbrace{\left(1\frac{3}{4} + 3\frac{3}{4}\right)}_{5\frac{1}{2}} + \frac{1}{2} \quad + \quad \underbrace{\left(\frac{5}{8} + 1\frac{5}{16}\right)}_{\text{just under } 2} + \frac{7}{8} = \text{just under } 9$$

Before we move on, take a few moments to reflect on the mental math and estimation strategies that you used and that were discussed here. What strategies have been adapted from ones we developed with whole numbers, such as compatible numbers? What strategies are new, such as the algebra shortcut? Think about this and then read on. . . 

INVESTIGATION 5.11 Understanding Multiplication of Rational Numbers

The following multiplication problems have several purposes. First, they will enable you to connect the meaning of multiplication, which we developed with whole numbers, to fraction situations. Second, they will enable you to connect the meaning of fractions to fraction multiplication. Third, your ability to apply this knowledge to nonroutine and multistep problems should increase. Finally, you will be able to understand and explain things that teachers need to know—for example, why $3\frac{1}{2} \cdot 2\frac{1}{4} \neq 6\frac{1}{8}$.

For each of the following problems, represent the situation with a diagram and determine the answer from the diagram. As you are doing this, think back to the models we developed for whole-number multiplication. Which problems connect well to repeated addition? to a rectangular array?

to the Cartesian product? Which problems do not connect well? Why don't they? 

- A.** Julio runs four times a week. His route is $2\frac{1}{4}$ miles long. How many miles does he run each week?

DISCUSSION

This connects nicely to multiplication as repeated addition: 4 times $2\frac{1}{4}$.

We can represent this problem on a number line, as in Figure 5.25.

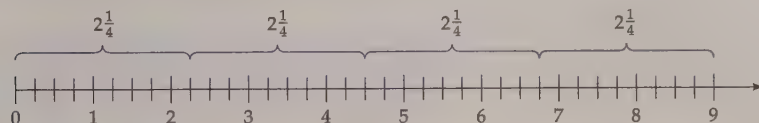


FIGURE 5.25

- B.** A group of investors purchased a rectangular parcel of land that is $\frac{3}{4}$ of a mile long and $\frac{2}{3}$ of a mile wide. How many square miles did they buy?

DISCUSSION

This problem is definitely not repeated addition, but it does connect to the area model of multiplication. Here we can say that we want $\frac{2}{3}$ of $\frac{3}{4}$ (of 1 square mile).

However, getting the answer by applying our knowledge of multiplication and fractions is not terribly easy. If the large square in Figure 5.26 represents 1 mile by 1 mile, the shaded region represents the area of the land that the investors bought: $\frac{3}{4}$ of a mile by $\frac{2}{3}$ of a mile. But how do we give a name to this amount? Try to do this yourself and then read on. . . 

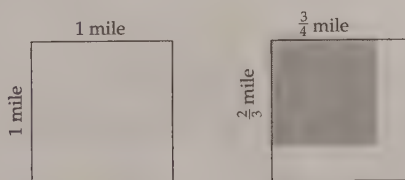


FIGURE 5.26

Just as we decomposed and recomposed numbers in working with whole numbers (when regrouping, for example), we need to decompose this (shaded) amount in such a way that we can recompose the amount as a set of equal-size pieces.

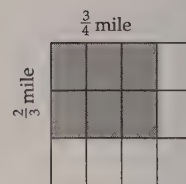
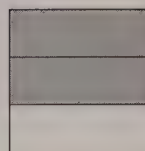
We can create equal-size pieces (needed to name a fraction) by dividing the square first into fourths (with vertical lines) and then into thirds (with horizontal lines). If we put the two diagrams together and shade in the area enclosed by $\frac{3}{4}$ times $\frac{2}{3}$, we have the figure on the right in Figure 5.27. Can you determine the answer now from the diagram? Can you justify that answer? Think and then read on. . . 

CHILDREN

There is another difference between multiplying whole numbers and multiplying fractions that bears noting, for it often baffles children when they encounter fraction computations for the first time. One notion of multiplication that many people (often unconsciously) get from working with whole numbers is that "multiplication makes bigger and division makes smaller," but just the opposite is true when we multiply proper fractions!



FIGURE 5.27



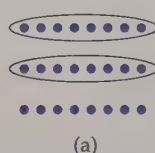
Because our unit (1 square mile) has been divided into 12 equal-size regions, each region has a value of $\frac{1}{12}$ square mile. The plot of land covers 6 of these rectangles, so its value is $\frac{6}{12}$ or $\frac{1}{2}$ square mile.

- C.** If $\frac{2}{3}$ of the class went on a field trip and there are 24 students in the class, how many students went on the field trip?

DISCUSSION

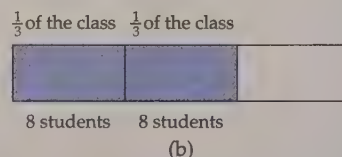
Problems like this one illustrate the third fraction context that we mentioned in Section 5.2: fraction as operator. The operator context can be thought of as a stretching or a shrinking process. In this case, we have a whole (24 students), and we are shrinking that whole—that is, we are considering a part ($\frac{2}{3}$) of that whole. When students memorize “of means multiply,” they are memorizing a procedure but not understanding this interpretation of fractions.

Let us now see how we can apply our knowledge of fractions and operations to solve this problem. Figure 5.28(a) represents the problem using a set model, with each dot representing one person. Because we have $\frac{2}{3}$ of the people, the $\frac{2}{3}$ tells us that we have to partition this amount into 3 equal groups and then take 2 of those groups. Figure 5.28(b) represents the problem using an area model. The whole rectangle represents the whole class. We have shaded in $\frac{2}{3}$ of the rectangle. If the whole class represents 24 students, then when we divide the class into 3 equal regions, each region must represent 8 students (that is, $8 + 8 + 8 = 24$ or $3 \cdot 8 = 24$). Both models quickly produce the correct answer of 16 students.



(a)

FIGURE 5.28



(b)

Connecting Multiplication of Fractions to the Multiplication Algorithm

As we did with multiplication of whole numbers, we will develop the algorithm with the rectangular model. Let us do so with the problem

$$\frac{3}{4} \cdot 2\frac{1}{2}$$

Let us first represent the problem with a diagram, determine the answer from that diagram, and then look to connect this solution to the algorithm.

First, check to make sure that you understand that Figure 5.29 represents $\frac{3}{4} \cdot 2\frac{1}{2}$. Then determine the answer from the diagram before reading on. . . . 

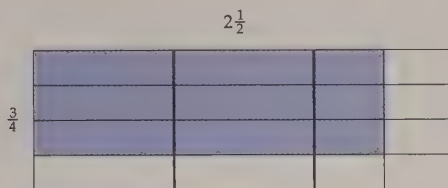


FIGURE 5.29

As we know, all the parts must be the same size in order for us to name the amount with a fraction. We can readily see (from Figure 5.30) that the numerator of our product must be 15 (that is, there are 15 parts). But what is the denominator? (Please resist the temptation to answer this question by doing the algorithm. In my own class, this question is actually a method I use to assess how deeply my students understand the meaning of denominator.) Then read on. . . . 

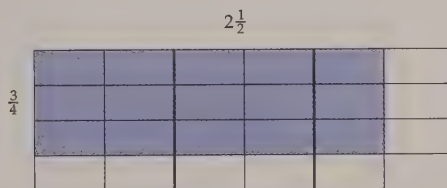


FIGURE 5.30

I usually get three different responses for the denominator: 24 because there are 24 parts in the “whole” diagram, 15 because there are 5 times 3 parts in the $2\frac{1}{2}$, and 8 because it takes 8 parts to make a 1 by 1 square (the unit).

The correct answer is 8. Why is this? Recall that the value of the denominator is determined by its relationship to the unit. That is, the denominator shows how many pieces it takes to have a value of 1, rather than how many pieces are in the whole.

Therefore, we have determined that

$$\frac{3}{4} \cdot 2\frac{1}{2} = \frac{15}{8} = 1\frac{7}{8}$$

Now let us examine the algorithm and look for connections between the algorithm and the concepts. We must first convert $2\frac{1}{2}$ to an improper fraction, and then we simply find the product of the numerators and the product of the denominators:

$$\frac{3}{4} \cdot 2\frac{1}{2} = \frac{3}{4} \cdot \frac{5}{2} = \frac{3 \cdot 5}{4 \cdot 2} = \frac{15}{8}$$

Can you justify this procedure? Can you explain the *whys* behind these *whats*? Think. Look back to Figure 5.30. Then read on. . . . 

If we look at the shaded region in Figure 5.30, we have 3 rows, each containing 5 regions; that is, we have 3 times 5 regions. Just as we found that the whole-number multiplication algorithm automatically regrouped for us, so too

the fraction multiplication algorithm automatically creates equal-size pieces. Similarly, we have $4 \cdot 2$ pieces in each unit.

Because of the space limitations of a textbook, I cannot present further examples here. However, I recommend that you do several problems on your own to ensure that you own rather than rent this connection between the rectangular model and the algorithm.

Justifying the Equality of Equivalent Fractions

In Section 5.2, we examined equivalent fractions. We can now justify the procedure we used earlier. The process is shown at the left for a specific case and at the right for the general case.

Specific case	Justification	General case
$\frac{3}{4} = \frac{3}{4} \cdot 1$	Multiplicative identity	$\frac{a}{b} = \frac{a}{b} \cdot 1$
$= \frac{3}{4} \cdot \frac{2}{2}$	Substitution	$= \frac{a}{b} \cdot \frac{c}{c}$
$= \frac{3 \cdot 2}{4 \cdot 2}$	Definition of multiplication of fractions	$= \frac{a \cdot c}{b \cdot c}$
$= \frac{6}{8}$		

The general case also emphasizes that the numerator and denominator of the new fraction are simply multiples of the numerator and denominator of the original fraction.

INVESTIGATION 5.12 Division of Rational Numbers



HISTORY

The Hindu mathematician Mahavira (A.D. 850) first stated a rule for dividing fractions: "After making the denominator of the divisor its numerator and vice versa, the operations to be conducted then are as in the multiplication [of fractions]." It was not until the sixteenth century that this procedure was "discovered" in Europe.⁵

As with most concepts being considered in this course, you probably remember the procedure. This recalls a famous saying of unknown origin: "Ours is not to reason why, just invert and multiply."

$$\frac{3}{4} \div \frac{5}{6} = \frac{3}{4} \times \frac{6}{5} = \frac{18}{20} = \frac{9}{10}$$

Now, though, ours *is* to reason why. Exploration 5.12 examined an alternative algorithm first invented by another Indian mathematician, Brahmagupta, more than a thousand years ago. Let us begin our investigation of the fraction division algorithm by first examining fraction division problems in context.

Recall the two contexts for division with whole numbers: partitioning and repeated subtraction.

Make up a story for each of the two division problems below.

First, solve the problem using what you know about division and fractions.

⁵ Louis Karpinski, *The History of Arithmetic* (New York: Russell & Russell, 1965), p. 128.

Then respond to the following questions: Which of your stories were from the partitioning model, and which were from the repeated subtraction model? If you made up a story connected to one model, could you make up another story for the same problem using the other model? 

- A. The divisor is a whole number: $3 \div 4$
- B. The divisor is a proper fraction: $3 \div 2/5$

DISCUSSION

A. Most people make up a story for $3 \div 4$ using the partitioning model, and most of the stories involve sharing. In fact, for many (if not most) children, the first experience with fractions has to do with sharing; for example, you each get $1/2$ of the candy bar. Children seldom need to be taught a central fraction concept: The parts must be equal!

Let us consider one such story and examine a solution: Basil has 3 pints of ice cream that he wants to share equally with 3 friends—that is, among 4 people. How much is each person's share?

Figure 5.31 shows a partitioning (dealing) solution. That is, if we divide each pint into 4 equal parts, each person gets 1 part per pint. Each person's share thus consists of 3 parts, each of which is $1/4$ pint. Therefore, each person's share is $3/4$ of a pint.

There are two aspects of this problem that bear noting.

First, $3 \div 4 = \frac{3}{4}$. As mentioned at the beginning of Section 5.2, fraction as division is another of the rational number contexts. That is, the division problem $a \div b$ and the fraction a/b are equivalent. This relationship between fractions and division is often problematic for children, and many do not realize that this is not a commutative relationship. For example, research has shown that many middle school children cannot accurately translate $5 \div 6$ into $5/6$ and vice versa.

Second, some people literally interpret this problem as a multiplication problem. If we are dividing the ice cream among 4 people, then each person gets $1/4$ of the ice cream. This connection between the two operations produces the following equality: $3 \div 4 = 3 \cdot \frac{1}{4}$ which we just mentioned. We will return to this relationship between multiplication and division shortly. However, first let us examine a story for the other problem.

B. Story for $3 \div 2/5$: Jake is stranded in the middle of the desert. He has 3 quarts of water, and he figures that he will drink $2/5$ of a quart each day. How many days' supply does he have?

Solve the problem using only your knowledge of fractions and your knowledge of division. Then read on. . . . 

This is a repeated subtraction problem because we have an amount (3 quarts) and we are specifying the size of the group ($2/5$ of a quart) to be repeatedly subtracted. The answer will be the number of groups we have. Virtually all stories for this problem involve repeated subtraction. In the first edition of this textbook, I stated in the next line "In fact, I cannot make up a realistic story for this problem using the partitioning model of division." Then I read a book called *Knowing and Teaching Elementary Mathematics* by Liping Ma. In her book, Ma asked a number of U.S. and Chinese elementary school teachers to make up a story problem for $1\frac{3}{4} \div \frac{1}{2}$. I was stunned to read that most of the



FIGURE 5.31

Chinese teachers' stories involved the partitioning model. While my conception of this model is not exactly the same as Ma's—who describes it as “finding a number such that $1/2$ of it is $1\frac{3}{4}$.” (p. 74). I found her example and her book to be very educational. I recommend the book to you.

I have chosen to represent this problem with a number line, although an area or set model could also have been used. Each day Jake uses $2/5$ of a quart. We see from Figure 5.32 that he is able to drink $2/5$ of a quart each day for 7 days. However, we have a small problem: We have repeatedly subtracted $2/5$ of a quart, but we have $1/5$ of a quart left over. Does this answer contradict the answer that comes from using the algorithm, which tells us that $3 \div \frac{2}{5} = 7\frac{1}{2}$? How do we reconcile the difference between the answer from the diagram and that from the algorithm? Think and then read on. . . . 

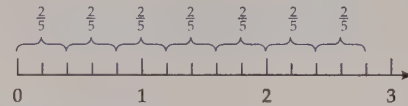


FIGURE 5.32

It turns out that both answers are correct—they represent different ways of describing the remainder; recall the remainder discussion in Chapter 3 and Exploration 5.13. The $1/5$ tells us that the remainder is $1/5$ of a quart; thus, we can say that at the end of 7 days, he will have $1/5$ of a quart left. The $7\frac{1}{2}$ tells us that the remainder is equivalent to $1/2$ day's water. This underscores the point made in Investigation 1.15 (How Many Pieces of Wire?) that the key to a problem is often knowing what the computation means.

Making Sense of the Standard Division Algorithm

Although using the invert and multiply algorithm is rather straightforward, understanding why it works is another matter. Unlike most of the other algorithms we have examined, it does not lend itself to a diagrammatic representation. Therefore, we will begin to build our understanding of it by examining the similarities between how division and multiplication are connected and how subtraction and addition are connected.

We need to introduce a new concept to do so. Just as we discovered in the first section of this chapter that every integer has an additive inverse, every nonzero rational number has a **multiplicative inverse**. These parallel concepts are shown side by side below:

$-a$ is the additive inverse of a	$\frac{1}{a}$ is the multiplicative inverse of a , $a \neq 0$
$a + -a = 0$ for any integer a	$a \cdot \frac{1}{a} = 1$ for any rational number, $a \neq 0$

Let us now recall the way in which we defined integer subtraction in terms of addition, and then we will define fraction division in terms of fraction multiplication.

$$a - b = a + ^{-}b$$

That is, to subtract b , we add the (additive) inverse of b .

In other words, one way to interpret subtraction is to add the *additive inverse* of the second number.

$$a \div b = a \cdot \frac{1}{b}$$

That is, to divide by b , we multiply by the (multiplicative) inverse of b .

In other words, one way to interpret division is to *multiply* by the *multiplicative inverse* of the second number.

Revisiting properties Let us now examine how the properties that we developed in Chapter 3 extend to operations with fractions. The commutative and associative properties hold for addition and multiplication but not for subtraction and division. The distributive properties hold for fractions. Both identity properties hold: 0 is the identity for addition and 1 for multiplication. The zero property of multiplication holds. We still have additive inverses, and we have now acquired the multiplicative inverse for all numbers except 0. With this in mind, let's investigate the four operations with fractions.

INVESTIGATION 5.13 Estimating Products and Quotients

As you discovered in our discussion of adding and multiplying fractions, many everyday situations that ask us to compute with fractions require estimates rather than exact answers. Let us now examine some estimation strategies for multiplication and division.

In each of the problems below, do the following: First obtain a rough estimate (5 to 10 seconds), and then try to get as close as you can to the actual answer—either a refined estimate or, in some cases, an exact answer (computed mentally). 

- A.** Anastasia walks at a pace of $3\frac{1}{2}$ miles per hour for 2 hours and 15 minutes. How far does she walk during this time?

DISCUSSION

Note: We first need to convert the time to fractional form; that is, we are estimating $3\frac{1}{2} \cdot 2\frac{1}{4}$.

STRATEGY 1: Bound the answer

One kind of estimating involves bounding the answer. For example, the answer will be at least 6. We get this by focusing only on the whole numbers. We call 6 a lower bound.

An upper bound can be obtained by rounding both numbers to the next higher whole number. The upper bound is thus $4 \cdot 3 = 12$.

STRATEGY 2: Get rid of one of the proper fractions (that is, rounding)

Another method of making a rough estimate involves getting rid of one of the proper fractions. We can more easily determine $3\frac{1}{2} \cdot 2 = 7$ or $3 \cdot 2\frac{1}{4} = 6\frac{3}{4}$.

STRATEGY 3: Use partial products

We can estimate and round the four partial products:

$$3\frac{1}{2} \times 2\frac{1}{4} = \left(3 + \frac{1}{2}\right) \times \left(2 + \frac{1}{4}\right) = (3 \times 2) + \left(3 \times \frac{1}{4}\right) + \left(2 \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right)$$

$$\approx \quad 6 \quad + \quad 1 \quad + \quad 1 \quad + \quad 0 \quad \approx 8$$

- B.** The Jones family finds that they spend about $\frac{2}{3}$ of their income on food and rent. If their monthly income is \$1635, how much is their monthly food and rent bill?

DISCUSSION

Rough estimate (using compatible numbers):

$$\frac{2}{3} \times 1500 = 1000$$

More refined estimate (using compatible numbers):

$$\frac{2}{3} \times 1650 = \frac{2}{3}(1500 + 150) = 1000 + 100 = 1100$$

- C.** Marvin has 23 yards of cloth with which to make costumes for the play. Each costume requires $3\frac{1}{4}$ yards of material. How many costumes can he make?

DISCUSSION

Rough estimate (using bounding):

$$23 \div 3 = 7\frac{2}{3}$$

$$23 \div 4 = 5\frac{3}{4}$$

The estimate is about 6 or 7 costumes.

Refined estimate (using repeated addition):

$$3\frac{1}{4} \times 2 = 6\frac{1}{2}, \text{ so}$$

$$3\frac{1}{4} \times 4 = 13, \text{ so}$$

$$3\frac{1}{4} \times 8 = 26$$

Because $26 - 3 = 23$, this estimate yields 7 costumes.

- D.** Shelly has 25 pounds of dog food, and each day she feeds her dog $\frac{3}{4}$ pound. How many days' worth of food does she have?

DISCUSSION

Rough estimate (using bounding):

$$25 \div 1 = 25 \quad \text{and} \quad 25 \div \frac{1}{2} = 50$$

Thus she has enough for more than 25 days and less than 50 days.
Refined estimate (using the algorithm and the commutative and associative properties):

$$25 \div \frac{3}{4} = 25 \times \frac{4}{3} = \frac{100}{3} = 33\frac{1}{3}$$

which is the actual quotient. Thus, she has 33 days' worth of dog food.

Applying Fraction Understandings to Nonroutine Problems

We have carefully examined the different meanings that fractions can have, and we have examined the four operations in terms of fractions. In the following nonroutine problems, you will need to determine which computation(s) to do and how to interpret the computation(s).

INVESTIGATION 5.14 When Did He Run Out of Gas?

Jeremy started on a trip from Mobile, Alabama, to New Orleans, approximately 120 miles away. His car ran out of gas after he had gone one-third of the second half of the trip. How far is Jeremy from New Orleans? Work on this problem yourself and then read on. . . .

**DISCUSSION**

This is a case in which a good diagram practically solves the problem by itself (see Figure 5.33).

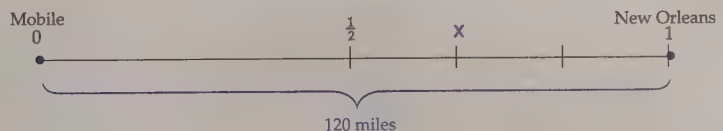


FIGURE 5.33

If half of the trip is equal to 60 miles, then each of the "thirds of the second half" is equal to 20 miles. X marks the spot of "one-third of the second half of the trip." So Jeremy is still 40 miles from New Orleans.

INVESTIGATION 5.15 Will She Be Late?

Can you help Jenny? She thinks she is going to be late to school, which starts at 8:15. She is not sure exactly how long it takes her to walk from home to school. However, she looked at her watch when she passed the

video store at 7:50, and then she looked at her watch when she passed the convenience store at 7:57. It just so happens that the video store is one-third of the way from home to school, and the convenience store is exactly halfway. If she continues walking at this rate, will she be on time? Work on this problem and then read on. . . .

DISCUSSION

Not surprisingly, this problem is much more easily represented with a number line than with any other model. Figure 5.34 shows a beginning representation. What else can we add to the diagram so that it will tell us the answer? Think and read on. . . .

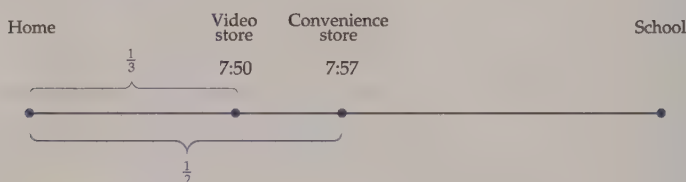


FIGURE 5.34

A key to the solution is to observe that it took her 7 minutes to go from $1/3$ of the way to $1/2$ of the way. If you recall from Chapter 3 that distance between two numbers is a subtraction problem, you can see that $1/2 - 1/3 = 1/6$ and that it takes her 7 minutes to go $1/6$ of the distance. This means that it would take her 42 minutes to go the whole way, but that in itself doesn't do us any good because we don't know when she started. However, if it takes her 42 minutes to go the whole distance, it takes her 21 minutes to go halfway, and 21 minutes after 7:57 is 8:18. Yes, she had better run!

INVESTIGATION 5.16 They've Lost Their Faculty!

Suppose you are reading the newspaper and you see that American State College has had to reduce its faculty by one-sixth because of an economic crisis in the state. The article says that American State College now has 350 faculty members. A curious reader might ask, "How many did they have before the cut?" Think about this problem before reading on. . . .

DISCUSSION

STRATEGY 1: Represent the situation with a diagram

The large rectangle in Figure 5.35 represents the original faculty.

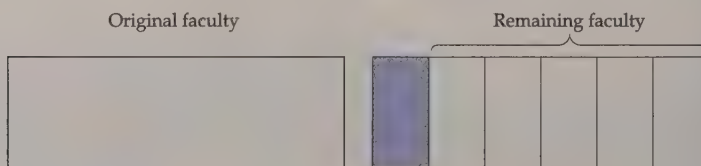


FIGURE 5.35

If the college lost one-sixth of its faculty, then we can divide that rectangle into 6 equal pieces and take away one piece, which represents one-sixth of the faculty. Thus the 5 remaining boxes represent the remaining $5/6$ of the faculty.

Because we know that there are 350 faculty members now, the value of those 5 boxes is 350. But if the value of 5 boxes is 350, then the value of 1 box is 70. Now we can answer the problem. Do you see why?

Because the value of the original faculty was represented by 6 boxes, there were $6 \times 70 = 420$ faculty.

STRATEGY 2: Use algebra

Can you find an algebraic solution? Work on it and then read on. . . .



One algebraic solution comes from letting $x =$ the number of the faculty before the cut. The remaining faculty represents $5/6$ of the original, so 350 is $5/6$ of x ; that is, $\frac{5}{6}x = 350$.

We solve the equation by multiplying both sides by $6/5$:

$$\left(\frac{6}{5}\right)\frac{5}{6}x = \left(\frac{6}{5}\right)350 \quad \text{and} \quad x = 420$$

Summary

Let us take a few moments to look back on this section. What have you learned? What fraction ideas make more sense now than they did before? What fraction ideas still feel difficult?

In this section, we have focused on connecting conceptual knowledge to procedural knowledge—on understanding the whys behind the how of the algorithms for operations with fractions. In order to be successful in the explorations, your focus should be more on understanding relationships among ideas and connecting ideas to one another than on solving “word problems” that have answers. Ultimately, we want you to become competent and confident when you encounter word problems in your everyday and work life. However, success in problem-solving first requires you to grapple with relationships within and between the mathematical ideas.

We have found that the contexts in which we first studied these four operations in Chapter 3 do not all apply when working with fractions; for example, the partitioning model of division does not readily apply when the divisor is not a whole number. In making sense of algorithms, we need to pay attention to the very fraction ideas that we developed in Section 5.2: that the parts in a fraction have the same value, that working with fractions involves parts and wholes, and that we have to pay attention to units and wholes (since they are not always the same). We also found that we have to make sure that what we are doing makes sense; recall Investigation 5.11C, concerning the students on the field trip.

When we explored mental arithmetic and estimation with fractions, we found that many of the strategies developed in Chapter 3 applied directly to computing with fractions—for example, the commutative and associative properties, compatible numbers, adding up, partial products, and expanded form.

When we applied our knowledge of operations to nonroutine and multi-step problems, we found that we needed to think about how we might represent the problem. We also had to make sure that we kept basic fraction ideas in mind—for example, the distinction between the unit and the whole.

Now that we have a deeper understanding of fractions, we are ready for our next extension of the number system—to decimals and real numbers.

EXERCISES 5.3

1. Perform the following computations.

a.
$$\begin{array}{r} 23\frac{7}{8} \\ +56\frac{5}{6} \\ \hline \end{array}$$

b.
$$\begin{array}{r} -3\frac{5}{12} \\ +1\frac{9}{40} \\ \hline \end{array}$$

c.
$$\begin{array}{r} 7\frac{3}{4} \\ 8\frac{3}{5} \\ +7\frac{3}{10} \\ \hline \end{array}$$

d.
$$\begin{array}{r} 213 \\ -21\frac{5}{16} \\ \hline \end{array}$$

e.
$$\begin{array}{r} 9\frac{1}{4} \\ -12\frac{5}{6} \\ \hline \end{array}$$

f. $12\frac{1}{3} \times 5\frac{3}{8}$

g. $6\frac{3}{4} \times 14\frac{2}{3}$

h. $36 \times 3\frac{3}{4}$

i. $\frac{3}{10} \times 45$

2. Determine each of the following mentally, and briefly explain your solution.

a.
$$\begin{array}{r} 5\frac{3}{4} \\ +7\frac{5}{8} \\ \hline \end{array}$$

b.
$$\begin{array}{r} 8\frac{4}{5} \\ +6\frac{2}{3} \\ \hline \end{array}$$

c.
$$\begin{array}{r} 16\frac{3}{4} \\ 15\frac{1}{2} \\ +7\frac{1}{4} \\ \hline \end{array}$$

d.
$$\begin{array}{r} 26 \\ -6\frac{3}{5} \\ \hline \end{array}$$

e.
$$\begin{array}{r} 7\frac{1}{3} \\ -3\frac{1}{2} \\ \hline \end{array}$$

f. $8\frac{2}{3} + 7\frac{5}{6} + 4\frac{2}{3} + 2\frac{2}{3}$

g. $3\frac{2}{3} \cdot 12$

h. $2\frac{1}{2} \cdot 3\frac{1}{4}$

3. For each of the following, determine which of the choices represents the better estimate. Then explain your choice.

a. $5\frac{5}{8} + 4\frac{3}{42}$ Greater than 10 or less than 10?

b. $\frac{7}{8} + \frac{3}{4} + \frac{1}{16}$ Greater than 2 or less than 2?

c. $\frac{1}{4} + 1\frac{9}{10}$ Greater than 2 or less than 2?

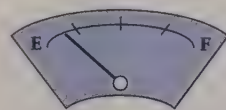
d. $8\frac{1}{2} - 2\frac{2}{3}$ Between which two numbers:
5 and $5\frac{1}{2}$ or $5\frac{1}{2}$ and 6?

e. $8\frac{3}{4} \times 2\frac{7}{8}$ Greater or less than 20?

f. $4\frac{7}{8} \div 8\frac{7}{8}$ Greater or less than $1/2$?

Finally, explain whether the “exact” answer is an exact number or still an estimate.

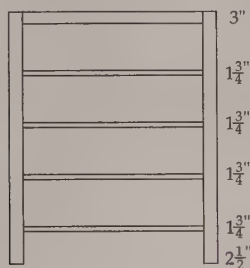
4. Sunflower seeds are sold in packages that weigh $3\frac{1}{4}$ ounces. If there is a supply of 66 ounces of sunflower seeds, how many packages of seed can be made? How many ounces of seeds will be left over?
5. What is the weight (in ounces) of a $4\frac{1}{2}$ -inch by $7\frac{3}{4}$ -inch rectangle of sheet metal if one square inch weighs $1/8$ ounce?
6. Betty is stacking boxes at a factory. Each box is $13\frac{3}{4}$ inches high. The ceiling is 16 feet. How many boxes can she stack in one pile?
7. Javier is $2/3$ as tall as his dad. Javier is 49 inches tall. How tall is his dad?
8. A recent survey concluded that $3/4$ of the teachers in a certain school district had at least 10 years of teaching experience. If there are 152 teachers in the district, approximately how many have at least 10 years of teaching experience?
9. We find from the *Unofficial U.S. Census* that more than 136,800,000 Americans admit to doodling. Of the doodlers, slightly less than $2/3$ said they doodle while talking on the telephone. Approximately how many Americans doodle while talking on the telephone?
10. Below is a picture of the gas gauge in Jackie’s car. If Jackie figures that her car averages 34 miles per gallon and her car has a 12-gallon tank, about how many miles does she have left on this tank before she runs out of gas?



11. John is making a bookcase. If the bookcase is $72\frac{3}{8}$ inches tall and he wants 5 shelves, equally spaced, tell him where to drill the holes on the side. Each shelf is $3/8$ inch thick. Your solution needs to contain a description that is clear enough so that he can understand it.
12. Julianna has $22\frac{1}{2}$ yards of material. If it takes $3/4$ yard of material to make a pair of shorts, can she make enough shorts for 30 children?
13. Johanna is making a chest of drawers. She has decided for aesthetic reasons that the base bar will be $2\frac{1}{2}$ inches above the ground, and the top will be beveled and will be 3 inches in height. She decides to have the four boards that support the drawers be $1\frac{3}{4}$ inches thick. How much space should she leave for each drawer? The

For Exercises 4 to 12, first estimate the answer, next explain your estimate, and then determine the “exact” answer.

chest of drawers will be 48 inches tall. [Note: The diagram is not drawn to scale.]



14. An analysis of freshmen at a college revealed that $\frac{1}{4}$ of the freshman women were from homes where both parents were professionals. Of these, $\frac{3}{5}$ were interested in the same profession as one or both of their parents. If this latter group is made up of 18 students, how many freshman women are there?
15. A small office purchases bottled drinking water. The water comes in containers that hold 5 gallons. Two-thirds of the office container was drained last week to put out a fire in the wastebasket.
 - a. How much water (in ounces) is left in the container?
 - b. How many 12-ounce glasses can be filled with the water that remains in the container?
16. Let's say that a certain antibiotic kills approximately $\frac{1}{2}$ of the germs on the first day, $\frac{1}{2}$ of the remaining germs on the second day, and so on.
 - a. After 10 days, what fraction of the germs remain?
 - b. If you had 20 million germs in your body at the beginning, how many of them will be alive at the end of the tenth day?
17. A fruit juice drink is $\frac{5}{7}$ water and $\frac{1}{6}$ fruit juice (by weight). Other additives (primarily sugar) make up what fraction of its weight?
18. In a certain town, $\frac{2}{3}$ of the women are married and only $\frac{1}{2}$ of the men are married. What fraction of the community is single? What assumptions do you make in order to solve the problem?
19. The Jacksons just bought some land in the country. Their deed shows that they bought $4\frac{1}{2}$ acres and that there is a $1\frac{1}{3}$ -acre pond on their land. The pond covers approximately what fraction of their property?
20. Four people order a pizza, which is cut into four pieces. Afterwards, three of the people are still hungry and order another pizza, which is also cut into 4 pieces. One person decides to eat only 1 piece, and the other two share what remains. What fraction of the total (two pizzas) does each person eat?
21. A manufacturing company finds that approximately 3 out of every 1000 spark plugs it produces are defective.

Last year the company produced 186,000 spark plugs. Approximately how many were defective?

22. In 1991, 207,610 rapes were reported to police. Estimates of the fraction of rape victims who report the rape range from $\frac{1}{2}$ to $\frac{1}{6}$. If $\frac{1}{2}$ of all rape victims reported the rape to police, approximately how many women were raped in 1991? If only $\frac{1}{6}$ of all rape victims reported the rape to police, approximately how many women were raped in 1991?
23. This is the kind of problem that really does happen—some data are lost, but other data can be used to retrieve them. Here are the employment data for a state:
 - $\frac{5}{8}$ of the population is urban, $\frac{3}{8}$ is rural.
 - $\frac{1}{10}$ of the urban population is unemployed, and $\frac{1}{6}$ of the rural population is unemployed.
 What fraction of the work force is unemployed? Illustrate this problem and solution with a diagram.
24. The Petersons have an apple-pressing machine that they use to make apple juice. The first pressing gets about $\frac{1}{3}$ of the total juice in the apples. Each succeeding pressing extracts about $\frac{1}{3}$ of the remaining juice in the apples. How many pressings will it take to get at least $\frac{3}{4}$ of the apples' juice? How many pressings will it take to get at least $\frac{9}{10}$ of the juice?
25. Make up a realistic story problem for $2\frac{1}{2} \div \frac{3}{4}$ and solve the problem using the meaning of fractions and division (that is, without the algorithm).
26. Make up a realistic story problem for the following multiplication-of-fractions contexts:
 - a. A proper fraction times a proper fraction
 - b. A proper fraction times a whole number
 - c. A proper fraction times a mixed number
 - d. A mixed number times a mixed number

27. Find the difference: $\frac{bc}{a} - \frac{bd}{ac}$

28.
 - a. How would you figure $26 \times \frac{11}{12}$ on a calculator? Show two ways.
 - b. How would you figure $26 \div \frac{11}{12}$ on a calculator?
29. Ginger has $8\frac{1}{3}$ yards of silk cloth from India. She has decided to divide the cloth equally into 3 parts, one for each of her siblings. How much cloth does each person get? *Do this problem without using an algorithm*—that is, using reasoning. You may want to use a diagram, but you are not required to do so.
30. Use a number line model or a set model to find $\frac{2}{3} + \frac{1}{4}$. Justify each step in the process.
31. When we convert $\frac{13}{5}$ to $2\frac{3}{5}$, which model of division do we use? Explain your answer.
32. Irene has discovered a rule that works for some addition problems. For example, consider $\frac{1}{5} + \frac{1}{2}$. The rule is this: The numerator of the answer is equal to the sum

of the denominators of the two fractions, and the denominator of the answer is equal to the product of the denominators of the two fractions. What do you think of her rule? Describe the set of fractions for which this rule works. Can you modify her rule so that it works for a larger set of problems?

33. A student says that she has found a quick way to divide fractions. She demonstrates her method as follows:

$$9\frac{1}{4} \div 3\frac{3}{4} = 3\frac{1}{3}, \text{ because } 9 \div 3 = 3 \text{ and } \frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$$

$$20\frac{3}{5} \div 4\frac{3}{5} = 5\frac{1}{1}, \text{ because } 20 \div 4 = 5 \text{ and } \frac{3}{5} \div \frac{3}{5} = 1$$

Unfortunately, her idea is not correct. Where is the conceptual error?

34. Demonstrate how to obtain the following product two different ways, neither of which can be the standard algorithm: $3\frac{1}{2} \times 2\frac{1}{2}$. Then explain why $6\frac{1}{6}$ is not the correct answer, as though you were talking to a student for whom $6\frac{1}{6}$ is the obvious answer.

35. What is wrong with the students' work in these problems?

a. $\frac{3}{5} + \frac{2}{5} = \frac{5}{10}$

b. $\frac{1}{5} + \frac{2}{3} = \frac{3}{8} + \frac{10}{8} = \frac{13}{8} = 1\frac{3}{8}$

c. $\frac{2}{3} \cdot \frac{3}{8} = \frac{9}{16}$

d. $8\frac{1}{8} \div 2\frac{1}{4} = 4\frac{1}{4}$

36. Can you explain the following mistake?

$$\begin{aligned} 7\frac{1}{8} &= 6\frac{11}{8} \\ -3\frac{5}{8} &= 3\frac{5}{8} \\ \hline 3\frac{6}{8} &= 3\frac{3}{4} \end{aligned}$$

37. Locate and explain the error in the following:

$$\frac{ab + c}{a} = \frac{ab + c}{a} = b + c$$

38. Selecting among the numbers 1 through 9 and repeating none of them, fill in the boxes below to make the sum as close as possible to 1, but not equal to 1.

$$\frac{\square}{\square} + \frac{\square}{\square}$$

39. Use each of these numbers exactly once to make the following equation true: 1, 2, 3, 5, 11, 15.

$$\frac{\square}{\square} + \frac{\square}{\square} = \frac{\square}{\square}$$

40. Selecting among the numbers 1 through 9 and repeating none of them, fill in the boxes below to make the smallest possible difference greater than zero.

$$\frac{\square}{\square} - \frac{\square}{\square}$$

41. Selecting among the numbers 1 through 9 and repeating none of them, make the following equation true.

$$\frac{\square}{\square} \times \frac{\square}{\square} = \frac{1}{2}$$

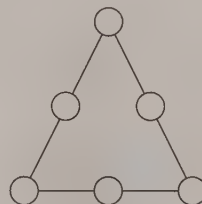
42. Selecting among the numbers 1 through 9 and repeating none of them, make the largest possible quotient.

$$\frac{\square}{\square} \div \frac{\square}{\square}$$

43. Selecting among the numbers 1 through 9 and repeating none of them, make the quotient closest to one-half.

$$\frac{\square}{\square} \div \frac{\square}{\square}$$

44. Can you make a magic triangle consisting only of unit fractions so that no fraction occurs more than once?



SECTION

5.4

BEYOND INTEGERS AND FRACTIONS: DECIMALS, EXPONENTS, AND REAL NUMBERS

In this section, we will extend the system of numbers yet again—to decimals and to irrational numbers (that is, numbers that cannot be expressed as the quotient of two integers, such as $\sqrt{2}$ and π). There is a name for the set of numbers that includes both rational and irrational numbers: the set of real numbers (an interesting choice for this set!). There are more sets of numbers beyond the real numbers, but they are not considered until high school.

WHAT DO YOU THINK?

- What would Alhabitian decimals look like?
- Why is there no “oneths” place?

Decimals

Technically, the set of decimals is not an extension of the set of fractions. Rather, our use of decimal numbers comes out of an alternative way of representing fractions, using the advantages of our base 10 system. In fact, the word *decimal* comes from the Latin word *decem*, which means “ten.” Thus, decimal numbers are any numbers that are written in terms of our base 10 place value. Conventionally, we refer to decimals as fractions that have been written using a decimal point.

The history of decimals If you stop to think about it, decimals could not have been invented before our base 10 system. Credit for bringing decimals into everyday life goes to Simon Stevin, who was the first to try to explain them. His book *The Art of Tenths or Decimal Arithmetic* was first published in 1585.

We will speak freely of the great utility of this invention . . . for the astronomer knows the difficult multiplication and divisions which proceed from the progression with degrees, minutes, seconds, and thirds. . . . This discovery . . . teaches (to tell much in one word) to compute easily, without fractions, all computations which are encountered in the affairs of human beings, in such a way that the four principles of arithmetic which are called addition, subtraction, multiplication, and division . . . are able to achieve this end.” He went on to say that this work treats of “something so simple that it hardly merits the name of invention.”⁶

The use of decimals was spurred primarily by a desire to make computation easier. For example, which computation would be easier to do: $24\frac{7}{8} \times 6\frac{4}{5}$ or 24.875×6.8 ? You may want to do both. How much faster was the decimal computation?

Even today there is not one universal notation. For example:

United States	England	Continental Europe
54.23	54·23	54,23

No decimals? Imagine our world without decimals! As you discovered earlier in this course, the base 10 numeration system has been in widespread use for less than 500 years. The widespread use of decimals is even more recent.

What would life in the United States be like if no one had invented decimals—that is, if we expressed amounts using only whole numbers and fractions? Would things just look different (for example, 43 dollars and 26 cents instead of \$43.26), or would some things be impossible?

You might find it helpful to look at cases in which we use decimals and to explore alternatives, as shown above for \$43.26. Spend some time writing your answers to these questions before reading on. . . . 

As you may have realized as you thought about these questions, much of our use of decimals involves measurement:

- We measure mass: for example, the package weighs 2.4 ounces.
- We measure distance: for example, Joan lives 3.7 miles from school.

⁶Louis Karpinski, *The History of Arithmetic* (New York: Russell & Russell, 1965), p. 131.

- We measure time: for example, the world record for the 100-meter dash is 9.79 seconds.

Decimals are used in many aspects of daily life to communicate. For example,

- The U.S. birth rate is 14.4 per 1000 population.
- The average life expectancy in the United States is 76.5 years.
- Interest rates are expressed with decimals, such as 7.5 percent.
- The new city budget is \$3.48 million.

Developing decimals The next 14 investigations look at decimals from several perspectives: examining different characteristics of the decimal system, why the algorithms work, rounding, scientific notation, and using decimals when solving problems. Using your increasing problem-solving, communicating, and reasoning abilities and your increasingly better connected network of mathematical concepts, you will be able to strengthen your own understanding of these concepts.

When children possess a sound understanding of fraction and decimal concepts, they can use this knowledge to describe real-world phenomena and apply it to problems involving measurement, probability, and statistics. . . . It is critical in grades K–4 to develop concepts and relationships that will serve as a foundation for more advanced concepts and skills.

(Curriculum Standards, p. 57)

Connecting decimals to integers and fractions I have found that many students who plan to teach elementary school do not feel that investigations with decimals are relevant because they believe that they will never teach decimals. In fact, although formal work with decimals is generally taught in middle school, informal work with decimals is an important part of the elementary school curriculum. What the NCTM is stressing is that in order to prepare elementary students for the decimal instruction they will receive in middle school, the elementary teacher needs to understand the conceptual relationships between decimals and whole numbers and between decimals and fractions.

The set of decimals has important connections to both the set of fractions and the set of integers. Much research indicates that these connections are not understood by a majority of students, and with severe consequences—limitations in ability to solve nonroutine problems and to work confidently in higher mathematics. The following investigations will help you to understand both similarities and differences between decimals and these other two sets.

Connecting Decimals and Integers

There are many important similarities and differences between the set of decimals and the set of integers. Before reading on, stop to think about the two sets of numbers. What similarities and differences can you see? 

From one perspective, the set of decimals can be seen as an extension of base 10 counting numbers.

... thousands	hundreds	tens	ones	tenths	hundredths	thousandths ...
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
1000	100	10	1	0.1	0.01	0.001

That is, the system of decimals and the system of integers are both base 10 place value systems. The value of each digit depends on its place, and the value of each place to the right is one-tenth the value of the previous place. Exploration 5.17 helps to develop your decimal sense and give you a better “feel” for the relative size of tenths, hundredths, thousandths, and so on. Rather than state the differences, we will let them emerge through investigation.

INVESTIGATION 5.17 Base 10 Blocks and Decimals

One of the themes of this book has been the importance of multiple representations for understanding mathematical concepts. In order to develop your understanding of decimals, we will investigate two representations: base 10 blocks (see Exploration 5.14 for more work on this model) and number line models (after Investigation 5.20). Each representation highlights different aspects of decimals and their relationship to other systems.

Let’s say that a sample of gold weighs 3.6 ounces. How would you use base 10 blocks (shown in Figure 5.36) to represent this amount? Note that the “small cube” was formerly called a “unit.” Work on this and then read on. . . .

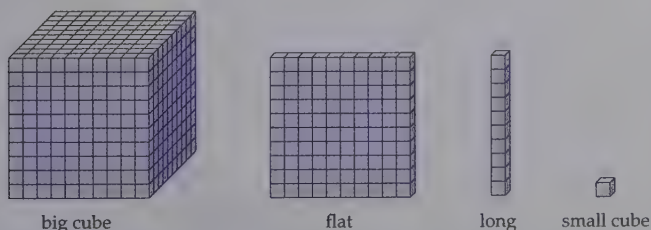


FIGURE 5.36

DISCUSSION

Any solution requires us to designate one of the pieces as the unit. For example, if we designate the big cube as having a value of 1, then we would represent 3.6 ounces as shown in Figure 5.37. Do you see why?



FIGURE 5.37

However, we could also have selected the flat to be the unit. If we had, then we would represent 3.6 ounces as shown in Figure 5.38.



FIGURE 5.38

Zero and Decimals

We will examine the role of zero in decimals in the following two investigations.

INVESTIGATION 5.18 When Two Decimals Are Equal

What if we added one or two zeros at the end of 0.2? Would that change its value? That is, do 0.2, 0.20, and 0.200 have the same value? Many sixth and seventh graders (and a surprising number of adults) believe that 0.2, 0.20, and 0.200 not only look different but also have different values because of the added zeros. How would you convince such a person that the value is not changed? Recall that in the context of the NCTM standards, this involves justifying the reason—that is, explaining the *why*. Work on this and then read on. . . .

DISCUSSION

There are several ways to justify the equality. We will consider two below.

STRATEGY 1: Connect decimals to fractions

- 0.2 is equivalent to the fraction $2/10$.
- 0.20 is equivalent to the fraction $20/100$, which can be shown to be equivalent to $2/10$.
- 0.200 is equivalent to the fraction $200/1,000$, which can be shown to be equivalent to $2/10$.

This strategy uses an important mathematical property. The **transitive property** is stated below for any three numbers, a , b , and c .

Transitive property of equality:

$$\text{If } a = b \text{ and } b = c, \text{ then } a = c.$$

Transitive property of inequality:

$$\text{If } a < b \text{ and } b < c, \text{ then } a < c.$$

Applying the transitive property of equality, we say that because 0.2, 0.20, and 0.200 are all equal to $2/10$, they are equal to each other.

STRATEGY 2: Use manipulatives

In this case, let a square represent a value of 1. Such manipulatives are sold commercially as Decimal Squares®. We can divide this square into 10 equal pieces, 100 equal pieces, or 1000 equal pieces, as in Figure 5.39.

Shade in 0.2 of the first square, 0.20 of the second square, and 0.200 of the third square.

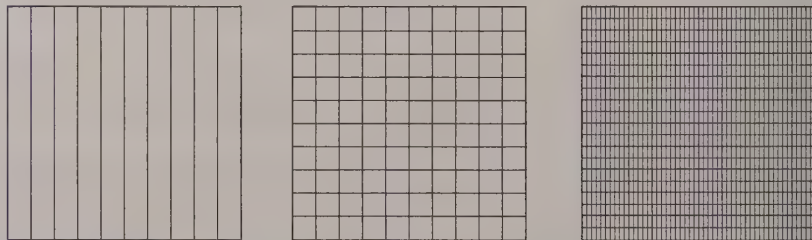


FIGURE 5.39

We find that in each case, the same area is shaded. Connecting this result to the area model of fractions, we conclude that 0.2, 0.20, and 0.200 are all equivalent decimals.



MATHEMATICS

The idea of multiple representations connects to our work with whole numbers. For example, just as

$$24 = 20 + 4$$

can be represented by 2 tens and 4 ones or by 24 ones, so too

$$0.24 = 0.2 + 0.04$$

can be represented by 2 tenths and 4 hundredths or by 24 hundredths.

INVESTIGATION 5.19 When Is the Zero Necessary and When Is It Optional?

An important difference between decimals and whole numbers has to do with our old friend zero. Examine each number below. In each case, explain whether you think the zero in the number is necessary, optional, or incorrect. If you think it is necessary, explain why. If you think it is optional, explain why the zero doesn't matter. If you think the use of the zero is incorrect, explain why the zero should not be there. 

- A. 2.08 B. 0.56 C. .507 D. 20.6 E. 3.60

DISCUSSION

- A** With 2.08, the zero is necessary; 2 wholes, 0 tenths, and 8 hundredths.
- B** With 0.56, the use of the zero is optional; it is more of a convention, like shaking hands with the right hand instead of the left.
- C** With .507, the zero is necessary: 5 tenths, 0 hundredths, and 7 thousandths.
- D** With 20.6, the zero is necessary: 2 tens, 0 ones, and 6 tenths.

E In one sense, the zero here is optional. Mathematically, $3.6 = 3.60$; you can verify this using the strategies in Investigation 5.18. In another sense, however, it depends on how the number is being used. For example, if we ask how long the room is, what is the difference between a response of 3.6 meters and a response of 3.60 meters?

A response of 3.6 meters implies that the room has been measured to the nearest tenth of a meter; that is, the length of the room is closer to 3.6 meters than to 3.5 or 3.7 meters. If the person doing the measuring has used an instrument that makes possible only this amount of precision but nevertheless reports the length as 3.60, I call this an example of pseudo-precision.

Some differences between decimals and integers This investigation illustrates some important differences between decimals and integers. With decimals, when we add a zero at the right end of the number, the value is unchanged; for example, $48.6 = 48.60$. As you know, this is not true with integers; for example, $48 \neq 480$. Two other differences between the two systems are worth noting. One has to do with language: With whole numbers, the suffix for each place is *-s*; with decimals, it is *-ths*—for example, the hundreds place versus the hundredths place. Another difference is that there is a ones place but not a oneths place; this lack of symmetry has been noted by more than one youngster. How would you explain it?



BEYOND THE CLASSROOM

We see examples of pseudo-precision every day. For example, the winter of 1993–1994 was one of the most severe winters in the Northeast in many years. A poll reported that 83.4 percent of people in the Northeast described that winter as “the worst I have ever experienced.” Given that polls generally have an error margin of several percent, this reported degree of precision is an example of pseudo-precision. It would have been more valid to report that about 83 percent of the persons polled agreed with that statement.

INVESTIGATION 5.20 Connecting Decimals and Fractions

As stated before, each decimal can be translated into a fraction. You may be familiar with common translations between decimals and fractions: $0.5 = 1/2$, $0.25 = 1/4$, and so on.

Many problem-solving situations require us to convert a fraction to decimal form or vice versa.

- A.** How many of the following common fractions can you convert to decimals immediately? How can you determine the decimal equivalent of others without using a calculator? 

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{8} \quad \frac{1}{10} \quad \frac{1}{16} \quad \frac{1}{50}$$

$$\frac{1}{100} \quad \frac{1}{1000}$$

DISCUSSION

Either by hand or with a calculator, all but $1/3$ and $1/6$ can be readily converted to decimal form. When we convert $1/6$ to a decimal (by dividing 1 by 6), we find that it repeats; that is, the value is $0.1666666\ldots$. There are two ways to write repeating decimals. We can write $1/6$ in decimal form as $0.166\ldots$ or as $0.1\overline{6}$. The bar shows the part that repeats. For example, we would write $348/999$ as $0.348484848\ldots$ or as $0.34\overline{8}$.

- B.** Now let us examine translating decimals into fractions. Translate the decimals 0.1 , 0.007 , and $0.\overline{18}$ into fractions before reading on. 

DISCUSSION

Most people have no trouble with the first two: $1/10$ and $7/1000$. The last decimal is equivalent to $2/11$. Someone (unknown to history) discovered the process that enables us to conclude that there is a unique fraction associated with every repeating decimal. The process works like this:

$$\text{Let } x = 0.18181818\ldots \text{ (that is, } 0.\overline{18}\text{).}$$

Now multiply both sides by 100, so that we have

$$100x = 18.\overline{18}$$

Do you see how these two equations allow us to determine the exact value of $0.\overline{18}$? Think and then read on. 

The key is to subtract the first equation from the second. The repeating part drops out, and we have $99x = 18$. When we solve the equation for x and simplify the fraction, we find that $0.181818\ldots$ is simply $2/11$ in disguise! You can check this by doing the division $2 \div 11$.

Decimals, Fractions, and Precision

We have used number lines as a model to represent whole numbers and fractions. The number line can also be used to represent decimals, using the analogy of a zoom lens. Suppose you had never heard of decimals and had to determine the length of the wire in Figure 5.40 as accurately as you could. Do this and then read on. 

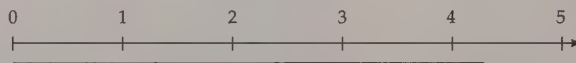


FIGURE 5.40



BEYOND THE CLASSROOM

Many mathematical concepts are misused in our society. One classic example of misuse of decimals occurs in baseball. Statistics for pitchers include the total number of innings pitched. There are 3 outs per inning, so if the pitcher pitched 7 innings and got 1 out in the eighth inning before being relieved, we would record his total innings pitched as $7\frac{1}{3}$ innings. If he had gotten two outs before being relieved, we would have recorded his total innings pitched as $7\frac{2}{3}$ innings. However, what the newspapers print is 7.1 and 7.2. Explain why this is mathematically incorrect. Why do you think they do this?

**MATHEMATICS**

Do you see how the number line model connects to the density principle for fractions that we discussed in Section 5.2?

A visual inspection of Figure 5.40 shows that the wire appears to be closer to $4\frac{1}{3}$ inches than to $4\frac{1}{4}$ inches. With decimals, we do not look at the most appropriate unit fraction; instead, we look at the fractional length in terms of tenths, hundredths, thousandths, and so forth. Now imagine using a zoom lens (see Figure 5.41). To the nearest tenth of an inch, how long is the wire?

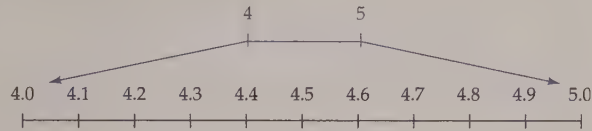


FIGURE 5.41

Because the length is closer to 4.3 inches than to 4.4 inches, we say that the length is 4.3 inches. If we want more precision, we can keep zooming in, in which case we move more decimal places to the right. Theoretically, we can continue this magnification process indefinitely. Realistically, electron microscopes are able to measure distances of about one ten-millionth of an inch, or 0.0000001 inch.

INVESTIGATION 5.21 Ordering Decimals

Just as young children need time to develop proper ordering relationships with whole numbers (for example, that 30, not 31, comes after 29), older children need time to develop ordering relationships with fractions and with decimals. (Exploration 5.14 provides more decimals to order.)

Order the following decimals, from smallest to largest: 0.39, 0.046, and 0.4. Justify your solution. There are several different strategies. If you answer the question using one strategy, go back and try to use a different strategy. Do this before reading on. . . .

**DISCUSSION****STRATEGY 1: Use equivalent fractions**

$$0.39 = \frac{39}{100} = \frac{390}{1000}$$

$$0.046 = \frac{46}{1000}$$

$$0.4 = \frac{4}{10} = \frac{400}{1000}$$

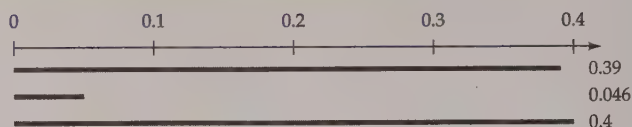
Thus, the order is 0.046, 0.39, 0.4.

STRATEGY 2: Line up the decimal points and add zeros

$$0.39 = 0.390$$

$$0.046 = 0.046$$

$$0.4 = 0.400$$

STRATEGY 3: Represent them on a number line**INVESTIGATION 5.22 Rounding with Decimals**

If you stop and think about it, rounding of decimals occurs frequently. For example, the rate I am charged for electricity is 11.146 cents per kilowatt-hour. If someone asked me what the electric rate is, however, I would say about 11 cents per kWh. When doing tax forms, the IRS allows people to round some amounts to the nearest dollar. Thus, I might record postage expenses as \$75 instead of \$74.68.

The process of rounding with decimals is very similar to the process of rounding with whole numbers. In fact, if you *really* understand place value, then rounding with decimals is very straightforward. As you may have found with the Right Bucket game (Exploration 5.16), being able to round decimals is important when estimating.

Round each of the numbers to the nearest hundredth and to the nearest tenth:

- A.** 3.623 **B.** 76.199 **C.** 36.215 **D.** 2.0368
- E.** As you work on these, take mental stock of your confidence level. If you do not feel 100 percent confident, what models (discussed earlier) might you use to increase your confidence? 

DISCUSSION

- A.** 3.623 to the nearest hundredth is 3.62, and to the nearest tenth is 3.6.
- B.** 76.199 to the nearest hundredth is 76.20, and to the nearest tenth is 76.2.
- C.** 36.215 to the nearest hundredth is 36.22, and to the nearest tenth is 36.2.
- D.** 2.0368 to the nearest hundredth is 2.04, and to the nearest tenth is 2.0.
- E.** Some students find the number line or physical models useful in determining how to round. One other common strategy looks like this: Look at 76.199. If we look at only two decimal places, we have 76.19. Therefore, the question can be seen as: Is 76.199 closer to 76.19 or 76.20?

If we insert zeros so that all three decimals are given in thousandths, we have:

76.200
76.199
76.190

Do you see how this strategy helps?

The number 36.215 requires us to deal with a 5 in the relevant place. To the nearest hundredth, this question becomes a decision between 36.21 and 36.22.

Remember that with whole numbers, the decision was not to round up whenever we encounter a 5, but rather to round so that the last nonzero digit is an even number. We will apply that rule here. Thus, we round 36.215 up to 36.22, but we round 36.245 down to 36.24.

Many students round confidently except when they encounter a zero. For example, many students feel comfortable stating that 2.0368 to the nearest hundredth rounds to 2.04 but do not feel comfortable stating that 2.0368 to the nearest tenth rounds to 2.0. If you feel this discomfort, I suggest using a number line, the base 10 blocks, or the strategy mentioned at the beginning of this discussion so that your understanding of rounding becomes more owned than rented!

INVESTIGATION 5.23 Decimals and Language

I saw the following line in a newspaper: "The School Board has proposed a budget of \$24.06 million." Express the School Board's proposed budget as a whole number. Do this before reading on. . . . 

DISCUSSION

I have received many different answers from students in my own class, including the following:

24 million 600 thousand dollars

24 million 60 thousand dollars

24 million 6 hundred dollars

24 million and 6 dollars

24 million dollars and 6 cents

Let us look at several strategies that students have used to determine the correct answer.

STRATEGY 1: Connect decimals to fractions

Some students who have successfully used this strategy did not know how to represent 24.06 million at first, and so they began with decimal/fraction connections that they did know.

24.5 million = 24,500,000 because $0.5 = 1/2$

24.25 million = 24,250,000 because $0.25 = 1/4$

24.1 million = 24,100,000 because $0.1 = 1/10$ [This is often a hard place for many students to start, but it makes sense now, in light of the first two steps. Does it make sense to you?]

24.06 million = 24,060,000 because it must be less than 24,100,000

STRATEGY 2: Connect decimals to expanded form

Some students who have successfully used this strategy did not know how to represent 24.06 million at first and found the solution by analyzing the meaning of 0.06 million.

24.06 million means 24 whole millions and six hundredths of another million.

Now we have to find the value of six hundredths of a million.



LANGUAGE

The language we use when we write checks is connected to place value. How would you write this amount: \$206.45?

It would be incorrect to say "two hundred and six dollars and 45 cents." Can you see why? To reduce confusion, the convention is to use *and* only once—just before giving the cents. Thus, the conventional way to write \$206.45 is "two hundred six dollars and 45 cents."

Using a calculator or computing by hand yields

$$0.06 \times 1,000,000 = 60,000$$

Thus, applying our knowledge of expanded form, we get

$$24.06 \text{ million} = 24,000,000 + 60,000 = 24,060,000$$

Once again we come back to the importance of being able to compose and decompose numbers: whole numbers, integers, fractions, and now decimals.

STRATEGY 3: Connect decimals to place value

If your understanding of place value is powerful, you know that the *places* of the digits will not change. Thus, when we represent 24.06 million as a whole number, we simply “add zeros” and we have 24,060,000. The *place* of the 6 does not change.

Operations with Decimals: Addition and Subtraction

With the widespread use of calculators, it is not common for most people to do most decimal computations longhand any more. However, it is crucial to understand why the basic algorithms work because it is often necessary to interpret what the calculator displays; recall Investigation 1.15; the “answer” on the calculator display didn’t indicate how much wire was wasted.

Few people make mistakes when adding or subtracting decimals if all the numbers have the same number of places, such as $3.24 + 5.56$. However, a majority of middle school students have difficulty with a problem like the following: $7.8 + 0.46$. Can you explain how to determine the sum and justify each step? You may recall being taught to “line up the decimal places,” but why does that rule work?

We can justify the lining up of decimal places in terms of both whole-number computation and fraction computation. When adding whole numbers, we begin with the ones place and move to the left, regrouping as necessary. Adapting this procedure to decimals, we begin with the smallest place, in this case, the hundredths place. We have 6 hundredths. Moving to the tenths place, we have 12 tenths (regrouped to 1 and 2 tenths). Moving to the ones place, we have 7 plus the regrouped 1, giving us a sum of 8 ones, 2 tenths, and 6 hundredths—that is, 8.26.

From the perspective of fractions, we need common denominators. Thus, representing the decimals in expanded form and applying the commutative and associative properties, we have

$$7.8 + 0.46$$

$$\begin{array}{r} 7.8 \\ +0.46 \\ \hline \end{array}$$

$$7.8 + 0.46$$

$$= \left(7 + \frac{8}{10}\right) + \left(\frac{4}{10} + \frac{6}{100}\right)$$

$$= 7 + \left(\frac{8}{10} + \frac{4}{10}\right) + \frac{6}{100}$$

$$= 7 + \frac{12}{10} + \frac{6}{100}$$

$$= 7 + 1 + \frac{2}{10} + \frac{6}{100}$$

$$= 8 + \frac{20}{100} + \frac{6}{100}$$

$$= 8 + \frac{26}{100}$$

$$= 8.26$$

Multiplication with Decimals

Children often find multiplication with decimals confusing. Let us examine two problems that arise:

- How do we know where to put the decimal point?
- How can you multiply tenths by tenths and get hundredths—for example, $3.2 \times 2.6 = 8.32$?

Note: There are three ways to represent multiplication: 3.2×2.6 , $3.2 \cdot 2.6$, and $(3.2)(2.6)$. The dot, which we use to represent whole-number multiplication, can be confusing when we are showing decimal multiplication, and so we shall use the $\times$ or parentheses.

Let us examine the problem 3.2×2.6 . Stop and look at the precision in this problem. We multiply two numbers that are accurate to the tenths place, but our answer includes a number in the hundredths place. For example, let's say the dimensions of a room are 3.2 meters by 2.6 meters, but its area is 8.32 square meters. If both the multiplier and the multiplicand are in tenths, why isn't the product in tenths also?

We look to a diagram for the explanation. We know from Chapter 3 that any multiplication problem can be represented by a rectangle; that is, the product of the two numbers is equal to the enclosed area. In this case, let us make a rectangle that is 3.2 meters long and 2.6 meters wide [Figure 5.42(a)].

Before reading on, please work on the following questions:

- How can you determine the area of the rectangle directly from the diagram?
- What are the connections between the diagram and the standard algorithm for multiplying decimals?

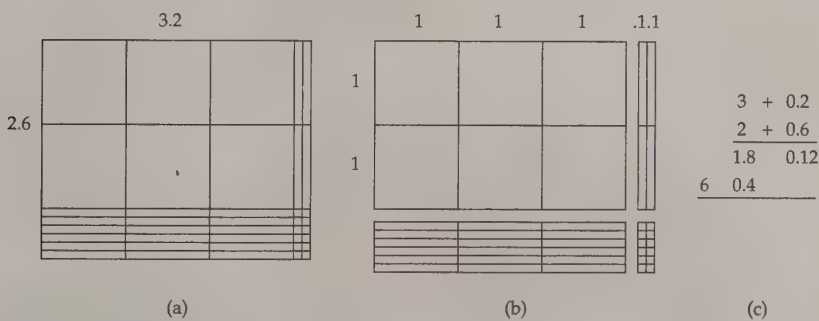


FIGURE 5.42

In this diagram, we have four distinct regions that correspond to the four partial products [see Figures 5.42(b) and (c)]. The top left region consists of 6 squares (each of which measures 1 meter by 1 meter). Thus, the top left region has a value of 6 square meters.

The top right region consists of 4 longs, each of which measures 1 meter by 0.1 meter. Thus each long has a value of 0.1 square meter. The value of the top right region, then, is 0.4 square meter.

The bottom left region consists of $3 \times 6 = 18$ longs and has a value of 1.8 square meters.

The bottom right region consists of 12 little squares, each of which is 0.1 meter by 0.1 meter. Thus the value of each square is 0.1×0.1 , or 0.01 square

meter. Therefore, the value of this region is 0.12 square meter, because we have $2 \times 6 = 12$ of these little squares.

Adding up the value of the four regions, we have $6 + 0.4 + 1.8 + 0.12 = 8.32$ square meters.

The calculation at the left below shows the sums of the four regions, and the calculation at the right shows the standard multiplication procedure. Do you see the connections?

3.2	3.2
<u>$\times 2.6$</u>	<u>$\times 2.6$</u>
1.8	192
.12	<u>64</u>
6	8.32
<u>.4</u>	
8.32	

We are still applying the distributive property. Going back to the meaning of whole-number multiplication, the multiplication algorithm simply tells us the sums of the four regions. The first row of the algorithm (the digits 192) tells us the product of 3.2×0.6 , and the second row (64) tells us the product of 3.2×2 . The beauty of the procedure is more evident when we realize that $3.2 \times 6 = 1.92$ and $3.2 \times 2 = 6.4$ don't "line up." However, when we move 64 over one place, the two partial sums line up just right. We are not generally aware of this lining up; we just remember how many decimal places to move over in the answer column.

Division with Decimals

Most of you know the procedure for dividing with decimals. For example, let us say that you travel 247.9 miles on 7.4 gallons of gas. To determine the miles-per-gallon, you divide 247.9 by 7.4. If you don't have a calculator, you know to move the decimal points over one place; you also know that the answer is 33.5 because you move the decimal point in the dividend straight up. This procedure is illustrated below. Can you provide a mathematical justification for moving the decimal point? Think and then read on. . . . 

$$\begin{array}{rcl}
 & & 33.5 \\
 7.4 \overline{)247.9} & \longrightarrow & 74 \overline{)2479.0} \\
 & & \underline{222} \\
 & & 259 \\
 & & \underline{222} \\
 & & 370 \\
 & & \underline{370} \\
 & & 0
 \end{array}$$

One key is the idea of multiple representations, which has been presented many times. We can also represent this division problem in the following form: $247.9/7.4$. This form connects to the idea of equivalent fractions. If we multiply this decimal fraction by $10/10$, we have

$$\frac{247.9}{7.4} \times \frac{10}{10} = \frac{2479}{74}$$

In other words, $247.9/7.4$ has the same value as $2479/74$. By moving the decimal point over, we create an equivalent computation in which the divisor (denominator) is a whole number.

INVESTIGATION 5.24 **Decimal Sense: Grocery Store Estimates**

An understanding of how the algorithms work increases one's ability to estimate, as you may have found in Exploration 5.16. As we have stressed throughout this book, the development of number sense is crucial if people are to be able to apply their knowledge to solve multistep and non-routine problems. In this and the next investigation, we will examine some estimation problems. We will pay attention to adapting estimation strategies developed with whole numbers and fractions and to using properties developed in Chapter 3 when appropriate.

In doing math outside the classroom, we generally find that we are more likely to solve decimal problems by estimating than by doing pencil-and-paper mathematics. Let's say that it is Friday afternoon. You drop by the supermarket to get a few items. Suddenly you realize that you didn't go to the bank, and you find that you have only \$7.45. The milk is \$1.95. The ice cream is \$1.87. Grapes are 89¢ a pound, and you have just over a pound. The bag of potato chips is \$1.69. The loaf of bread is \$1.29. Do you have enough money? How would you estimate this amount in your head? Try it (see the column of numbers at the left) and then read on. . . .

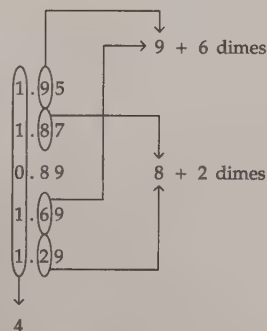
1.95
1.87
89¢ a pound
1.69
1.29

DISCUSSION**STRATEGY 1: Use leading digit and compatible numbers**

Looking at the "dollars column" (that is, the ones column), we have 4. Next, we look at the "dimes" (tenths) column.

This is what that strategy sounds like:

- 8 (dimes) + 2 (dimes) is one dollar.
- 9 (dimes) + 6 (dimes) is a dollar fifty, so that's two fifty.
- 8 (dimes) more makes three thirty, plus the 4 (dollars). That's seven thirty plus the cents. You don't have enough money!

**STRATEGY 2: Round up and use compatible numbers**

\$1.95 is almost \$2

\$1.70 + 1.30 = \$3

\$0.89 + \$1.87 ≈ \$0.90 + \$1.90 = \$2.80

Thus, we need almost \$7.80, and we know that the actual sum is lower than this estimate. In situations like this, people often intentionally overestimate because they don't want to find out at the checkout counter that they don't have enough money.

Just a reminder that there are many ways to estimate. The focus should not be on "getting" the "right" way, but rather on developing ways that make sense to you and that are reasonably sophisticated.

This next investigation offers several ways to apply ideas from different concepts we have studied.

INVESTIGATION 5.25 Decimal Sense: How Much Will the Project Cost?

Let's say a contractor needs 308 sheets of plywood for a project. He can get them for \$7.55 each at one store. He could also get them for \$7.39 at another store, but he doesn't particularly like the owners, and more paperwork would be required. How much money would he save if he went to the second store? Estimate the answer as closely as you can without using a calculator or pencil and paper. Then read on. . . . 

DISCUSSION

We could try to estimate 308×7.55 and 308×7.39 , but this would be quite tedious. With a bit of reflection, we realize that what we need to know is not the approximate total cost but the difference. Because the difference is 16¢, or \$.16, per sheet, we find that we really have to estimate $308 \times $.16$ or $308 \times 16\text{¢}$.

STRATEGY 1: Convert to fractions

$$0.16 \times 308 \approx \frac{1}{6} \times 300 = \$50$$

STRATEGY 2: Use substitution and the associative property

$$\begin{aligned} 0.16 \times 308 &\approx 0.16 \times (100 \times 3) \\ &= (0.16 \times 100) \times 3 \\ &= 16 \times 3 \\ &= \$48 \end{aligned}$$

INVESTIGATION 5.26 How Much Did They Pay for Their Home?

Now that we have explored the mechanics of estimation, let us examine estimation in context. This adds the dimension of deciding what operation to use, deciding how to interpret the result, and the like.

The Hill family is moving. The value of their home is two and a half times what they paid for it, and it is now worth \$123,250. Approximately how much did they pay for it? 

DISCUSSION

Before we can estimate, we have to translate the problem into a form that enables us to estimate.

We can translate this problem as

$$\text{Buying price} \times 2.5 = \$123,250$$

This is a division (missing factor) problem, and the answer will be $123,250/2.5$.

STRATEGY 1: Guess-check-revise

What number times 2.5 is equal to \$123,250?

First guess: 40,000; check: $40,000 \times 2.5 = 100,000$; revise.

Second guess: 50,000; check: $50,000 \times 2.5 = 125,000$.

The Hills paid about \$50,000 for their house.

STRATEGY 2: Use compatible numbers

$$\frac{120,000}{2} = 60,000 \quad \frac{120,000}{3} = 40,000, \text{ so } \frac{120,000}{2.5} \approx 50,000$$

**LEARNING**

As stated throughout the book, there is considerable evidence that many of our children enter high school with little ability to apply their mathematical knowledge to nonroutine problems. Consider the following examples from the National Assessment of Educational Progress. Students were given the following problem and asked to circle the correct product among five choices.

Select the correct answer to 3.04×5.3 .

Many of you might say, "Well, 72 percent of all eleventh graders is not that bad (see Table 5.1). After all, a lot of eleventh graders don't go on to college, and a lot of eleventh graders aren't taking any math and may be a bit rusty." But now look at a very similar problem, which actually should have been easier because the students were asked only to *estimate* the product.

TABLE 5.1

PERCENT CORRECT	
Grade 7	Grade 11
57	72

Which of the following is the best estimate of 3.04×5.3 ?

As Table 5.2 shows, barely one out of five seventh graders was able to see that 16 is the best estimate of 3.04×5.3 ! In fact, more seventh graders chose 1.6, probably because it was the only answer that had a decimal point. Barely one out of three eleventh graders recognized that 16 is the best estimate of 3.04×5.3 !

TABLE 5.2

PERCENT RESPONDING		
	Grade 7	Grade 11
1.6	28	21
16	21	37
160	18	17
1600	22	11
I don't know	9	12

INVESTIGATION 5.27 How Long Will She Run?

A runner is running at 7.6 meters per second. At this rate, about how long will it take her to run 100 meters? 

DISCUSSION

This problem is a classic example of how decimals can obscure simple relationships. The problem-solving strategy of using simpler numbers is powerful

here. For example, suppose the runner was running at 5 meters per second. How long would it take her to run 100 meters? With these simpler numbers, it is almost immediately apparent that it will take 20 seconds; that is, divide 100 by 5. We can now apply this understanding to this problem and see that it will take her $100/7.6$, or about $100/8$, seconds, which is about 12 or 13 seconds. Over my teaching career, I have seen many students be very intimidated by “ugly” numbers.

INVESTIGATION 5.28 Exponents and Bacteria

With the development of negative numbers and decimals, we are now poised to understand better the operation of exponentiation, which is an operation that we have already referred to (for example, in describing place value relationships) and that we will use in future investigations.

If a kind of bacteria doubles every hour, how many bacteria will there be after 24 hours if we begin with 1 bacterium? Think and then read on. . . . 

DISCUSSION

There are several ways to address this problem. For example, we can make a table to represent the growth. The third column simply represents the number of bacteria, using exponents.

Does this help you? Using exponents, how would you represent the number of bacteria after 24 hours?

Number of hours	Number of bacteria	Number of bacteria
0	1	
1	2	
2	4	2^2
3	8	2^3
4	16	2^4
5	32	2^5
.	.	.
.	.	.
.	.	.

The number of bacteria after 24 hours will be 2 to the 24th power. Do you see why?

Exponents and multiplication Just as multiplication can be seen as repeated addition, exponentiation can be seen as repeated multiplication, as in Figure 5.43.

$$b^x = \overbrace{b \cdot b \cdot b \cdot b \cdots b \cdot b \cdot b}^{b \text{ occurs } x \text{ times}}$$

FIGURE 5.43

In mathematical language, b is called the **base** and x is called the **exponent**

We say

$$b^2 = b \text{ squared}$$

$$b^3 = b \text{ cubed}$$

$$b^4 = b \text{ to the fourth power}$$

$$b^x = b \text{ to the } x\text{th power}$$

Computing exponents This then brings us to another problem: How do we determine the value of 2^{24} ? What do you think? If you remember anything about exponents, explore alternatives for a while before reading on. If you don't recall anything about exponents, read the following discussion first and then try. . . 

In one sense, we want to know how to make the following computation less tedious:

$$2 \cdot 2$$

One (of several) possibilities is to translate this computation to an equivalent computation, using the associative property of multiplication:

$$(2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2)$$

That is, it is much quicker to press $16 \cdot 16 \cdot 16 \cdot 16 \cdot 16 \cdot 16$ on a calculator than to press 2 twenty-four times.

If you have a scientific calculator, pressing the following entries yields the answer:

$$2 \quad [y^x] \quad 24 = 16777216, \text{ that is, } 16,777,216.$$

Negative exponents When the exponent is 2 or greater, we can interpret it as repeated multiplication. However, what if the exponent is less than 2? For example, what are the meanings and value of the following amounts: 2^1 , 2^0 , 2^{-1} , 2^{-2} ?

One way to understand the meaning and value of such exponents is to look at the connections among all exponents. For example, what is happening to the value of the expression each time we decrease the exponent? Try to verbalize this and then read on. . . 

$$2^4 = 16$$

$$2^3 = 8$$

$$2^2 = 4$$

$$2^1 = ?$$

$$2^0 = ?$$

$$2^{-1} = ?$$

One way to verbalize the pattern is to say that each time the exponent decreases, the value of the expression is divided by 2. It is reasonable to assume that this pattern will continue, and thus it is reasonable to conclude that the value of 2^1 is $4 \div 2 = 2$. This line of reasoning enables us to continue the progression. In this manner, we deduce that the value of 2^0 is $2 \div 2 = 1$.

Continuing this pattern, $2^{-1} = 1 \div 2 = 1/2$, and so $2^{-2} = 1/2 \div 2 = 1/4$.

Can you verbalize the pattern now so that you can give the value of 2^{-n} ? Work on it and then read on. . . 

Once again, we rely on multiple representations. We can represent 2^{-2} as $1/4$, or we can represent 2^{-2} as $1/2^2$, because $4 = 2^2$. This now leads us to the generalization that

$$2^{-n} = \frac{1}{2^n}$$



HISTORY

In the 1500s, Francois Viète and Michael Stifel introduced symbols for unknowns and powers, for example, x and x^2 . Before this time, writing algebraic expressions was very tedious. For example, the expression we write as $x^2 + 2x = 8$ would have been written as Zp 2Rm 8 in the 1500s.⁷ To decipher the sixteenth-century code, you need to know that they let R represent the unknown amount;

however, to represent the square of the unknown amount, they chose a different letter, Z (for zensus or census); then they used p for plus and m for minus. With this code, we can now translate Zp 2Rm 8 as “some unknown to the second power plus 2 of those unknowns minus 8.” Can you imagine solving equations or factoring in the sixteenth century? With this primitive notation, it was very cumbersome.

INVESTIGATION 5.29 Scientific Notation: How Far Is a Light-Year?



One of the ways in which we use exponents is to express very small and very large numbers. In many cases, the numbers that we use are so large or so small that computation becomes very cumbersome. For example, the U.S. federal debt at one point in the mid-1990s was \$5.12 trillion, which is 5,120,000,000,000. The wavelength of red light is 0.0000000000000586 meter. The following situation is a good introduction to the need for an alternative notation, which we call *scientific notation*.

The size of the universe is so huge that it is almost beyond the ability of the human mind to comprehend. One way in which astronomers address this hugeness is to measure astronomical distances not in miles or kilometers but rather in a unit called “light-years”—that is, the distance light travels in one year. The speed of light is 186,000 miles per second. Determine the length of a light-year. Work on this yourself before reading on. . . .



DISCUSSION

In this problem, dimensional analysis can greatly simplify the computations.

$$1 \text{ light-year} = \frac{186,000 \text{ miles}}{1 \text{ second}} \times \frac{60 \text{ seconds}}{1 \text{ minute}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} \times \frac{24 \text{ hours}}{1 \text{ day}} \times \frac{365.25 \text{ days}}{1 \text{ year}}$$

If you have an inexpensive calculator, you probably got a big E, representing “Error.”

Many of you will have something like this:

$$5.8697 \text{ } 12 \quad \text{or} \quad 5.8697136 \text{ } 12$$

⁷H.A. Freebury, *A History of Mathematics* (New York: Macmillan, 1961), p. 92.

The more sophisticated calculators represent the computation in scientific notation, which allows us to represent very large or very small numbers without having to count all the zeros. The 5.8697 12 is the calculator's code for 5.8697×10^{12} .

Let's work with smaller and simpler numbers to ensure that you can understand how scientific notation works. Consider the number 34,000. One way to represent this amount that connects to place value comes from the realization that this number means 34 thousands. Symbolically, we can thus say

$$34,000 = 34 \times 1000$$

We could also represent 34 as 3.4×10 . Substituting 3.4×10 for 34, we have

$$3.4 \times 10 \times 1000$$

We can simplify this as

$$3.4 \times 10,000$$

Now we can substitute 10^4 for 10,000, and we have

$$3.4 \times 10^4$$

Formally, we say that a number is in **scientific notation** if it is in the form $a \times 10^b$, where a is a number between 1 and 10 and b is an integer.

The reasoning behind restricting a to a number between 1 and 10 is based on convention rather than mathematical structure. This convention makes it easier for us to compare amounts. For example, if we have 6.4×10^7 and 5.6×10^8 , we immediately know that the second amount is larger. However, if we have 0.64×10^8 and 56×10^7 , it is not as quickly evident that the second amount is larger.

Now let us return to our problem of representing the length of a light-year. Translate this amount into a whole number and then read on. . . . 

One strategy is to write the decimal, add "a bunch of zeros," and then move the decimal point 12 times:

$$5.86970000000000000000$$

When we erase all the extra zeros and insert the commas, we have 5,869,700,000,000. How would we say this amount?

We would say 5.9 trillion miles.

Irrational Numbers

Thus far, our investigation of numbers has proceeded from counting numbers to whole numbers to integers to fractions and to decimals. There is one more set of numbers that we shall consider in this course. The discovery of this set of numbers is also one of the more dramatic stories in the history of mathematics. Many important contributions to mathematics came from a group of people who called themselves Pythagoreans; you first encountered them in Chapter 4. As you may recall, one of their core beliefs was that all the laws of the universe could be represented using only whole numbers and ratios of whole numbers. Now recall the Pythagorean theorem (for any right triangle with legs a and b and hypotenuse c , $a^2 + b^2 = c^2$) and our mention that the Greeks were the first people to prove this relationship. Ironically, this proof was part of the undoing of the Pythagoreans. Here is how it happened.

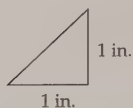


FIGURE 5.44

Consider a right triangle in which the length of each of the two sides is one inch, as shown in Figure 5.44.

If we call the hypotenuse x and apply the Pythagorean theorem, we have

$$1^2 + 1^2 = x^2$$

$$x^2 = 2$$

That is, x is the number that, when multiplied by itself, equals 2.

This problem was perplexing to the Pythagoreans, because try as they might, they could not find a rational number that, when multiplied by itself, came to *exactly* 2.

Using modern notation and base 10 arithmetic, we know that we can get *very* close:

$$1.4 \times 1.4 = 1.96$$

$$1.41421 \times 1.41421 = 1.9999899$$

However, there is no rational number (that is, a number that can be expressed as the quotient of two integers) that will produce a product of exactly 2. In modern language, we say $x = \sqrt{2}$.

Finally, one member of the sect proved that there is no rational number that will solve this problem. This discovery was like a child finding out that Santa Claus is not real or an adult finding out that there really are people from other planets. What it meant for the Pythagoreans was that one of their cornerstone beliefs—that all the laws of the universe could be represented as whole numbers and ratios of whole numbers—was not true.

What happened to the person who discovered the proof? Legend has it that he was set out to sea in a leaky rowboat! This is possibly the source of the saying, “Don’t kill the messenger who bears the bad news.” It is also interesting to note that the word *irrational*, even in our present time, means “contrary to reason.”

INVESTIGATION 5.30 Square Roots

Students in elementary school may come upon situations in which they will encounter square roots that are not rational numbers, so it is important to investigate square roots briefly. Technically, any positive number has two square roots. For example, the square roots of 81 are $+9$ and -9 . Because, in elementary school, square roots are explored only in the context of measurement, and because all lengths are positive, we will focus our explorations with the positive square roots of numbers, also called the **principal square root**.

Before the development of hand-held calculators, schoolchildren had to learn an algorithm for approximating the square root of a number. This algorithm was a standardized procedure that was taught and practiced in much the same way that long division was taught and practiced.

Without using the square root button on your calculator, determine the square root of 800 to the nearest hundredth. As you do so, use the tools that seem most appropriate. Then read on. . . . 

DISCUSSION

To determine $\sqrt{800}$, a little estimating and mental math helps. For example,

$$30 \times 30 = 900$$

Therefore, a reasonable starting point would be 29. However, if you apply the mental math skills developed in Chapter 3, you realize that 29^2 is still too high. Can you mentally compute 29×29 ?

We can represent 29×29 as $(30 - 1)(30 - 1)$ and apply the distributive property $900 - 30 - 30 + 1 = 841$. In any case, since we know that $30^2 = 900$ and $29^2 = 841$, a reasonable first guess is 28. If you want to follow the discussion below actively, please cover the table and think about *your* next guess and your reasoning before reading on. . . . 

Guess	Result	Analysis
28	784	Too low. Because 800 is closer to 784 than to 841, the next guess should be closer to 28 than to 29.
28.4	806.56	Not bad! Let's try 28.3.
28.3	800.89	Clearly 28.2 will be too small. Next guess? 28.29? 28.28? Why?
28.29	800.3241	
28.28	799.7584	

Clearly $28.28 < \sqrt{800} < 28.29$. Without getting out your calculator or pencil and paper, which is closer? How did you figure it out?

The Real-Number System

With the set of irrational numbers, we can now extend our system of numbers to the set of **real numbers**, which is defined as the union of the sets of rational and irrational numbers. This is as far as we will go with number systems in this course. The next major expansion, which students encounter in high school, is imaginary and complex numbers—for example, the square root of -1 .

Let us look back at the numbers we have examined in this course with respect to how they are related to each other. We began our study of numbers with the set of natural numbers, which is the first set of numbers young children encounter. Our first extension of the number line was to add zero. The union of the set of natural numbers and zero is called the set of whole numbers. We then examined two kinds of numbers that children will encounter in elementary school: integers and rational numbers. Finally, we encountered numbers that cannot be represented as the ratio of two whole numbers—that is, irrational numbers. If you were to represent the relationships among these number systems visually, what kind of a diagram might you draw? Do this before reading on. . . . 

Look at Figure 5.45. Could you explain this chart to someone who was familiar with the terms but not the relationships? Try to do so before reading on. . . . 

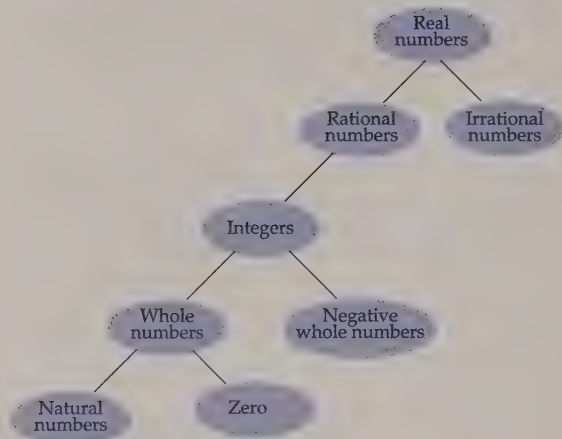


FIGURE 5.45

Properties of Real Numbers

As we have explored each of these sets of numbers, we have examined the properties of these number systems. These properties are not simply “technical stuff” that children should memorize because their teachers say so. When students understand and “own” these properties, their ability to work successfully with higher mathematics in general and algebraic concepts in particular greatly increases. We have used these properties to understand algorithms, and we have used these properties when estimating.

Now that we have finished our investigations of different sets of numbers, it is worthwhile to stop and reflect on what we have learned.

I hope you will witness students discovering some of these properties (for example, “It doesn’t matter whether you add 6 and 7 or 7 and 6, you get the same thing!”) and that your students will be curious enough in your class to ask you questions like “Will you always get a number when you divide?” Which property is this student asking about?

This question connects to the closure property. The operations of addition, subtraction, and multiplication are closed for real numbers. That is, for any two real numbers, a and b :

$a + b$ is a real number, $a - b$ is a real number, and $a \cdot b$ is a real number.

Why do we not say that division is closed for real numbers?

If we were to say that division is closed, we would have to make one qualification. If we say that for any two real numbers a and b , a/b is a real number, we would simply have to note that b cannot equal zero.

Recall also that the closure properties for the four operations do not hold for all the number systems we have encountered. For example, the set of natural numbers is closed for addition and multiplication only. Do you remember why? The absence of closure with the natural numbers, whole numbers, and integers can be disturbing for youngsters. Do you see why the following two questions have to do with the closure property?

- “What if you subtract a big number from a little number?”
- “How can you divide 10 cookies among three people?”

Another property that is important for you to know about has to do with the density property of rational numbers, which represents a qualitative differ-

ence between the set of rational numbers and the three previous sets. However, as dense as the rational numbers were, we found that the number line was not yet complete, because there are points on the real number line that have no rational name. When we add the set of irrational numbers to the set of rational numbers, our number line is complete. This is known as the **completeness property**:

There is a real-number name for every single point on the number line.

Summary

To understand decimals, we explored their relationship both to the set of integers and to the set of rational numbers. In doing so, we examined different ways to represent decimals, including base 10 blocks and number lines.

When examining connections with decimals, we found that zero presents special issues, just as zero did with integers. When we are representing an amount with a decimal, sometimes the placement of a zero is necessary and sometimes the placement is optional (a convention).

When examining connections with fractions, we found that some decimals terminate ($1/2 = 0.5$) but some do not ($1/3 = 0.333\ldots$).

Understanding decimal computations relied on applying connections with whole numbers and with fractions. When adding and subtracting, we must add digits that represent the same place value, just as we did with whole numbers. To understand why we put the decimal point in a particular place when multiplying, we needed to examine decimal multiplication in the context of an area model, just as we did with multiplication of whole numbers. To justify the movement of the decimal when dividing, however, we made use of the relationship between decimals and fractions.

The operation of exponentiation was introduced as repeated multiplication. Exponentiation enables us to understand a use of decimals that is important in many fields: scientific notation, which is used to represent very large and very small numbers.

Finally, we examined one more extension of the number system, irrational numbers, thus completing our journey through number systems in this course. The set of rational numbers is dense, but the set of real numbers is complete and sufficient for most everyday and business uses of elementary mathematics.

EXERCISES 5.4

- Express each of the following as directed.
 - $3/40$ as a decimal
 - $4/5$ as a decimal
 - $3/100$ as a decimal
 - 0.36 as a fraction in simplest form
 - 0.005 as a fraction in simplest form
 - 1.6 as a mixed number
- Express each decimal in expanded form and as a fraction.
 - 4.6
 - 0.75
 - 1.234
 - 4.06
- Write the following decimal numbers in words.
 - $.32.04$
 - 0.00004
 - 508.604
- Round the following decimals as directed:
 - 2.36 to the nearest tenth
 - 6.04 to the nearest tenth
 - 2.398 to the nearest hundredth
 - 5.0006 to the nearest hundredth
 - 0.9876 to the nearest thousandth
 - 6.1237 to the nearest thousandth
- Circle the smaller of the pairs of numbers:
 - 0.4 0.42
 - 1.03 0.98899
 - 0.05 0.058
 - 0.302 0.087
- Order the following decimals from smallest (at the left) to largest (at the right).
 - $0.56, 0.058, 0.0084, 0.6$
 - $1.04, 0.065, 0.0086, 0.9$
- Name a decimal between 0.999 and 1 .
 - Name a fraction between 3.44 and 3.45 .

8. In each part, draw a number line and place the given numbers on it.
 - a. 3.2, 3.8, 2.9, 3.04
 - b. 2.43, 2.4325, 2.4396, 2.441
 - c. 0.004, 0.00397, 0.00394
9. Using grid paper, represent each of the problems below and explain how to determine the product by interpreting the grid.
 - a. 0.3×0.2 b. 1.2×0.4 c. 3.4×2.6
10. Recall the question about the proposed school board budget in Investigation 5.23. What if the superintendent recommended adding \$380,000 to the proposed budget of \$24.06 million? How much would the new budget be, both in whole-number form and in decimal form?
11. Recall the discussion of reporting of innings pitched in baseball on page 282. Why do you think this is reported in a way that is mathematically inaccurate?
12. Our system of money is not a pure decimal system. Why not? Actually, this question has two parts:
 - a. What aspects of our money system are not aspects of a pure decimal system?
 - b. Why don't we use a pure decimal system?
13. Perform the following computations mentally and describe your strategy.
 - a. $6.47 - 3.95$ b. $2.3 + 6.9 + 4.7$
 - c. $65.38 \div 1000$ d. $0.75 \times 23 \times 16$
 - e. $8(8.5)$
14. Estimate each of the following; then briefly explain and justify your estimate.
 - a. $19.4 + 136 + 4.825$ b. $23.4 + 24.6 + 5.7 + 34.4$
 - c. $23.34 - 22.56$ d. $\$10.00 - \$4.34$
 - e. 12.5×0.034 f. 2.3×6.7
 - g. 34.2×0.007 h. $7.8 \div 3.12$
15. Represent the following numbers in scientific notation:
 - a. 123,456,789 b. 3,000,000,000,000,000
 - c. 0.00000000056 d. 0.000000302
16. Perform the following computations using scientific notation:
 - a. $123,000,000,000 \times 34,000,000$
 - b. $123,552,000,000 \div 23,400,000$
 - c. $0.0000000034 \times 0.0000000045$
 - d. $(0.00043)^3$
17. The width of a certain cell is 3×10^{-6} cm. If we placed 250 of these cells side by side, how long would that line be? Express this length without using scientific notation.
18. a. Explain how you would determine the length of a light-year if you had a calculator that did not "know" scientific notation—that is, that just displayed an E sign for answers that were too big. Justify your work. What properties that we developed in Chapter 3 did you use?
 - b. Determine the following product with such a calculator:

$$14,000 \times 3,356,000 \times 7,890,000$$
19. a. Suppose a particular bacterium can divide into two bacteria every 45 minutes. If this process continues for 48 hours, how many bacteria will there be? Express your answer in scientific notation.
 - b. What if we had started with 100 bacteria instead of 1? Predict the answer without doing any pencil-and-paper or calculator computation. Justify your reasoning.
 - c. What if we had started with 1 bacterium that doubled every 16 minutes. How many would there be after 48 hours?
20. Sherry is a hard-working student who is suffering from fatigue. The doctor says that she is not getting enough sleep. Below are her bedtimes for one work week. What is her average bedtime?
 10:30 P.M. 11:15 P.M. 11:45 P.M. 1:15 A.M. 11:00 P.M.
21. People frequently mispunch the calculator when doing problems.
 - a. Let's say you were multiplying 24 boards @ \$4.50 per board and got \$1080. What is the right answer? What did you punch?
 - b. Let's say you are adding $\$3.45 + \$12.45 + \$16.23 + \$34.45 + \$23.45$. You have just entered the last number and hit the = sign, and you realize that you punched \$3.45 for the last number. What could you do with the number that now appears on the calculator rather than starting all over?
 - c. Let's say you are finding the cost of 12 boards @ \$3.45 per board. You have just multiplied $12 \times \$3.45$ to get \$41.40. Now you decide that you want to buy 14 boards. There are several alternatives to multiplying $14 \times \$3.45$. Can you find them?
22. The national debt at one point in the mid-1990s was just over \$5.12 trillion. How could we represent this amount in a way that people could relate to?
23. Earlier in the chapter, we discussed the misuse of decimals with respect to baseball. When I was reading *Immunity* by John Paulos, I came across another misuse of decimals that astonished me. Most people know that "normal body temperature" is 98.6° Fahrenheit. How do you think this number was determined?
 Paulos tells us that the number was determined in the following manner. Many years ago (he doesn't say when), researchers kept track of the "normal" body temperature of many people. The temperatures were recorded in Celsius rather than Fahrenheit. The researchers determined that the "normal" body temperature varies between 36.2 and 37.5 . That is, some people's normal body temperature is as low as 36.2°C , whereas other people's normal body temperature is as high as 37.5°C . However, most people want an exact number for normal body temperature, not a range. Let's say you were an American who wanted to represent normal

body temperature in degrees Fahrenheit, and you wanted one number rather than a range.

- a. If you were given this task, what would be your answer for normal body temperature in Fahrenheit? Justify your decision.
 - b. Can you give a plausible explanation for how someone came up with 98.6?
24. Let's say a company receives medicine in 1-gallon (128-ounce) jars and then sells the medicine in vials that hold 1.25 ounces. If we were to ask how many vials can be made from 1 gallon, we would divide 128 by 1.25 and obtain 102.4. What does the 102 mean? What does the .4 mean?
25. Without doing any computation, determine which is bigger, $(0.8)^3$ or $(0.8)^2$? Briefly explain how you arrived at your decision.
26. Think back to the Alphabitian system that we explored in Chapters 2 and 3. Answer each of the following questions without translating the amounts to base 10. That is, answer them as though you were an Alphabitian and that was the only numeration system you knew.
- a. Draw a picture to represent the value of 0.A.
 - b. Add $A.B + C.D$.
27. At one point the telephone rates between Los Angeles and Tokyo were \$9.50 for the first 3 minutes and \$1.85 for each additional minute. What would be the cost of a 20-minute phone call?
28. Judy recently took a trip in her car. The trip took 11 hours, including a lunch break of $1/2$ hour. She filled the gasoline tank (11.3 gallons) and noted that the odometer read 38329.8. At the end of the trip, she refilled the tank (12.4 gallons), and the odometer read 38735.4. She paid $\$1.46\frac{9}{10}$ per gallon.
- a. How many miles per gallon did the car get on the trip (to the nearest tenth of a mile per gallon)?
 - b. What was the cost of the gas?
 - c. What was her average speed?
29. Most gasoline stations price gasoline out to $9/10$ of a cent—that is, $\$1.46\frac{9}{10}$ per gallon rather than just \$1.46 per gallon. How much extra money per day might a gasoline station expect to make from this extra $9/10$ cent per gallon?
30. A manufacturing plant used 4.2 centimeters of wire on each item. If 3549 cm of wire are used in a day, how many items were manufactured? If the cost of the wire is 3.4¢ per meter, what is the cost per day of the wire?
31. A group of students held a car wash to raise money. They charged \$2.50 for regular cars and \$4.00 for trucks. They washed 150 cars and 68 trucks. If they spent \$15 on soap, sponges, and other supplies, how much profit did they make?
32. In the *Guinness Book of World Records*, the record for dominoes toppling was 281,581 in 12 minutes 57.3 seconds. How many dominoes fell per minute? per second?
33. In the *Guinness Book of World Records*, the greatest distance a baby carriage was pushed in 24 hours was 342.25 miles. What was the average speed of the baby carriage? What was the average time per mile to the nearest second?
34. Let's say your electric rate is 9.23¢/kWh, and you have 5 different 100-watt lights in the house that run from 5 P.M. until 11 P.M. (1 in the kitchen, 2 in the living room, 1 in the dining room, and 1 in the bedroom). How much do you pay per day for running these lights?
35. Oil companies argue that using super unleaded rather than regular unleaded pays for itself because you get better mileage. If this were the only benefit from using super unleaded and you get 20 miles per gallon with regular unleaded that costs \$1.09 per gallon, how many miles per gallon would you need to get with super unleaded at \$1.29 per gallon to make buying super unleaded worthwhile?
36. You have decided to open a checking account with Xanadu County Savings and Loan. They offer you two options: (a) pay a flat \$5.00 per month fee or (b) pay a monthly fee of \$2 and 10¢ per check.
- a. If you write about 15 checks per month, which option is cheaper?
 - b. How many checks per month would give you the same fee in both cases?
37. The length of the tropical year is 365.24220 days, as compared to the length of 365.2425 days used by the Gregorian calendar.
- a. What is the difference (in seconds) between the tropical year and the Gregorian calendar year?
 - b. How many years will it take for this difference to amount to 1 day?
38. A laser printer prints characters at a density of 300 dots per inch, both horizontally and vertically.
- a. How many dots can be printed across the 8.5-inch width of a standard piece of paper?
 - b. If every dot on a standard 8.5×11 inch piece of paper was printed, how many dots would that be? Express your answer in scientific notation.
39. Your school has just purchased a new high-tech mimeograph machine called a risograph. The accountant has determined that the school pays 5¢ per copy using the copier machine but only 2¢ per copy on the risograph. The school has kept track of the usage of the risograph for a week and has found the average number of copies per day is 1375. How much money per year will the school save with the purchase of the new machine? What assumptions did you make in order to solve this problem?
40. Betty has arrived! She just traded in her compact car for a midsize sedan. The payments on her old car were \$135.45 per month, it averaged 34 miles per gallon, and car insurance was \$450 per year. The payments on her new car are \$234.67 per month, it averages 25 miles per gallon, and car insurance will be \$675 per year. How

much will her monthly car expenses increase? Assume that she drives 12,500 miles per year and pays \$1.24 per gallon for gas.

41. Henry is currently working at a job that pays \$9.45 per hour, and he regularly works 40 hours per week. Another company has offered to pay him an annual salary of \$24,000. How does this offer compare to his present yearly income? What assumptions did you make in order to solve this problem?
42. Let's say the cost of gas just went up by 5¢ per gallon. Approximately how much will this affect the average citizen?
43. Let's say you were on vacation, traveling through a small town, and you saw that the temperature was 21°C

but you were not familiar enough with Celsius to know offhand what this meant. However, you knew that the formula for conversion from Celsius to Fahrenheit is $F = 1.8C + 32$.

- a. Estimate the Fahrenheit equivalent of 21°C in your head and explain your reasoning. What properties and/or estimation strategies that we developed in Chapter 3 did you use?
 - b. Estimate the Fahrenheit equivalent of 28°C in your head and explain your reasoning.
 - c. Determine the Celsius temperature, to the nearest degree, that would be closest to 100°F.
44. Why is there no one's place?

CHAPTER SUMMARY

1. The set of integers is simply an extension of the set of whole numbers.
2. Operations with positive whole numbers can be adapted to work with integers.
3. A fraction is not simply a number; rather, a fraction expresses a relationship between two quantities. The numerator and denominator can be seen as a code that tells us the relative size of the fraction.
4. A fraction can be interpreted in four ways: as measure, as quotient, as operator, and as ratio.
5. Certain important ideas abide in all the rational-number contexts:
 - Something is to be partitioned into parts of equal size (value).
 - The something can have a value of 1, in which case the unit = the whole.
 - The something can have a value $\neq 1$, in which case the unit $\neq$ the whole.
6. The set of decimals has important connections with the set of integers and with the set of rational numbers.
7. All the sets of numbers that children will study in elementary school are subsets of the set of real numbers.

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Proportional Reasoning

- ◆ 6.1 Ratio and Proportion
- ◆ 6.2 Percents

In one respect, this chapter represents the culmination of the first five chapters. Over the course of the first five chapters, we have explored fundamental mathematical concepts and we have investigated applications of those concepts in real-life settings. In this chapter, we will focus explicitly on the idea of proportional reasoning. This is not a new idea—it is inherent in the concept of multiplication—but it has remained implicit up to now. The concept of proportional reasoning is a powerful one. In fact, some have said that proportional reasoning is both the capstone of [children’s] elementary [school] arithmetic and the cornerstone of all that is to follow.¹ The NCTM asserts that “the ability to reason proportionally develops in students throughout grades 5–8. It is of such great importance that it merits whatever time and effort must be expended to [ensure] its careful development” (*Curriculum Standards*, p. 82). Let us examine why.

Additive Versus Multiplicative Comparisons

Much of our use of mathematics involves comparisons (How much more does this car cost than that one?) and change (How has the standard of living changed in the past 20 years?). To compare amounts and express change, we can use the mathematical tools called ratio, proportion, and percent. We shall see that these tools involve *multiplicative comparisons* between amounts.

In this chapter, we will examine two fundamentally different ways in which we can compare amounts and describe changes. For example, let’s say we are comparing the cost of two cars, one of which costs \$10,000 and the other \$15,000.

¹ Richard Lesh, Thomas Post, and Merylyn Behr, “Proportional Reasoning,” in *Number Concepts and Operations in the Middle Grades*, ed. James Hiebert and Merylyn Behr (Reston, VA: NCTM, 1988), p. 94.

We can say that the second car costs \$5000 *more than* the first car, that it costs $1\frac{1}{2}$ *times as much as* the first car, or that it costs 50% *more than* the first car.

The first and third descriptions both use the word *more*, but the first expresses the relationship in additive terms, and the second and third express the relationship in multiplicative terms. The equations below illustrate why we use the terms *additive* and *multiplicative* to contrast the two ways to compare the amounts. To make the mathematics more visible, let b = the cost of the second car and a = the cost of the first car.

There are many ways to express this relationship, using addition, subtraction, multiplication, division, and ratios (see the equations below). However, the first two are equivalent, and we call them both **additive comparisons**. The last three representations are equivalent, and we call them **multiplicative comparisons**.

$$\begin{array}{lll} b = a + 5000 & b - a = 5000 & \\ b = 1\frac{1}{2}a & \frac{b}{a} = \frac{3}{2} & b:a = 3:2 \end{array}$$

We tend to use multiplicative comparisons more than additive ones. For example, we hear on television that a paper towel absorbs “50% more liquid than the leading brand.” If you did Exploration 6.2, did you use additive or multiplicative comparisons in Step 2?

In the first section, we will develop the concepts of ratio and proportion and investigate real-life applications. Many real-life problems involving whole numbers or fractions actually have to do with ratios and proportions. In the second section, we will develop the concept of percent, the use of which pervades virtually everyone’s daily life. The concept of percents rests firmly on the concept of ratio and proportion. Therefore, understanding how ratios and proportions operate is very important.

SECTION

6.1

RATIO AND PROPORTION

WHAT DO YOU THINK?

- Are all ratios fractions? Are all fractions ratios? Why or why not?
- What do we mean by proportional reasoning?
- How do the concepts of ratio and proportion give us tools for making comparisons?

The Unit Concept Matures

A noted educator and scientist once wrote that “it seems odd to refer to a relationship as a quantity.”² For example, if we say that there are 25 students in a class, we can see or visualize them. Similarly, if we say that a “large” drink is 20 ounces, we can see or visualize that amount using our knowledge of measurement. We might use 1 ounce as our referent unit, or we might use a standard 12-ounce can as our referent unit; that is, “20 ounces is not quite 2 cans of soda.” However, it is much harder to “see” rates—for example, this car gets 35 miles per gallon. That is, when we say “35 miles per gallon,” the 35 actually expresses a *relationship* between two amounts (miles traveled and gallons consumed). Yet, as Schwartz notes, we “refer to [this] relationship as a quantity.” However, 35 miles per gallon is much more abstract than 35 ounces or 35 people.

In the primary grades, children mostly work with whole numbers and with the operations of addition and subtraction. As their understanding of mathematical ideas grows, they move from counting physical objects to counting numbers themselves. In both cases, however, the unit is still a single whole entity (for example, 1 ounce or 1 person). With the introduction of multiplication

²Judah Schwartz, “Intensive Quantity and Referent Transforming Arithmetic Operations,” in *Number Concepts and Operations in the Middle Grades*, ed. James Hiebert and Merylyn Behr (Reston, VA.: NCTM, 1988), p. 43.

and division, and then the introduction of rational numbers, both operations and numbers become more complex, as you saw in Chapters 3 and 5.

“Underneath all of the surface level changes is a fundamental change with far-reaching ramifications: a change in the nature of the unit. It is difficult to overestimate the significance of this basic shift. Many of the important differences in the subject matter between the primary and middle grades can be traced back to a change in the nature of the unit. Given the difficulty of mastering the concept of unit in whole number situations, it is not surprising that changes in the nature of the unit in the middle grades bring new cognitive demands and renewed difficulties for students.”³ In the example above, we talked about 35 miles per gallon. In this case, our unit is 1 mile per gallon—a more complex unit than 1 ounce or 1 student. As noted in Chapter 5, one reason why multiplication and division problems are generally more difficult than addition and subtraction problems is that they involve a more complex unit.

These observations point out that in this and the next section, it is crucial that we ask ourselves questions about the unit as we solve problems and answer questions.

Ratios, Rates, and Proportions

A **ratio** is a relationship between two amounts or quantities. It can be expressed in the following equivalent ways: $a:b$, a/b , or $\frac{a}{b}$.

When the two amounts in a ratio represent different units, we often refer to such ratios as **rates**—for example, 30 miles/gallon or 55 miles/hour.

When we set two ratios equal to each other, we have a **proportion**.

That is, $a:b = c:d$ iff $\frac{a}{b} = \frac{c}{d}$ and $b \neq 0$, $d \neq 0$.

Ratios, Rates, and Proportions in Mathematics and Real Life

Ratios and rates pervade mathematics. When two fractions are equivalent, we have a proportion. Ratio is inherent in the concept of place value: The ratio of the value of each place to the value of the place to its right is 10:1. When we convert one unit to another, such as kilometers to miles, we use proportions. The concept of similarity involves equal ratios. When we make graphs, we use proportions. When we make scale drawings and scale models, we obey proportions. The essence of probability involves ratios. These are a few of the many aspects of ratio and proportion in mathematics.

Ratios, rates, and proportions show up in all kinds of real-world contexts.

- **Banking:** When I applied for a mortgage, the bank applied a ratio called the 28% rule: If the ratio of fixed monthly payments (mortgage, property tax, car payments, and so forth) to monthly income is more than 28:100, the bank is not likely to make the loan.
- **Botany:** If we represent the number of complete turns made around the stem by t and represent the number of leaves between the two points as n , then the fraction t/n is called a divergency constant for that species (see Figure 6.1). Table 6.1 gives divergency constants for a number of different trees. The numerators and denominators are all Fibonacci numbers (see Exercise 35 in Section 1.4). This divergency constant, which can be expressed as a fraction, is actually a ratio. Do you see why?



LANGUAGE

The term *ratio* comes from the Latin verb *ratus*, which means “to think or estimate.” Many mathematicians in the sixteenth and seventeenth centuries used the word *proportion* for ratio. Even today you hear the two terms used interchangeably; for example, instructions for making a certain color might say, “Mix the two colors in the following proportion—3:2.”



Source: Excerpted from *Symmetry: A Unifying Concept* © 1994 by Istvan and Magdolna Hargittai. Reprinted by permission.

FIGURE 6.1

³ James Hiebert and Merylyn Behr, “Introduction: Capturing the Major Themes,” in *Number Concepts and Operations in the Middle Grades*, ed. James Hiebert and Merylyn Behr (Reston, VA: NCTM, 1988), p. 2.

TABLE 6.1

Tree	Divergency constant
Elm	$1/2$
Beech, hazel	$1/3$
Apple, oak, apricot, poplar	$2/5$
Pear, weeping willow	$3/8$
Willow, almond, pussy willow	$5/13$

- *Commerce:* Many companies use rates to determine how much to charge for their goods or services. For example, Allied Shipping Company charges \$7 for every 100 pounds of goods shipped.
- *Cooking:* If you want to make more than the amount given in the recipe, all the ingredients must be increased in the correct proportion.
- *Education:* One of the criteria used to determine the “quality” of a college is the student:teacher ratio.
- *Shopping:* If you get 250 tablets of generic aspirin for \$2.49 and 100 tablets of name-brand aspirin for \$3.89, how much cheaper is the generic aspirin?
- *Sports:* Football writers and announcers talk about the “turnover ratio,” determined by subtracting the number of times a football team has lost the ball because of a fumble or interception from the number of times it has recovered the ball because of a fumble or interception. If a team has caused a total of 30 fumbles and interceptions and lost the ball 25 times because of fumbles and interceptions, the team is said to have a +5 ratio. Unfortunately, this is not a valid use of the term *ratio*. Can you explain why not? Would you recommend that these writers and announcers convert the numbers to a ratio or use the same numbers but invent a different term?

In many instances, quantities “should be” but are not proportional. Try to think of a few before reading on. . . . 

Airline rates are often not proportional. Longer flights are often proportionally cheaper than short flights, and flights between major cities tend to be proportionally cheaper than flights between small cities. Phone rates are often not proportional in similar ways.

The affirmative action debate for the past 30 years has been based on a belief that many people hold: that the ratio of a minority group in society and the ratio of members of that minority group in the work force or the schools should be equal. For example, the ratio of women:men is approximately 1:1, but the ratio of female college presidents to male college presidents is not 1:1, and so we say that a disproportionate number of college presidents are men.

Delving Further into Ratios

Look at the following ratio statements. If the statement can be expressed as a fraction statement, do so. If it cannot, try to explain why. Then read on. . . . 

1. The ratio of males to females at Mountain State College is 3:2.
2. In order to get a stain out of a shirt, Lisa made a mixture of bleach and water in the ratio 3 parts water to 1 part bleach.
3. In the rectangle in Figure 6.2, the ratio of the width to the length is 2:3.

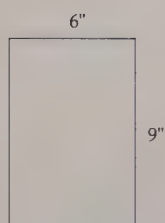


FIGURE 6.2

4. Fred's car gets 25 miles per gallon.

Statement 1: We can say $\frac{3}{5}$ of the students are male and that $\frac{2}{5}$ of the students are female.

Statement 2: The stain mixture Lisa made was a $\frac{1}{4}$ bleach solution. [Note: Many people would be more likely to call this a 25% bleach solution.]

Statement 3: There are two ways in which we can translate this statement into fraction language. We can say that the width is $\frac{2}{3}$ as long as the length. Alternatively, we can say that the length is $1\frac{1}{2}$ times as long as the width.

Statement 4: There is no way to translate this statement into fraction language. Do you see why? In Statement 3, the two amounts (parts) had the same unit: length. In this statement, the two amounts are measured with different units: miles and gallons.

Table 6.2 summarizes some of these points.

TABLE 6.2

Statement	Diagram	Whole?	Can be expressed as fraction statement?								
1	<table><tr><td>Males</td></tr><tr><td>Females</td></tr></table>	Males	Females	The student body	Yes						
Males											
Females											
2	<table><tr><td>Bleach</td></tr><tr><td>Water</td></tr><tr><td>Water</td></tr><tr><td>Water</td></tr></table>	Bleach	Water	Water	Water	The mixture	Yes				
Bleach											
Water											
Water											
Water											
3	<table><tr><td>6"</td></tr><tr><td>9"</td></tr></table>	6"	9"	Either number can be seen as the whole	Yes						
6"											
9"											
4	<table><tr><td>Miles</td><td>Gallons</td></tr><tr><td>25</td><td>1</td></tr><tr><td>50</td><td>2</td></tr><tr><td>75</td><td>3</td></tr></table>	Miles	Gallons	25	1	50	2	75	3	No whole	No
Miles	Gallons										
25	1										
50	2										
75	3										

When Are Ratios Like and When Are They Unlike Fractions?

Before we move on to problem-solving situations, let us summarize the ways in which ratios are like fractions and the ways in which they are unlike fractions.

Ratios are like fractions in that they can be expressed as an ordered pair of numbers representing two sets or two amounts. Ratios are different from fractions in that fractions are restricted to part-whole relationships, whereas ratios are not.

Let us illustrate this difference by returning to the example of the water-bleach mixture.

The ratio of water to bleach is 3:1. However, there are actually several possible ratios:

3 water:1 bleach	1 bleach:3 water	3 water:4 total	1 bleach:4 total
part:part	part:part	part:whole	part:whole

Because a part-whole relationship exists, the water-bleach mixture can be expressed as a fraction.

In the case of the miles per gallon, we cannot make a fraction statement. However, the ratio is 24 miles:1 gallon, which is a whole:whole relationship even though the two amounts are measures of different units, namely miles and gallons.

Note also that some ratios represent discrete amounts, such as students, whereas others represent measured amounts, such as bleach, water, length of a side, miles, or gallons.

Let us now examine these various concepts and ideas that underlie what we call *proportional reasoning*. This first investigation represents a classical problem with a history to it.

INVESTIGATION 6.1 Unit Pricing—Is Bigger Always Cheaper?

 Let's say you want to buy some laundry detergent. The small jug costs \$2.99 for 36 fluid ounces and the large jug costs \$3.79 for 48 fluid ounces. Which is the better buy? 

DISCUSSION

STRATEGY 1: Use fractions

In this case, the additive comparison is not the one that is useful. That is, we get 12 more ounces for 80 more cents. However, we can use this information to make a valid multiplicative comparison.

The larger size gives you $\frac{1}{3}$ more detergent, so if it costs $\frac{1}{3}$ more, then the value of both sizes will be the same. Does the larger jug cost $\frac{1}{3}$ more? Because $\frac{1}{3}$ of \$2.99 is \$1.00, the two sizes will be the same value if the large size costs \$4. Note that here we are rounding \$2.99 to \$3.00. Because \$3.79 is less than \$4, we conclude that the larger size is a better buy. Why is this?

Perhaps you used this strategy in Exploration 6.1 to determine which box of pancake mix was the better buy. This strategy, used intuitively by many students,⁴ is called the **factor-of-change method**. In this case, the factor of change was $\frac{1}{3}$. That is, if we multiply the 36 ounces and the 299 cents by the same factor, we produce an equivalent ratio. The idea of equivalent fractions helps us to understand why this method works.

$$\frac{299 \text{ cents}}{36 \text{ ounces}} \times \frac{1\frac{1}{3}}{1\frac{1}{3}} = \frac{400 \text{ cents}}{48 \text{ ounces}}$$

STRATEGY 2: Use ratios and a calculator

$$\begin{aligned} \frac{\$2.99}{36 \text{ ounces}} &= 0.0830556 \\ \frac{\$3.79}{48 \text{ ounces}} &= 0.0789583 \end{aligned}$$

⁴Kathleen Cramer, Thomas Post, and Sarah Currier, "Learning and Teaching Ratio and Proportion: Research Implications," in *Research Ideas for the Classroom: Middle Grades Mathematics*, ed. Douglas T. Owens (New York: Macmillan, 1993), pp. 159–178.

I have deliberately not placed labels on the two amounts. What do the two decimals mean? Think and read on. . . . 

The meaning of 0.083 is \$.083 per ounce, or 8.3 cents per ounce. From a meaningful interpretation of these ratios, we see that the larger size is a better buy because its unit price is less; that is, \$.079 per ounce is less than \$.083 per ounce.

This strategy, also intuitively used by many students, is known as the **unit-rate method**. That is, we determine the cost of 1 unit (in this case 1 ounce).

STRATEGY 3: Another way to express the ratios?

Maria says she used the previous strategy but “upside down.” What do these ratios mean? What do you think of her idea? Is this a variation of the unit-rate method?

$$\frac{36 \text{ ounces}}{\$2.99} \qquad \frac{48 \text{ ounces}}{\$3.79}$$

In this case, we have 12.0 ounces per dollar versus 12.7 ounces per dollar. This is a variation of the unit-rate method. In this case, the unit is 1 dollar instead of 1 ounce; that is, the ratios tell us how much detergent we get for a unit of money. Had we selected cents as our unit, we would have had the ratios 0.120 ounce per cent versus 0.127 ounce per cent.

STRATEGY 4: Solve a proportion

We can let x represent how much a 48-ounce jug of equivalent value would cost:

$$\frac{\$2.99}{36 \text{ ounces}} = \frac{x \text{ dollars}}{48 \text{ ounces}}$$

When we solve for x , we find that $x = \$3.99$; that is, if the 48-ounce jug cost \$3.99, the two jugs would have the same value. Because the 48-ounce jug costs less than \$3.99, it is a better buy.



MATHEMATICS

Many states now require grocery stores to show the unit price below each item. One of the reasons for this law is that many companies realized that people tended to believe that larger packages were cheaper. Even today with unit prices, you can find instances in most stores where the larger amount is not cheaper.

INVESTIGATION 6.2 How Much Money Will the Trip Cost?

Most real-life uses of ratios and rates involve proportions. For example, let's say you are planning a 2000-mile trip and want to estimate how much money you will spend on gas. How would you do that? Work on this problem on your own and then read on. . . . 

DISCUSSION

As you may have realized, the cost of gas for the trip would depend on two variables: how many miles per gallon your car gets and the cost of gasoline. Let's say your car averages 25 miles per gallon and you estimate that gas will cost \$1.39 per gallon on the trip. Now estimate the cost of gas for the trip and then read on. . . . 

The rate at which your car uses gas is expressed by the ratio 25 miles:1 gallon. The following proportion enables us to determine how many gallons of gas you will need at that rate of consumption:

$$\frac{25 \text{ miles}}{1 \text{ gallon}} = \frac{2000 \text{ miles}}{x \text{ gallons}}$$

In other words, what number do we need to divide 2000 by so that the ratio is still 25 to 1?

We find x by solving the proportion for x : $25x = 2000$; therefore $x = 80$ gallons.

We can now use another proportion to find the cost of 80 gallons.

$$\frac{\$1.39}{1 \text{ gallon}} = \frac{y}{80 \text{ gallons}}$$

We find y by solving the proportion for y : $y = (\$1.39)(80)$; therefore $y = \$111.20$.

We could have solved the entire problem in one step using dimensional analysis. The units for the answer will be dollars. Therefore, we begin with the amount that is represented by dollars: the cost of gasoline. If we multiply the ratio representing the cost of gasoline by the ratio representing the consumption of gasoline, look what happens:

$$\frac{\$1.39}{\text{gallon}} \times \frac{1 \text{ gallon}}{25 \text{ miles}} \times 2000 \text{ miles} = \$111.20$$

Rates as Functions

You may recall working with proportional functions in Chapter 2. If you did Exploration 6.3, you explored several different proportional functions. Rates express a functional relationship between two variables. For example, if my car averages 25 miles per gallon, there is a functional relationship between the miles I drive and the gallons of gas my car consumes (see Figure 6.3). In this case, we say that these variables are *directly proportional* to each other. Do you see why?

If we take any two points on the graph in Figure 6.3—for example, the points (1, 25) and (4, 100)—and represent them as ratios—25 miles/1 gallon and 100 miles/4 gallons—the ratios are equal; that is, $25/1 = 100/4$. When two ratios are equal, by definition, we have a proportion.

Now that we have worked with ratios and proportions, let us revisit some investigations from previous chapters for which ratios and proportions are appropriate strategies for answering the questions.

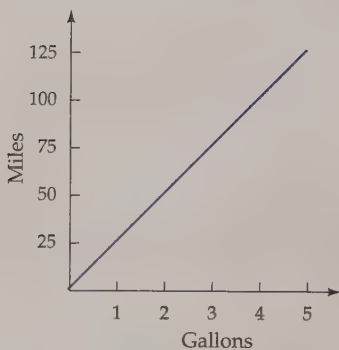


FIGURE 6.3

INVESTIGATION 6.3 Reinterpreting Old Problems

- A.** In Investigation 5.6, Josh finds that $1/8$ of the balloons are blue and that there are 120 more red balloons than blue balloons. How many blue balloons are in the box? In Section 5.2, we explored several different ways to solve this problem. How could you solve this problem using ratios and proportions? Work on this and then read on. . .



DISCUSSION

If $1/8$ of the balloons are blue, then the ratio of blue balloons to red balloons is 1:7. Why is this?

We need to focus on the ratio of blue to red balloons; that is, blue:red = 1:7. If there are 120 more red balloons than blue balloons, we can express this relationship by letting x = the number of blue balloons and $x + 120$ = the number of red balloons. Since the ratio is still 1:7, we have the proportion

$$\frac{x}{x + 120} = \frac{1}{7}$$

Solving the proportion for x , we have

$$7x = x + 120$$

from which we find that $x = 20$.

Thus, there are 20 blue balloons and 140 red balloons.

- B.** In Section 5.2, Investigation 5.5, Jose paid \$6 for a $3/4$ -pound box of chocolates. What is the price of 1 pound (at this rate)?

DISCUSSION

Many students find this problem challenging as a fraction problem but can solve it more easily if they use decimals and proportions:

$$\frac{\$6}{0.75 \text{ pound}} = \frac{x \text{ dollars}}{1 \text{ pound}}$$

- C.** In Investigation 5.13, Marvin has 23 yards of cloth to make costumes for the play. Each costume requires $3\frac{1}{4}$ yards of material. How many costumes can he make?

DISCUSSION

In Section 5.3, we solved this problem by interpreting it as division (repeated subtraction). However, it can also be interpreted as a proportion. In this case, 3.25 yards can be seen as the unit—that is, the number of yards needed for 1 costume.

$$\frac{1 \text{ costume}}{3.25 \text{ yards}} = \frac{x \text{ costumes}}{23 \text{ yards}}$$

INVESTIGATION 6.4 Using Estimation with Ratios

The following investigations consist of problem-solving situations that can be answered with an estimate (rather than an exact answer). Assume that you have no calculator or pencil and paper.

- A.** It took Rene 24 minutes to go 15 miles from Hingham to Marshfield. What was Rene's average speed for the trip? What is your first impression (5-second estimate)? What is your best estimate? Can you get the "exact" answer? Why is the "exact" answer probably not exact? Work on these questions and then read on. . . 

DISCUSSION

The problem can be represented as the following proportion: 15 miles is to 24 minutes as x miles is to 60 minutes (that is, 1 hour).

$$\frac{15 \text{ miles}}{24 \text{ minutes}} = \frac{x \text{ miles}}{60 \text{ minutes}}$$

STRATEGY 1: Use equivalent fractions

We can see that $15/24$ is just under $16/24$, which is equivalent to $2/3$, and $2/3$ of $60 = 40$. Thus, a quick estimate (if you connect this question to equivalent fractions) is "slightly under 40 miles per hour."

STRATEGY 2: Use the missing factor model of division

We also can get a decent estimate by asking: 24 times what is equal to 60? This strategy is related to the compatible numbers strategy we have used before; it would not have been so straightforward had the number of minutes been, say, 22.

Because 24 times $2\frac{1}{2}$ is equal to 60, we now have to determine 15 times $2\frac{1}{2}$, but that is relatively simple using multiplication as repeated addition: $15 + 15 + 7\frac{1}{2} = 37\frac{1}{2}$, which is the exact answer—well, sort of.

The exact answer of $37\frac{1}{2}$ miles per hour is itself an approximation. Why is this? The 24 minutes and 15 miles are rounded numbers. In this case, the “exact” numbers were 23 minutes and 45 seconds and 15.2 miles. Even if we now computed Rene’s “exact” average speed, that is,

$$\frac{15.2}{23.75} = \frac{x}{60}$$

and solved for $x = 38.4$ miles, the term *average speed* is itself a mental construct, not something that exists in an observable sense. At times Rene was going 55 miles per hour; at times Rene was going 20 miles per hour. We will study the concept of average in more detail in Chapter 7.

B. It is Christmas time, and virtually every store in town is selling wrapping paper. If Zoe doesn’t care about looks and just wants the cheapest, which should she buy? Work on this and then read on. . . . 

- Store A: 36 square feet for 88¢.
- Store B: 50 square feet for \$1.79.
- Store C: 150 square feet for \$2.99.

DISCUSSION

If we divide the cost (in pennies) by the square feet, the resulting ratio will tell us the cost per square foot. The ratio that is the smallest number will represent the cheapest paper. Try this on your own and then read on. . . . 

In this case, if the prices are not terribly close, you may need only one run-through.

- Store A: $88/36 =$ more than 2—that is, more than 2¢ per square foot.
- Store B: $179/50 =$ more than 3—that is, more than 3¢ per square foot.
- Store C: $300/150 = 2$ ¢ per square foot. We can use 300 instead of the exact 299¢ because 1¢ makes virtually no difference in this case. There is no need to do more arithmetic here, because store C clearly has the “best” ratio.

C. Let’s say Ellie bought a new car on August 16. On November 1 the odometer read 2650. Ellie’s insurance company gives a discount if she puts less than 10,000 miles per year on the car. Will she qualify? Work on this and then read on. . . . 

DISCUSSION

The time between August 16 and November 1 is very close to $2\frac{1}{2}$ months, so we can represent this problem by the following proportion:

$$\frac{2650 \text{ miles}}{2\frac{1}{2} \text{ months}} = \frac{x \text{ miles}}{12 \text{ months}}$$

We can once again obtain a quick estimate by using the idea of equivalent fractions: What multiple of $2\frac{1}{2}$ will be closest to 12? A bit of mental arithmetic yields the realization that $2\frac{1}{2} \times 5 = 12\frac{1}{2}$. Therefore, the estimated miles per year will be 2650×5 . How can you calculate that amount mentally?

Because $2650 \times 10 = 26,500$, 2650×5 will be half that amount, or 13,250. At this rate, Ellie will travel about 13,250 miles in just over 12 months and won't qualify for the discount.

Another quick estimate can be obtained by connecting to compatible numbers: 2650 miles is just over 2500 miles, which is $\frac{1}{4}$ of 10,000 miles. Thus, if she had traveled 2500 miles in 3 months, she would be on target to travel 10,000 miles in 12 months. Because she has traveled more than 2500 miles in less than 3 months, her ratio of miles:months is higher than 2500 miles:3 months.

Real-Life Applications

The next three investigations are not quite as simple and tidy as the previous ones and require a careful application of concepts we have studied thus far. I find that most real-life applications of proportion are more like the following investigations than like the previous ones. Just a reminder: Work on these actively—that is, get out a pencil and paper and a calculator and work on them yourself, as opposed to just reading what I have written. By doing so, you will not only deepen your understanding of proportional reasoning but also improve your computational ability, including estimation and mental math.

INVESTIGATION 6.5 Is the School on Target?

 An all-female college first began admitting men three years ago. Its goal for this, the third year, is to have a 2:1 ratio of females to males in the incoming freshman class. The admissions department has just reported a problem. When the college sent out letters of acceptance to this year's freshman class, the ratio of females to males was 2:1. However, the ratio of females to males in the 1000 students who indicated that they will be coming this fall is 7:3. If the college still wants to meet the 2:1 goal, how many more males will it need to accept to reduce the ratio of females to males to 2:1? Try to solve this problem on your own and then read on. . . . 

DISCUSSION

The desired ratio of females to males is 2:1. If the ratio of females to males among the 1000 incoming freshmen is 7:3, that means that there are 700 women and 300 men. One way of representing the dilemma is to let x be the number of additional males the college needs to admit.

In other words, if the college admits x more males, the ratio of females to males will be $700:(300 + x)$, and this ratio needs to be equal to 2:1. Because the two ratios are equal, we have the following proportion:

$$\frac{700}{300 + x} = \frac{2}{1}$$

We find that $x = 50$.

This problem could also have been solved using common sense. Targeting a ratio of 2 females to 1 male can be interpreted as wanting the number of males to be one-half the number of females. Because 350 is half of 700, the college needs a total of 350 males—that is, 50 more males than it has presently accepted.

INVESTIGATION 6.6 Transferring Credits

I attended the University of Tennessee in my freshman year and then transferred to the University of Arizona. At that time Tennessee ran on a quarter system (the academic year consisted of three 10-week quarters and then summer school), and Arizona followed the more common fall and spring semesters, each being about 15 weeks long. At the University of Tennessee, I took 15 credits in the first quarter, 15 in the second, and 19 in the third, giving me a total of 49 credits for the year. When I transferred to Arizona, how many semester-hour credits did my 49 quarter-hour credits convert into?

DISCUSSION

STRATEGY 1: “See” the implied proportion

If we see that two ratios must be equal, we can immediately create a proportion: 49 quarter-hour credits is to x semester-hour credits as 3 quarters is to 2 semesters.

$$\frac{49}{x} = \frac{3}{2}$$

$$3x = 98$$

$$x = 32\frac{2}{3} \text{ semester-hour credits}$$

STRATEGY 2: Solve a simpler problem

We can solve a simpler problem and then generalize. If I had taken 16 quarter-hours for each of the three quarters, then that would have been equivalent to taking 16 semester-hours for each of two semesters. That is, 48 quarter-hours is equivalent to 32 semester-hours. This means that a conversion ratio is 48:32, which simplifies to 3:2. This can be stated in fractions as saying that we have to multiply the quarter-hour credits by $2/3$ to get the semester-hour credits, because $2/3$ of 48 is 32. In this case, $2/3$ of 49 is $32\frac{2}{3}$.

INVESTIGATION 6.7 Stuck Behind a Truck

One day I was driving to see a student teacher who was teaching in a nearby town. I was late, and I got stuck behind a truck. The speed limit was 45 miles per hour, but I was stuck for 5 miles behind the truck, which went only 20 miles per hour. How much time did I lose?

DISCUSSION

As is true with many problems, we need to define the question. In this case, I am assuming that I would have averaged 45 miles per hour as opposed to 20 miles per hour for those 5 miles.

STRATEGY 1: Use ratios

At the rate of 20 miles per hour (that is, 20 miles in 60 minutes), I would go 10 miles in 30 minutes or 5 miles in 15 minutes.

At the rate of 45 miles per hour (had I not been stuck behind the truck), I would go 5 miles in $1/9$ of an hour. Why is this? If we round $1/9$ to the

nearest tenth of an hour, we have 0.1 hour, and $0.1 \text{ hour} \times 60 \text{ minutes per hour} = 6 \text{ minutes}$.

Instead of taking 6 minutes, it took me 15 minutes, so I lost about 9 minutes.

STRATEGY 2: Set up proportions

$$\frac{45 \text{ miles}}{60 \text{ minutes}} = \frac{5 \text{ miles}}{x \text{ minutes}}$$

$$x = 6.7 \text{ minutes}$$

$$\frac{20 \text{ miles}}{60 \text{ minutes}} = \frac{5 \text{ miles}}{x \text{ minutes}}$$

$$x = 15 \text{ minutes}$$

Thus, I lost $15 - 6.7 = 8.3$ minutes, or about $8\frac{1}{2}$ minutes.

INVESTIGATION 6.8 Electric Bill—What Do All Those Numbers Mean?

Can you help Malcolm understand how the electric company figures his bill? He knows that there are two meters by the side of his house, but when he looks at the bill (Figure 6.4), he can't understand how it is determined. How did the electric company get the \$95.81?

SERVICE PERIOD: JUL 3 TO AUG 4 1992 32 DAYS				PREVIOUS BILL PAYMENT THROUGH AUG 4 ADJUSTMENTS		\$ 77.27 \$ 77.27CR
				BALANCE FORWARD		\$.00
	METER NUMBER	METER PRESENT	READING PREVIOUS	MULTI PLIER	KWH USE	
RESIDENTIAL SERVICE	22721385	8955	8534		421	
RATE D						
				CUSTOMER CHARGE @ \$ 7.39/MONTH		\$ 7.39
				250 KWH X 9.241¢	23.1025	
				171 KWH X 12.616¢	21.5768	\$ 44.68
QUICK RECOVERY	19693023	9027	8590	437		
				METER CHARGE @ \$ 1.58/MONTH		\$ 1.58
				437 KWH X 9.647¢	42.1574	\$ 42.16
WATER HEATING						
CHARGES THIS PERIOD						\$ 95.81
- THE ABOVE AMOUNTS INCLUDE A NUCLEAR DECOMMISSIONING CHARGE OF \$ 0.33 BASED ON A RATE OF 0.038 CENTS/KWH MULTIPLIED BY 858 KWH. - FOR YOUR CONVENIENCE, TIME PAYMENT PLANS ARE AVAILABLE. FOR MORE INFORMATION, PLEASE CALL YOUR CUSTOMER SERVICE OFFICE LISTED BELOW. - LEND A HELPING HAND BY CONTRIBUTING \$1 TO THE NEIGHBOR HELPING NEIGHBOR FUND. BY ADDING \$1 TO YOUR BILL PAYMENT, YOU CAN HELP PSNH CUSTOMERS WHO DO NOT QUALIFY FOR FUEL ASSISTANCE. SERVICE ADDRESS						
						PLEASE PAY \$95.81 TOTAL AMOUNT

FIGURE 6.4

DISCUSSION

First, let's see where the \$95.81 comes from: $\$7.39 + \$44.68 + \$1.58 + \$42.16$. Do you understand what each of the four numbers represents, or where each came from? Think and read on. . .

- The \$7.39 represents the "customer charge." What does that mean?
- The \$44.68 is actually a sum: $\$23.10 + \$21.58$. We will explore it further in a moment.

- The \$1.58 represents the meter charge. What does that mean?
- The \$42.16 is the charge for the electricity used to make hot water.

The math involved in the \$44.68 is interesting:

$$\begin{array}{rcl} 250 \text{ KWH} \times 9.241\text{¢} & 23.1025 \\ 171 \text{ KWH} \times 12.618\text{¢} & 21.5768 & \$44.68 \end{array}$$

At this point, could you explain in meaningful terms how the \$44.68 is obtained? If so, try it. If not, read on.

The electric company counts one's usage of electricity in kilowatt-hours.⁵ The company charged Malcolm's first 250 kilowatt-hours at the rate of 9.241¢ per kilowatt-hour. After that, the rate went up to 12.618¢ per kilowatt-hour. Malcolm gets hot water from an electric hot-water heater. The company charges him yet another rate for the electricity used by the hot-water heater: 9.647¢ per kilowatt-hour. It's the same kind of electricity. For reasons not known to me, the company charges different rates for "regular" electricity and electricity for hot-water heaters.

Going back to Figure 6.4, the electric company has actually shown us its computations:

$$\begin{array}{rcl} 250 \text{ KWH} \times 9.241\text{¢/kWh} & \$23.1025 \\ \text{Similarly, } 171 \text{ KWH} \times 12.618\text{¢/kWh} & \$21.5768 \end{array}$$

When we add $\$23.1025 + \$21.5768$, we get \$44.6793, which the company has rounded to \$44.68.

INVESTIGATION 6.9 How Much Does That Extra Light Cost?

 Let's say you are considering the installation of a 300-watt outdoor security light for your home. By how much will your electric bill increase? 

DISCUSSION

In order to answer this question, what assumptions do we need to make? What data do we need?

Let us assume that the light will be on an average of 12 hours per day over the course of a year—less than 12 in the summer, more than 12 in the winter. We also need to know how much electricity costs. For this problem, use the rate of 12.325¢/kWh. See whether you can do the problem on your own now, and then read on. . . . 

STRATEGY 1: Multiply step by step

- 0.3 kilowatt per hour times 12.325 cents/kWh = 3.6975 cents per hour to run the light.
- 3.6975 cents per hour times 12 hours per day = 44.37 cents per day to run the light.
- 44.37 cents per day times 365 days per year = 16195.05 cents per year to run the light.

Note: This problem illustrates the need to make sure constantly that each step makes sense. Many of my students give me an answer of \$16,195.05.

⁵ A kilowatt is 1000 watts. For example, if you run a lamp with a 100-watt light bulb for 10 hours, you have used 1000 watt-hours or 1 kilowatt-hour.

However, the number 16195.05 refers to cents, not dollars. How do you convert cents into dollars?

STRATEGY 2: Use dimensional analysis

By aligning the rates correctly, we can solve the problem in one step:

$$\frac{0.3 \text{ kW}}{\text{hour}} \times \frac{12.325 \text{ cents}}{\text{kW}} \times \frac{12 \text{ hours}}{\text{day}} \times \frac{365 \text{ days}}{\text{year}} \times \frac{\$1}{100 \text{ cents}} = \frac{\$161.9505}{\text{year}}$$

Instead of doing each computation separately, what we do on the calculator is simply

$$.3 \ (\times) \ 12.325 \ (\times) \ 12 \ (\times) \ 365 \ (\div) \ 100 \ (=)$$

What appears on the calculator is 161.9505. Dimensional analysis tells us that the meaning of this number is dollars per year.

Dimensional Analysis Explained

Dimensional analysis was first introduced in Chapter 1. Can you now justify this method?

The justification lies in the concept of equivalent fractions and rates. Usually, we think of finding equivalent fractions by multiplying the numerator and denominator by the same number. However, we can generalize this notion of equal number to equal amount. When we use the light bulb, we are using electricity at the rate of 300 watts per hour—that is, 0.3 kilowatt/hour. Because the electric company charges us 12.325 cents per kilowatt-hour, we can determine the cost of 1 hour of usage of this light bulb with a proportion:

$$\frac{12.325 \text{ cents}}{1 \text{ kWh}} = \frac{x \text{ cents}}{0.3 \text{ kWh}}$$

Solving the proportion for x , we find $x = 3.6975$ cents. That is, this represents the cost to run the light bulb for 1 hour.

When using dimensional analysis, we are treating the units as though they were numbers:

$$\begin{aligned} & \frac{0.3 \text{ kW}}{\text{hour}} \times \frac{12.325 \text{ cents}}{\text{kW}} \\ &= \frac{0.3}{1} \times \frac{\text{kW}}{\text{hour}} \times \frac{12.325}{1} \times \frac{\text{cents}}{\text{kW}} \\ &= \frac{0.3}{1} \times \frac{12.325}{1} \times \frac{\text{kW}}{\text{kW}} \times \frac{\text{cents}}{\text{hour}} \\ &= \frac{3.6975}{1} \times \frac{\text{kW}}{\text{kW}} \times \frac{\text{cents}}{\text{hour}} \\ &= \frac{3.6975}{1} \times 1 \times \frac{\text{cents}}{\text{hour}} \\ &= \frac{3.6975}{1} \times \frac{\text{cents}}{\text{hour}} \\ &= \frac{3.6975 \text{ cents}}{\text{hour}} \end{aligned}$$

Our initial problem.

Our knowledge of multiplication of fractions tells us that this is an equivalent representation.

The commutative and associative properties tell us that we can change the order.

Our knowledge of multiplication of fractions tells us that we have the amount 3.6975.

Any amount divided by itself is equal to 1.

The identity property of multiplication tells us that when we multiply an amount by 1, the value is unchanged.

Our knowledge of multiplication of fractions tells us that this is an equivalent representation.

Of course, when we use dimensional analysis, we simply compute and “cancel” units when appropriate.

Summary

In this section, we have explored the last interpretation of fractions that was introduced in Section 5.2: fraction as ratio. We have seen that not all ratios can be expressed as fractions—only those that are given in part-whole terms or can be translated into part-whole terms. We have seen that rates are a special kind of ratio, one that cannot be expressed in fractional terms. Rates also express a functional relationship between two variables.

In this section, we have made explicit the difference between additive and multiplicative comparisons. If you did Exploration 6.5, you probably realized how tempting it was to use additive comparisons to compare slopes. Like fractions, ratios and rates signify multiplicative relationships between two amounts. When two ratios are equal (or when two rates are equal), we have a proportion. Thus, the concept of a proportion connects to the notion of equivalent fractions, which we studied in Chapter 5. In the next section, we will build the concept of percent upon the concept of proportion.

We have examined several real-life applications of ratios, rates, and proportions. In doing so, we found that it is critical to make sure that the computations make sense. We have given names to two methods that students often invent: the factor-of-change method and the unit-rate method. Dimensional analysis is one tool that helps us to solve proportion problems. Finally, we have seen that many estimation problems involve proportions.

EXERCISES 6.1

Exercises 1–16 present you with fairly straightforward ratio and proportion situations. In each problem, first obtain a rough estimate (5 to 10 seconds, entirely in your head); then obtain a refined estimate (30 or so seconds, with a minimum of paper and pencil) or the “exact” answer mentally. Then explain how you obtained the rough estimate and the refined estimate (or exact answer). Finally, explain whether the exact answer is, itself, actually exact or an approximation.

1. A quart (32 ounces) container of yogurt contains 920 calories. Approximately how many calories would there be in a 5-ounce serving?
2.
 - a. If the Dow Jones Industrial Average in the stock market began the day at 12,000 and lost 800 points, how much would a proportional fall be if the stock market began the day at 1000?
 - b. If the stock market was at 2700 and rose 75 points, how much would a proportional rise be if the stock market began the day at 10,000?
3. An advertisement says that 5 out of 8 dentists recommend the new zigzag toothbrush. If 264 dentists were interviewed, how many recommended the toothbrush?
4. An intravenous solution needs 2 liters (L) of glucose mixed with 7 units of blood. How much glucose is needed for 40 units of blood?
5. An employee making \$24,000 was given a raise of \$1000. All employees were given proportional raises.
 - a. How much of a raise would an employee making \$18,000 receive?
 - b. How much of a raise would an employee making \$30,000 receive?
 - c. How much of a raise would an employee making \$23,450 receive?
6. If $1\frac{3}{4}$ cups of flour are required to make 30 cookies, how many cups of flour (to the nearest $\frac{1}{4}$ cup) are required for 96 cookies?
7. In the summer of 1991, Israel airlifted 14,000 Ethiopian Jews to Israel as immigrants. If the United States were to receive a proportional number of refugees at one time, how many would we receive? Say the population of Israel is approximately 4.4 million and that of the United States is 270 million.
8.
 - a. On one map, $\frac{1}{3}$ inch represents 18 miles. If two cities are $2\frac{1}{2}$ inches apart on the map, what is the actual distance between them?
 - b. On another map, 1 inch represents 65 miles. Los Angeles is about 1000 miles from Portland. How many inches apart would Portland and Los Angeles be on this map?
9. You can use proportions to estimate the height of a tree. John is 6 feet tall, and his shadow is $10\frac{1}{2}$ feet long. How high is a tree whose shadow is 90 feet long?
10. Jane finds that she can read 36 pages of a book in 40 minutes. At this rate, how long, to the nearest minute, will it take her to finish a book 473 pages long?
11. The other day Janet was using the rowing machine in the Fitness Center. Her goal was to row 2200 meters in

- 10 minutes. At exactly 7 minutes, she saw that she had rowed 1560 meters. Is she going to make her goal if she continues at this rate?
12. Sheila and Dora worked $3\frac{1}{2}$ hours and $4\frac{1}{2}$ hours, respectively, on a programming project. They were paid \$176 for the project. How much did each earn?
 13. A photograph is 3 inches high and 5 inches wide. If it is enlarged to be 7 inches high, how wide will it be (to the nearest $\frac{1}{4}$ inch)?
 14. The recommended dosage for a particular medication is 6.8 milligrams (mg) per pound of body weight per day, not to exceed 1000 mg per day. For a 125-pound patient, about what would you expect the doctor to recommend for a daily dosage?
 15. A car and a train both begin at the same time and are moving at constant speeds, but the train is moving faster. For every 2 miles that the car travels, the train travels 3 miles.
 - a. How far will the train have traveled when the car has traveled 30 miles?
 - b. How far will the car have traveled when the train has traveled 40 miles?
 16. What is the average speed in each of these situations?
 - a. You travel 11 miles in 16 minutes.
 - b. You travel 31 miles in 39 minutes.
 - c. You travel 85 miles in 90 minutes.
 - d. On a bicycle you travel $12\frac{1}{2}$ miles in 45 minutes.
 - e. On a bicycle you travel $3\frac{3}{4}$ miles in $10\frac{1}{2}$ minutes.
 17. Make up and solve a real-life problem that lends itself to needing a rough or refined estimate.
- Exercises 18–37 are nonroutine, multistep problems. If it is possible to estimate the answer, do so and briefly explain how you obtained your estimate. If you feel that making an estimate is not practical, describe what measures you took to ensure that your answer is reasonable.
18. a. If you are traveling 65 miles per hour, how fast are you traveling in kilometers per hour, to the nearest whole number? (Fifty miles per hour is equivalent to 80 kilometers per hour.)
 b. If you are traveling 35 kilometers per hour, how fast are you traveling in miles per hour, to the nearest whole number?
 19. It was reported that Ross Perot spent approximately \$40 million of his own money in the 1992 presidential election. His total worth is reported to be approximately \$4 billion. If your total worth were \$50,000 and you spent the same fraction of your worth on an election as Perot did, how much money would you have spent?
 20. In a healthy person, the ratio of red blood cells to other blood cells should be about 1 to 5000. Amy just got back a lab report that showed 300 red blood cells out of 230,000 blood cells. Is her red blood cell count low, high, or normal?
 21. A car travels 60 miles per hour, and a plane travels 15 miles per minute. How far does the car travel when the plane travels 600 miles?
 22. On a TV game show, a contestant makes \$700 for every correct answer but loses \$500 for every wrong answer. After answering 24 questions, Sarah broke even. How many questions did she answer correctly?
 23. Five cups of a certain flour weigh 1 pound, and 1 cup of cornstarch weighs $\frac{1}{4}$ pound. If the ratio of flour to cornstarch in a mixture is 2:1, how much would 1 cup of the mixture weigh?
 24. This is a problem given to students in Baghdad over 2000 years ago. Two men sat down to eat, one with five loaves and the other with three, all the loaves having the same value. Just as they were about to begin, a third man came along and proposed to eat with them, promising to pay eight cents for his part of the meal. If they ate equally and consumed all the bread, how should the eight cents be divided?
 25. In the women's downhill during the Olympics, the difference between first place and second place was 0.04 second. If the two skiers had been racing side by side, what would have been the distance between the two at the finish line? Assume that they were going 60 mph at the end of the race.
 26. Convert each of the following figures to miles per hour.
 - a. The world record (as of 1993) for the men's 100 meters was 9.86 seconds, held by Carl Lewis. What was his average speed over the course of that race?
 - b. The world record (as of 1993) for the men's 10,000 meters was 27:08.23, held by Artuor Barrios of Mexico. What was his average speed throughout that race?
 - c. The world record (as of 1993) for the women's 100 meters freestyle swimming was 54.48 seconds, held by Jenny Thompson. What was her average speed over the course of that race?
 27. In the *Guinness Book of World Records*, the record for handshaking is 16,615 in 7 hours and 25 minutes. How many is this per minute? The person shook one hand every ____ seconds.
 28. At a certain college, the ratio of men to women is 9:4. If there are presently 360 women, how many additional women would it take to reduce the ratio of men to women to 2:1?
 29. At a certain college, there are 7 men for every 5 women. If there are 420 more men than women, what is the total enrollment?
 30. Yosha is working on a small project that is due in 2 days. She has spent $4\frac{1}{2}$ hours on it and figures that she is about $\frac{3}{4}$ done. How many more hours will she need to spend?
 31. A painting crew of 4 takes 5 days to do an apartment building. There are 40 apartment buildings in their current project.

- a. If the foreman wants to do the project in 20 days, how many people should he hire? What assumptions do you make in order to solve this problem?
 - b. A crew of 3 could do one building in ___ days.
32. Ginger wants to fill her new swimming pool. She has two pumps; the large pump takes 40 minutes to fill the pool, and the small pump takes 60 minutes. How long will it take to fill the pool if both pumps are working?
 33. The ratio of physicians to inhabitants of the United States is 1 to 549. The ratio of prison inmates to inhabitants of the United States is 1 to 497. Are there more physicians or prison inmates in the United States? Approximately how many are there of each?
 34. Here are the directions from an oatmeal container:

Servings	1	2	3
Water or milk	$\frac{3}{4}$ cup	$1\frac{2}{3}$ cup	$2\frac{1}{3}$ cup
Cereal	$\frac{1}{2}$ cup	1 cup	$1\frac{1}{2}$ cup
Salt (optional)	Dash	$\frac{1}{8}$ tsp	$\frac{1}{8}$ tsp

- a. The proportions are not the same. Explain why not.
 - b. What amount of each ingredient would you use to make 10 servings? Explain your work.
35. A recipe for chocolate chip cookies follows.
 - $1\frac{1}{4}$ cups flour
 - $\frac{1}{2}$ cup sugar
 - $\frac{1}{2}$ teaspoon salt
 - $\frac{1}{2}$ cup butter
 - 6 oz chocolate chips
 - 1 teaspoon vanilla extract
 - 1 egg
 - $\frac{1}{2}$ teaspoon baking powder
 The recipe makes 4 dozen cookies.
 - a. How much of each ingredient would you need if you wanted to make 10 dozen cookies?
 - b. Unlike in the oatmeal problem, it is essential in this case that certain ingredients be in the "correct proportion." Explain what that phrase means.
 36. An advertisement says that 5 out of 8 doctors recommend one brand of aspirin over another. In actuality, 325 doctors were interviewed, and 199 of the doctors recommended the first brand over the other. Is the advertisement accurate? Explain your response.

37. If the ratio of boys to girls in a class is 3 to 8, will the ratio of boys to girls stay the same, become greater, or become smaller if 2 boys and 2 girls are added to the class? Justify your response.

Problems 38–44 require you to make some assumptions in order to determine an answer. Describe and justify the assumptions you make in determining your answer.

38. A worker estimates that she spends 75 minutes a day at the copying machine. How many hours would this be per year?
39. Two persons have the same yearly income, but one gets paid every other week, while the other gets paid twice a month. Which paycheck is larger?
40. One morning I was listening to National Public Radio, which was conducting its annual pledge drive. The announcer said, "Our goal this morning (from 6 A.M. to 9 A.M.) is to get 40 new pledges. We've received 15 pledges so far, so we're on track." I was puzzled because it was 7:30.
 - a. Explain why the announcer was inaccurate from a mathematical perspective.
 - b. Assuming that the announcer was aware that the math wasn't quite right, explain why he might have made that statement.
41. Ursula and Brad are going to buy Christmas wrapping paper. Their choices are a jumbo roll containing 200 square feet for \$3.99 or a jumbo package containing 8 different rolls (each containing 25 square feet) for \$3.99. Ursula says that even though the ratios are the same in both cases, the former is a better buy. Brad says that they're the same deal because you get the same amount of wrapping paper for the same amount of money. What do you think?
42. Sonja claims that she was 20 minutes late to work because she got stuck behind a truck the whole way. Is this plausible? If not, explain why. If so, make up some numbers to support your conclusion.
43. Determine the cost of flying from your nearest airport to two different cities. If we consider the fares as rates—that is, cost per mile—are the two rates proportional? If not, why do you think they are not proportional?
44. You are in charge of designing a 6-week summer school program at your college. Some classes will meet every day, and others will meet three times per week. How long will the classes that meet every day be? How long will the classes that meet three times a week be?

You first learned percents in the sixth or seventh grade, and you see them every day in the newspaper. Yet the confident use of percents as a tool eludes most adult Americans. The concept of percents is not as much one of the big ideas of

WHAT DO YOU THINK?

- What does percent mean?
- To what other mathematical concepts does percent relate?
- In what real-life contexts have you encountered percents?



MATHEMATICS

We can also interpret this problem from the perspective of rates. That is, if the student were to continue to get problems correct *at this rate* how many correct would she or he get out of 100 questions?



HISTORY

The concept of percent is related to societies' need to compute interest, profit and loss, and taxes.

When the Roman emperor Augustus levied a tax on all goods sold at auction . . . the rate was 1/100 . . . In the Middle Ages, as larger denominations of money came to be used, 100 became a common base for computations. Italian manuscripts of the 15th century contained such expressions as "20p100," "x p cento," and "vi p c" to indicate 20 percent, 10 percent, and 6 percent. . . . The percent sign, %, has probably evolved from a symbol introduced in an anonymous Italian manuscript of 1425. Instead of "per 100," "P100," or "P cento" which were common at that time, this author used P^o. By about 1650 the ^o had become ^o so per ^o was often used. Finally the per was dropped, leaving ^o or %.⁶

arithmetic as it is a tool that rests upon many big ideas of arithmetic. In this section, you will see how percents rest upon the notion of fractions (in terms of part-whole relationships) and proportions (in terms of equivalent ratios or equivalent fractions). You will learn to translate freely among percents, fractions, and decimals.

The Origin of Percent

The word **percent** literally means "per hundred" and comes directly from the Latin *per centum*.

There are direct and immediate connections among percent, equivalent fractions, and proportions. For example, let's say that a student correctly answered 12 out of 16 questions on a quiz. To determine the student's score as a percentage, we are asking what fraction with a denominator of 100 is equivalent to 12/16. This question can be stated as a proportion:

$$\frac{12}{16} = \frac{x}{100}$$

That is, what value of x makes $x/100$ and $12/16$ equivalent fractions? Solving for x , we find the students' grade is 75%.

Uses of Percent

We use percents for a variety of purposes:

- To communicate—for example, we hear on the news that a fire is 30% contained.
- To make sense of situations—a manufacturer may say that the germination rate of a grass seed is 80%.
- To make decisions—should I refinance my house if the interest rate is down to 7.5%?
- To compare—in one high school of 345 there were 12 dropouts, and in another high school of 567 there were 17 dropouts; which school has the lower dropout rate? Converting these numbers to percents essentially treats them as rates. The first school has a dropout rate of 3.5% and the second school has a dropout rate of 3.0%.

Let us use this basic understanding of percents to solve a few problems and then look at commonalities and differences among those problems.

INVESTIGATION 6.10 Who's the Better Free-Throw Shooter?

You are the coach of the girls' basketball team at a local middle school. It is the fifth game of the season, the game is tied, and there are only

⁶Historical Topics for the Mathematics Classroom: 31st Yearbook (Reston, VA: NCTM, 1969), p. 147.

5 seconds left on the clock. The referee has called a technical foul on the other team. You get to choose any one of your girls to take the shot. Basing your decision only on their free-throw shooting thus far this season, whom would you pick? 

- Becky has made 8 of 12 free throws.
- Rachel has made 15 of 20 free throws.

DISCUSSION

STRATEGY 1: Use fractions

Represent the players' free-throw shooting as fractions.

$$8/12 = 2/3$$

At this rate, Becky would make $2/3$ of 20 shots, which is $13\frac{1}{3}$. Thus, Rachel is doing better.

STRATEGY 2: Make a proportion

Some students feel more comfortable with proportions. How would you describe why this proportion will tell us the answer?

$$\frac{2}{3} = \frac{x}{20}$$

In this case, x represents how many baskets Rachel must have made in order to have the same ratio as Becky. When we solve for x , we get $x = 13\frac{1}{3}$. Because Rachel has made 15 out of 20, which is better than $13\frac{1}{3}$, she is doing better than Becky.

STRATEGY 3: Convert fractions to decimals or percents

$$\frac{8}{12} \approx 0.67 = 67\% \quad \text{whereas} \quad \frac{15}{20} \approx 0.75 = 75\%$$

In real life, what factors other than the players' free-throw percentages might a coach consider before making the decision?

LEARNING

When Investigation 6.10 is given to middle school students, many will pick Becky, saying that she has missed fewer shots. Do you see why? What is their misconception?

INVESTIGATION 6.11 Understanding a Newspaper Article

A newspaper story reports that 8% of the 7968 students at Midvale College work full-time. How many students work full-time? First try to estimate the number of students and then determine the exact answer. 

DISCUSSION

STRATEGY 1: Use 10% as a benchmark

A very rough estimate: 8% is close to 10%, which is $1/10$, a fraction that we can do mental multiplication with rather simply. If we round 7968 to 8000, then $1/10$ of 8000 is 800.

STRATEGY 2: Use 1% as a benchmark

We could mentally find 1% and use simpler numbers to build up to 8%:

$$1\% \text{ of } 7968 \text{ is about } 80.$$

$$8\% \text{ of } 8000 \text{ is } 80 \times 8 = 640.$$

STRATEGY 3: Find a close unit fraction

We could use $1/12$ for 8%. Do you see why?

In this case, we can use the compatible numbers strategy from Chapter 3.

$$\frac{7968}{12} \approx \frac{8400}{12} = 700$$

When we compare this to $7200/12 = 600$, we conclude that the actual number of students working full-time is between 600 and 700.

STRATEGY 4: Break the problem into parts

We could also adapt the decomposition strategy developed for whole numbers in Chapter 3.

10% of 8000 is 800.

5% of 8000 is 400.

Because 8% is between 5% and 10% (a little bit closer to 10%), we can conclude that 8% of 8000 is a little more than 600.

Converting among percents, decimals, and fractions When we estimate the answers to percent problems, it is helpful to know the basic conversions among percents, fractions, and decimals. Table 6.3 shows some of the more commonly used conversions.

TABLE 6.3

Fraction	Decimal	Percent
$1/2$	0.5	50%
$1/3$	0.333... or $0.\overline{3}$	$33\frac{1}{3}\%$
$1/4$	0.25	25%
$1/5$	0.2	20%
$1/10$	0.1	10%
$1/100$	0.01	1%

With this in mind, let us examine different ways to determine the “exact” number of students working full-time at Midvale College.

STRATEGY 1: Connect to the meaning of percent

We can represent the ratio of students working full-time to the total number of students as a fraction—that is,

$$\frac{\text{Number of students working full-time}}{\text{Total number of students}} = \frac{8}{100}$$

If we let x represent the number of students working full-time, we have

$$\frac{x}{7968} = \frac{8}{100}$$

Now we can solve for x and see that $x = 637.44$.

STRATEGY 2: Use an appropriate procedure

$$8\% \text{ of } 7968 = 0.08 \times 7968 = 637.44$$

About 637 students work full-time.

STRATEGY 3: Use a diagram

Figure 6.5 shows 8%. How can you use this diagram to solve the problem?

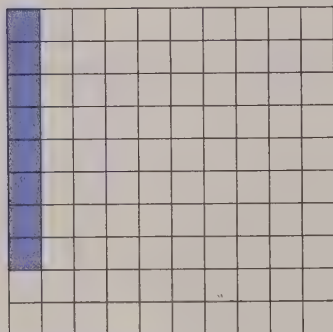


FIGURE 6.5

If the whole square represents the total student population, this means that 100 boxes represent 7968 students. The shaded area represents that fraction (percent) of the student body that works full-time.

To solve the problem using this diagram, we need to know the value of 8 boxes. If

$$100 \text{ boxes} = 7968$$

then

$$1 \text{ box} = 79.68$$

so

$$8 \text{ boxes} = 79.68 \times 8 = 637.44$$

An answer of 637.44 or even 637 is too precise; it is an example of what I call pseudo-precision. Why is this? Think before reading on. . . . 

When the college reports that 8% of its students work full-time, if it is using mathematics appropriately, it is saying that the percent of the students who work full-time is closer to 8% than it is to 7% or 9%. That is, when the person in the college determined that 8% of the students work full-time, he or she took the number of students who work full-time and divided by the total number of students, and the number that appeared on the calculator was closer to 0.08 than it was to 0.07 or to 0.09.

In other words, the actual number of students who work full-time is between 7.5% of 7968 and 8.5% of 7968. That is, the actual number is between 598 and 677. Do you see why?

We have just found that any number of full-time students between 598 and 677 will round to 8% of 7968. Because we know only that 8% was reported, it would be more accurate to say not that 637 students work full-time, but rather that the number of students who work full-time is between 598 and 677, or 637 ± 40 .

INVESTIGATION 6.12 Buying a House

The Benanders are going to buy a house. Before giving a family a mortgage to buy a home, banks generally require that the total monthly payment (including property taxes) be no more than 28% of the family's gross monthly income. What must the Benanders' monthly income be in order for them to buy a home on which the monthly payment will be \$800?

This is not an easy question. Please think about it, grapple with it, apply your understanding of percent and your problem-solving toolbox, and then read on. . . . 

DISCUSSION

STRATEGY 1: Use guess-check-revise

One way of paraphrasing the problem is to say that if the bank computes 28% of the Benanders' monthly pay, this number must be at least \$800 or the bank will turn them down. In other words, the bank takes their monthly income and multiplies it by 0.28. If the product is greater than \$800, they qualify.

For many students, this line of reasoning leads to guess-check-revise, as shown in Table 6.4.

TABLE 6.4

Guess	Computation	Analysis
\$3000	$\$3000 \times 0.28 = \$840$	Too much. Guess less.
\$2900	$\$2900 \times 0.28 = \$812$	Too much. When the guess went down by \$100, the payment went down by \$28. [Do you see the connection to ratios here?] If the next guess is \$50 less, the payment will be \$14 less, which will be too low (798).
\$2860	$\$2860 \times 0.28 = \$800.80$	So make a guess of \$40 less.

It would take several more trials to get the exact answer, and that is one limitation of guess-check-revise: It is not practical when you need an exact answer. However, unlike the electric company in Investigation 6.8, when determining eligibility, the bank is not interested in three decimal places. Banks have some flexibility; that is, if you are “close” to 28% and there are other factors in your favor (such as job stability, a promotion due, or an inheritance), then you are likely to qualify.

STRATEGY 2: Rewrite the problem as an equation

Another set of students paraphrases the problem something like this:

28% of their salary must be at least 800.

Changing the wording to be more “mathematical,” we have

28% of what is \$800?

The jump to an equation now is not as great a leap to make:

0.28 times $x = 800$, where $x =$ their monthly salary

That is, if $0.28x = 800$, solve for x , and see that $x = 2857.14$. That is, their monthly income must be at least \$2857.

STRATEGY 3: Use a diagram

We can also represent this problem with a diagram. Even if you “understand” how to get the answer, can you explain the “why” of Figure 6.6? Try to do so before reading on. . . .



In Figure 6.6, the whole box represents their monthly income, and the shaded area represents 28% of their income.

If we look at the problem from a part-whole perspective,

$$\frac{\text{Part}}{\text{Whole}} = \frac{28 \text{ boxes}}{100 \text{ boxes}} = \frac{\$800}{\text{total income}}$$

We can also interpret the figure in the following way:

If 28 boxes has a value of 800, then 100 boxes has a value of what?

If 28 boxes represents a value of \$800, then

1 box represents a value of \$28.571428 (Why is this?)

100 boxes represents a value of \$2857.14

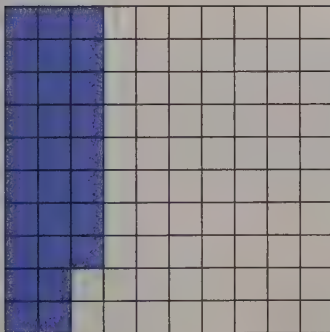


FIGURE 6.6

Connections Between Percent and Other Mathematical Topics

How would you represent the three previous investigations in terms of part-whole relationships? How would you interpret them in terms of proportions?

One of the keys to seeing the interconnectedness of all three problems is to realize that in percent problems there are two parts and two wholes.

For example, in Investigation 6.10, the 12 shots Becky attempted is the whole and 8 is the part she made. When we determine that this part-whole relationship is equivalent to 67%, we are saying that this is equivalent to 67 parts in a whole of 100 [Figure 6.7(a)].

To determine 8% of 7968 in Investigation 6.11, we need to find the part out of the whole of 7968 that is equivalent to 8 parts out of 100 [Figure 6.7(b)].

In Investigation 6.12, to determine the Benanders' monthly payment, we ask, 800 is 28% of what whole [Figure 6.7(c)]?

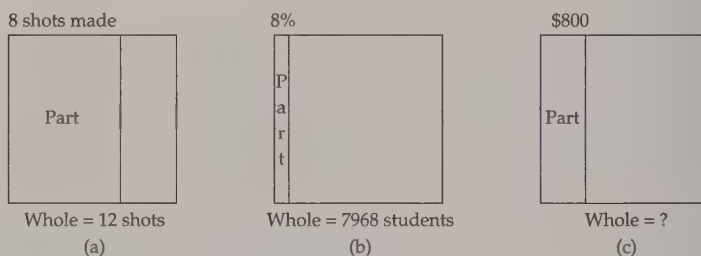


FIGURE 6.7

If we now examine these problems in terms of proportions, in each case 100 represents the whole. From this perspective, we can see that the differences among the three problems have to do with what part or whole is missing.

$$\frac{\text{Part}}{\text{Whole}} \qquad \frac{8}{12} = \frac{x}{100} \qquad \frac{x}{7968} = \frac{8}{100} \qquad \frac{800}{x} = \frac{28}{100}$$

Now that we have investigated some of the basic aspects of percents, the following investigations can serve both as stretching problems and as a self-assessment of how well you “own” the concept (that is, how well you can apply your knowledge to nonroutine problems). If you did Exploration 6.6, you grappled with understanding and justifying different ways of solving percent problems.

INVESTIGATION 6.13 Sale?

Jorge is excited. He just saw in the newspaper that Showroom Appliances is celebrating 20 years in business by offering 20% off on all merchandise. He just went there yesterday to buy a television set that was regularly priced at \$260. What is the sale price of the television? 

DISCUSSION

Probably the most common solution I see to this problem looks like this:

$$0.20 \times \$260 = \$52$$

$$\$260 - \$52 = \$208$$

That is, you determine the discount and then subtract the discount from the regular price.

Jorge will pay \$208.

However, Randi says that she solved the problem in one step:

$$0.80 \times \$260 = \$208$$

Do you understand why Randi did this? How would you explain it to someone else? Please reflect on these questions before reading on. . . . 

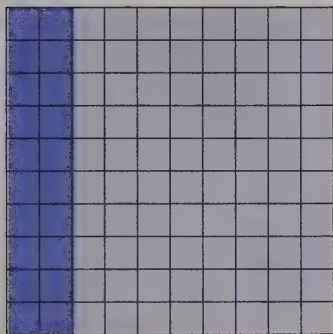


FIGURE 6.8

One way to get beyond the “how” of this shortcut and into the “why” is to represent this problem with a diagram. In Figure 6.8, if the whole box represents \$260 (the original price) what do the two shaded areas represent (that is, 20 boxes and 80 boxes)? Think before reading on. . . . 

The 20-box shaded region represents the **discount**—that is, the amount by which the store will reduce the price, or how much Jorge will save.

The 80 boxes therefore represents the **sale price** of the television. Because it is the sale price that we are looking for, we do not need to find 20% of the price and subtract it from the original price; we can determine the sale price directly.

We can show that the two procedures are mathematically equivalent:

$$\begin{aligned} 260 - 0.20(260) &= 260(1 - 0.20) && \text{We are using the distributive property.} \\ &= 260(0.80) && \text{Since } 1 - 0.20 = 0.80 \end{aligned}$$

Thus,

$$260 - 0.20(260) = 260(0.80)$$

Percent Change

Many problems involving percents in real life are not simple problems like the ones we have investigated thus far, but rather involve change. Such comparisons are called percent increase, percent decrease, percent change, percent error, percent faster or slower, or percent more or less.

We need to be able to understand change in order to interpret intelligently the changes in our society—in employment, the economy, AIDS, and other areas. *On the Shoulders of Giants: New Approaches to Numeracy*⁷ is a book that was written by mathematicians to help people to see the “big ideas” of mathematics. The six chapters discuss pattern, dimension, quantity, uncertainty, shape, and change. In one sense, the authors are saying that *this* is what mathematics is about, not just formulas and rules. Do you feel the presence of these six ideas in the work we have been doing thus far? Percent is one of many mathematical tools that can help us better understand and make sense of change.

The concept of percent change or difference is one of the more elusive concepts to own. I find that more students rent the concept of percent change than almost any other concept in the course. Therefore, it is even more imperative that you enter this section owning the knowledge in the previous discussions and that you engage in the investigations in this section and in the explorations. Explorations 6.7 and 6.8 require you to make sense of percent change and to examine ways of determining percent change.

⁷Lynn Steen, ed., *On the Shoulders of Giants: New Approaches to Numeracy* (Washington, D.C.: National Academy Press, 1990).

INVESTIGATION 6.14 What Is a Fair Raise?

Let's say I am the president of a small company, Bassarear's Bagels, that has done very well in the last year. I have decided that I want to share my good fortune with my hard-working and devoted employees. I announce that I will be giving a \$1-an-hour raise to everyone. Two days later I become aware of some grumbling. A number of employees are complaining that this is not fair. I am stunned. To me, giving everyone the same raise is the epitome of fairness. Why are some employees grumbling? How would you explain this to me if you were one of my dissatisfied employees? Think and then read on. . . . 

DISCUSSION

When I do this investigation with my own students, I receive many different (reasonable) reasons for the complaints:

- Workers with more seniority should receive "bigger" raises.
- Full-time workers should receive "bigger" raises.
- Hard-working workers should receive "bigger" raises.
- This raise is not fair to the people making more money.

For the concept of percent increase, I want to focus on the last reason. Why would a higher-paid employee feel that my raise was not fair to him or her? Imagine you are such an employee. How might you help me to see your point? Think and read on. . . . 

Let's say the janitor is making \$5 an hour and the manager of one of the stores is making \$15 an hour. From a proportional perspective, the janitor is pretty happy because \$1 an hour represents $1/5$ of her salary. The manager, however, is not very happy, because \$1 represents only $1/15$ of her salary. From the proportional perspective, a "fair" raise would be one in which the ratio of raise to present salary would be equal for everyone; that is, all the raises would be proportional.

Additive versus multiplicative increases The chart below shows the difference between *adding* the *same amount* to each person's wage (an additive increase) and *multiplying* each person's wage by the *same amount* (a proportional increase).

Original salary	Add the same amount	Multiply by the same amount
5	$5 + 1 = 6$	$5(1.2) = 6$
15	$15 + 1 = 16$	$15(1.2) = 18$

What if I were to be convinced that I should not add the same amount to each person's salary, but rather should multiply each person's salary by the same amount? How would I announce that to the company in terms of percent? Think and read on. . . . 

If we multiply the janitor's salary by 1.2, we are in effect increasing her salary by $1/5$. That is,

$$5(1.2) = 5 + 0.2(5) = 5 + 1 = 6$$

If we multiply the manager's salary by 1.2, we are in effect increasing her salary by $1/5$. That is,

$$15(1.2) = 15 + 0.2(15) = 15 + 3 = 18$$

Because $1/5$ is equivalent to $20/100$, we say that both employees have received a 20% raise; that is, their salaries are 20% greater than they were before.

In order to understand better the difference between additive and multiplicative changes, look at Figure 6.9, which shows the “before” and “after” salaries using both methods. What do you notice?

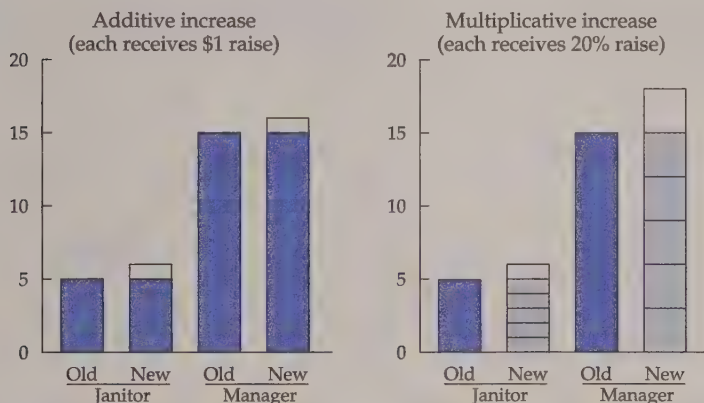


FIGURE 6.9

In the additive case (at the left in Figure 6.9), what is equal is the amount of the raise (represented by the white boxes). That is, both people received a raise of \$1 per hour.

In the multiplicative case (at the right in Figure 6.9), what is equal is the ratio of the increase to the wage. In both cases, the amount of the raise (represented by the white boxes) is equal to $1/5$ of the wage.

As I hope you are seeing, the idea of “equality” is as complex in mathematics as it is in history and political science. It is also important to know that not everyone would agree that the 20% raise is fair. Many people interpret this scenario as “the rich getting richer” because the difference between the janitor’s and the manager’s hourly pay was \$10 and is now \$12.

Now let us investigate a problem that is near and dear to the hearts, or at least the pocketbooks, of most students.

INVESTIGATION 6.15 How Much Did the Bookstore Pay for the Textbook?

When college bookstores purchase textbooks, they generally sell the books for 20% to 25% more than they paid for them. In other words, their markup is between 20% and 25%. Let’s say that you paid \$65 for a textbook and your bookstore marked up the price by 25%. How much did the bookstore pay for the textbook? Work on this problem before reading on. . . .



DISCUSSION

STRATEGY 1: Act it out

Let's say you are working in the bookstore. You would take the price of the book (which we don't know at this point) and find 25% of that number, and then you would add that to the price of the book. This sum would be \$65. We can solve the problem by using guess-check-revise (see Table 6.5) or by forming an equation.

STRATEGY 2: Use guess-check-revise

TABLE 6.5

Guess	Work	Reflection
40	$0.25(40) + 40 = 50$	Too low.
50	$0.25(50) + 50 = 62.50$	Increasing the guess by 10 increases the sum by 12.50.
53	$0.25(53) + 53 = 66.25$	Too high.
52	$0.25(52) + 52 = 65$	Done!

STRATEGY 3: Form an equation

If we let x = the price the bookstore paid for the book, many students can "see" the equation emerging by acting out the process. The bookstore employee finds 25% of x and then adds this amount to the amount the bookstore paid for the book; this sum must be \$65.

That is,

$$0.25x + x = 65$$

$$1.25x = 65$$

$$x = 52$$

Virtually every day, I see percents described in the context of a variety of situations. One of the difficulties many people have is translating percent ideas into words. The following situation is one of many instances of the need to translate.

Percents Less than 1 or Greater than 100

In real life, we frequently encounter percents greater than 100% or less than 1%. Researchers and teachers know that when students do not thoroughly understand the concept of percents, their success rate with such problems goes down dramatically.

INVESTIGATION 6.16 The Copying Machine

CClark Elementary School has decided to buy a new copying machine. One of the selling points of the new machine is that the manufacturer advertised that the school can expect about 2 paper jams per 1000 copies. After six months, the faculty wanted to see how the copier was actually doing. They had run 42,164 copies and had encountered 96 paper jams. Think about the following questions and then read on. . . . 

- A. What do you think? Is the school averaging about 2 paper jams per 1000 copies? Do you think the ratio of paper jams at Clark is consistent with the advertised paper jam rate?
- B. Represent the advertised paper jam rate as a ratio, a fraction, a decimal, and a percent.
- C. Which one would you use if you were writing the ad for the copier?
- D. How do you think the manufacturer determined the figure of 2 paper jams per 1000 copies?

DISCUSSION

A. One way (of several) to answer this question is to set up the following proportion:

$$\frac{96}{42,164} = \frac{x}{1000}$$

Could you explain this proportion to someone who didn't understand it? What does the x mean?

By setting the ratio $96/42,164$ equal to $x/1000$, we are saying the rate of 96 jams per 42,164 copies is equivalent to how many jams per 1000 copies?

Solving for x , we find $x \approx 2.3$, which means that Clark is averaging about 2.3 paper jams per 1000 copies. Because the paper jam rate for Clark is closer to 2 jams per 1000 than to 3, we would conclude that the copier is acting as advertised.

B. All of the following are mathematically equivalent to a paper jam rate of 2 per 1000:

1 in every 500 copies will jam.

$1/500$ of the copies will jam.

0.2% of the copies will jam.

One-fifth of 1% of the copies will jam.

C. As an advertiser, I think I would say that the paper jam rate is one-fifth of 1%. This sounds smaller than 2 copies per 1000. However, I also know that most people will understand 2 copies per 1000 much better than one-fifth of 1%.

D. The manufacturer probably divided the total number of jams by the total number of copies. I would hope that this was done with many different copiers rather than just one copier, and I would hope that these copiers were not all brand new. The ratio that resulted from this calculation was closer to 2 per 1000 than to 1 per 1000 or 3 per 1000.

INVESTIGATION 6.17 132% Increase?

In 1990, I read in a newspaper that "the percent increase since 1975 in births by Cesarean section in the United States is 132." The article went on to say that 640,000 Cesarean sections had been done in 1975. I immediately wondered how many had been done in 1990, and that is the question I want you to answer now. Think and then read on. . . . 

- A. Before you solve the problem, try to make a rough or refined estimate.
- B. Then solve the problem.

DISCUSSION

A. *A rough estimate:* Because a 100% increase is double the original number, the number of Cesarean sections more than doubled, so the answer will be more than 1,280,000 Cesarean births in 1990.

A refined estimate:

An increase of 132% = original amount + 100% + 32% (about $1/3$)
 $600,000 + 600,000 + 200,000 = 1,400,000$ Cesarean births in 1990

B. STRATEGY 1: Break the problem into parts

A 132% increase can be broken into a 100% increase and a 32% increase.

A 100% increase is equivalent to doubling, which gives us 1,280,000.

A 32% increase is $0.32(640,000) = 204,000$.

Therefore a 132% increase brings us to 1,484,800 Cesarean births in 1990.

STRATEGY 2: Connect to a similar, simpler problem

For example, if the problem had said that there had been a 25% increase, many students would have found the answer by multiplying 640,000 by 1.25. Do you see why?

If a 25% increase is determined by multiplying by 1.25, what would we do to determine a 132% increase? Think and then read on. . . . 

We would multiply 640,000 by 2.32.

One of the subtleties of estimating is to know when a rough estimate is fine and when a more accurate (refined) estimate is called for. What do you think in this case? .

Without an estimate, many students arrive at the wrong answer of 844,800 because they multiply 640,000 by 1.32 or they add 32% of 640,000 to 640,000. In this case, even a very rough estimate serves many students nicely: It tells them that the number of Cesarean births in 1990 must be at least 1.2 million. Students who do a very rough estimate and then obtain an actual answer of 844,800 see a contradiction between the estimate and the computed answer that causes them to go back and reexamine their work. This is one of the many useful aspects of estimating.

Interest

One of the ways in which almost everyone encounters percents is with interest—when you buy a car, when you buy a house, and when you don't pay off your entire credit card balance at the end of the month, you pay interest on the amount that you owe.

Most people have a very limited understanding of the consequences of interest. Let's say you buy a house for \$110,000 and make a down payment of \$10,000. You go to the bank and take out a 30-year mortgage for \$100,000, and you are told that your monthly payment will be \$767. Multiply this monthly payment by 360 (there are 360 months in 30 years) to determine how much you actually pay for the house (the \$100,000 you borrowed plus all the interest). Then read on. . . . 

No, you did not mispunch the calculator. Yes, you really will pay that much for the house! You actually paid \$286,120 for the house! Thus, over 30 years, you pay \$176,120 of interest!

Let us examine how interest is determined, first in a simple case and then in some instances that are more realistic. Several variables affect how much interest you receive (on an investment) or pay (on a loan): the original amount, called the principal (P), the annual interest rate (r), specified in percent, and the time of the loan (t), specified in years.

Although simple interest is not common, a good starting point for understanding how interest is determined is the case of simple interest—when the principal does not change over the course of the loan. For example, let's say that Jenna borrows \$2000 from a relative and agrees to repay the loan after 1 year at the rate of 6%. At the end of the year, she pays \$2000 + 6% of \$2000; that is, $\$2000 + 120 = \$2120$.

More commonly, interest is compounded; that is, the interest is determined at specified intervals and added to the principal at those times. For example, if Jenna's relative had specified that the loan be compounded semiannually, to determine how much Jenna would owe at the end of the year, the interest would be determined every 6 months and added to the principal. Thus, after 6 months, she would owe $\$2000 + 0.03(2000) = \$2060$. Why is this?

The annual interest rate is 6%, so the semiannual rate is $1/2$ of 6%, or 3%, because 6 months is $1/2$ year. We determine how much she owes after the next 6 months as follows: $\$2060 + 0.03(2060) = 2060 + \$61.80 = \$2121.80$. In this particular case, a 1-year loan compounded only semiannually, the difference between simple and compound interest is not huge. However, on most deposits and loans, the interest is compounded daily; that is, the annual interest rate is divided by 365.

In the following investigation, we will develop the formula for determining compound interest. In the second investigation, we will examine a very relevant situation.

INVESTIGATION 6.18 Saving for College

When Emily was born, her grandparents decided to contribute toward her college education by opening a savings account for \$1000. If they add no other money, and if the account earns 6% compounded annually, how much money will there be in the account after 18 years? 

DISCUSSION

This is a nonroutine, multistep problem. Therefore, let us go one step at a time.

After 1 year How much money would there be in the account after 1 year? Do this and then read on. . . . 

There are several different methods students will use.

We can find 6% of \$1000 and then add that to 1000; that is, $\$1000 + 0.06(\$1000) = \$1060$. Or, as we saw earlier in this chapter, we can obtain \$1060 in one step: $\$1000(1.06) = \$1060$. Do you see why?

This connection between procedures is important because in order to understand this problem, you need to be able to understand the second procedure; the first one is too cumbersome.

After 3 years Now determine how much money would be in the account after 3 years and then read on. . . . 

$$\$1000(1.06) = \$1060 \text{ after 1 year}$$

$$\$1060(1.06) = \$1123.60 \text{ after 2 years}$$

$$\$1123.60(1.06) = \$1191.02 \text{ after 3 years}$$

Let us stop and analyze this strategy. In the first step, we multiplied 1000 by 1.06 to get 1060. In the second step, many students clear the calculator and then multiply 1060 by 1.06. Then, in the third step, they clear the calculator and then multiply 1123.60 by 1.06. If we examine this carefully, we can see a shortcut. See whether you can find it yourself before reading on. . . . 

We could simply do this on the calculator:

$$1000 \quad (\times) \quad 1.06 \quad (\times) \quad 1.06 \quad (\times) \quad 1.06$$

Do you see why? If you don't believe this, do it on the calculator to see that it does work. Then try to explain why it works before reading on. . . . 

We can represent what we have learned as

$$\$1191.02 = 1000(1.06)(1.06)(1.06)$$

We can now use our knowledge of exponents to say

$$\$1191.02 = 1000(1.06)^3$$

The third, and most efficient, method for finding the amount after 3 years uses this knowledge. How would we enter the last expression in a scientific calculator? Try it before reading on. . . . 

Simply enter $1000 \times 1.06 \quad y^x \quad 3$ to get the same answer.

Before we move on to solve the original question, this is a good time to understand the basic formula for use with interest. Look at how we determined the value (amount) in the bank after 3 years. Try to represent that procedure with a formula using the interest symbols introduced above, and then read on. . . . 

$$\begin{aligned} \text{The money after 3 years} &= (1000)(1.06)^3 \\ &= 1000(1 + 0.06)^3 \end{aligned}$$

That is,

$$A = P(1 + r)^t$$

After 18 years This discovery makes the original problem much less tedious to solve.

STRATEGY 1: Use the y^x button

The most straightforward solution is to use the y^x button on your calculator:

$$1000 \quad (\times) \quad 1.06 \quad (y^x) \quad 18 = \$2854.34$$

STRATEGY 2: Use the memory key

If your calculator lacks an exponentiation key, you could use the memory key $M+$ by doing the following:

First, activate the memory by pressing 1.06 and then $M+$.

Now press $1000 \times MR =$ (you should see 1060, the amount after 1 year).

To get the amount for each succeeding year, you simply press $\times$ MR that many times.

That is, you press $1000 \times \overbrace{\text{MR} \times \text{MR} \times \text{MR} \cdots \times \text{MR}}^{18 \text{ times}} =$

	A	B
1	0	1000
2	1	1060
3	2	
4	3	
5	4	
.	.	
.	.	
.	.	
.	.	
.	.	
.	.	
.	.	
.	.	
19	18	

FIGURE 6.10

STRATEGY 3: Use a spreadsheet

A spreadsheet (on a computer) can also be used. When you open the spreadsheet, the columns are marked with letters—A, B, C, and so on—and the rows are marked with numbers. In column A, we enter the numbers 0 through 18 (see Figure 6.10). In column B, we enter 1000 into the first row, signifying the amount we have at the beginning.

In the B2 cell, we now enter “= B1*1.06”; that is, we tell the computer to multiply the amount in the B1 cell by 1.06. Now we will see 1060 in the B2 cell.

Now, we highlight the B column from row 2 through row 19 and select “Fill down.” This command essentially tells the computer to repeat the computation—that is, to multiply the previous amount by 1.06. After we do this, the computer will display the amount at the end of each year!

One advantage of the spreadsheet is that we can play the “what if” game: what if they had started with \$3000, what if the interest rate had been 7%, and so on.

INVESTIGATION 6.19 How Much Does That Credit Card Cost You?

Most adults in the United States have credit cards. Usually, credit cards require a minimum monthly payment each month. If the customer is not able to pay the entire balance (for example, after a vacation or after Christmas), the customer pays at least this minimum amount; then a fee called a finance charge is determined by the credit card company on the basis of the interest rate the company is charging.

Let's say George found himself in just that situation. His VISA bill has come in, and the balance is \$761.34. He decides he can pay \$61.34. If the bank issuing the VISA card determines the finance charge at the annual rate of 18%, called the APR (annual percentage rate), what will his finance charge be? Think and then read on. . . . 

DISCUSSION

Because the balance is compounded monthly, the bank will determine George's finance charge by multiplying his unpaid balance (\$700) by 0.015. Do you see where the 1.5% comes from?

To find the monthly rate, we divide the yearly rate by 12 (months): $18\% \div 12 = 1.5\%$, and $1.5\% = 0.015$.

Thus, George's finance charge will be $\$700(0.015) = \$10.50$.

Summary

In this section, we began with the basic concept of percent, a part expressed as a hundredth, and then we connected this to the concepts of equivalent fractions and proportions. We found, however, that this simple notion of percent needs to be refined when we are dealing with percents greater than 100, with percent increase and decrease, and with nonroutine problems. This need for a richer concept of percent was also underscored in the explorations.

In examining percent problems, you saw applications of percents in several areas, including interest. You also saw the importance of estimating—sometimes because the question calls for an estimate instead of an exact answer, and sometimes because estimation helps you to check your solution or your reasoning.

EXERCISES 6.2

1. For each of the following questions, estimate first, and briefly explain how you determined your estimate. Then determine the actual answer. Compare your estimate to the actual answer. If your estimate was close, move on; if not, you might want to reread part of the section and/or consult with a friend or the instructor.
 - a. What is 4% of 450?
 - b. What is 23% of 85?
 - c. What is 83% of \$1460?
 - d. What is 3.2% of 1700?
 - e. What is 0.25% of 345?
 - f. What is 120% of 200?
 - g. 1200 is what percent of 1500?
 - h. 30 is what percent of 35?
 - i. 56 is what percent of 657?
 - j. 1.2 is what percent of 4.6?
 - k. 75 is what percent of 400?
 - l. 175 is what percent of 120?
 - m. What is 3% more than 45?
 - n. What is 20% more than 400?
 - o. What is 16% more than 6.2?
 - p. What is 100% more than 35?
 - q. What is 10% less than 36?
 - r. What is 3% less than 45?
 - s. What is 1.2% less than 1200?
 - t. What is 80% less than 800?
 2. Missy Adams's gross pay is \$1500 per paycheck. If her total payroll deductions are \$340, her take-home pay is what percent of her gross pay?
 3. All three second-grade teachers at Uphill Elementary School asked their students how many have pets at home. The response was that 23 out of 65 of the students have pets. What percent of the students have pets?
 4. A victim won damages of \$2.4 million from a company. If the lawyer's fee was 35%, how much did the lawyer get?
 5. If 30% of the patients in a hospital have heart problems and there are 210 patients with heart problems, how many patients are there in the hospital?
 6. Last year Mr. Rich paid \$27,000 in income taxes because he was taxed at the rate of 38% of his gross income. What was his gross income?
 7. a. Between 1980 and 1987, the price of an average house in Keene increased from \$42,000 to \$97,000. What is the percent increase (rounded to the nearest whole number) in the average selling price?
 - b. What is the percent increase for a house that sells for \$60,000 and then later for \$75,349?
 - c. What is the percent increase for a house that sells for \$71,450 and then later for \$102,500?
 - d. What is the percent increase for a house that sells for \$62,000 and then later for \$144,000?
 8. Two dresses are on sale. The first was selling for \$119 and is being marked down 40%. The second was selling for \$79.99 and is being discounted by 20%. Which dress costs less now?
 9. At birth a baby weighed 8 pounds 4 ounces. Two days later, the baby weighed 7 pounds 12 ounces. What percent of its weight had it lost? (This weight loss is normal—part of the adjustment to the living outside the mother's womb.) If a 160-pound adult lost the same proportion of its weight in 2 days, how much weight would the adult have lost?
 10. *The Unofficial U.S. Census* reports that 34,177,500 of America's 139,500,000 cars are washed at least once each week, and that 6,556,500 cars are never washed. Assuming that these numbers are relatively accurate:
 - a. What percent of cars are washed each week?
 - b. What percent of cars are never washed?
 11. In 1990, about 248 million people lived in the United States, and the world population was about 5.13 billion. Approximately what percent of the world population lived in the United States?
 12. A 1991 news story said that 16% of the elderly are served by the Medicare prescription program, in which they receive assistance when their monthly prescription bill exceeds a certain amount. The story went on to say that about 5 million people are served by the program. What is the total number of elderly people in the United States?
 13. It has been estimated that 0.8% of Americans are homeless. Now 0.8% is a small percentage, but there are approximately 270 million people in the United States, and 270 million is a large number. Approximately how many Americans are homeless?
 14. In 1988 only 900,000 of the 2.6 million eligible voters aged 18 to 24 actually voted. What percent voted?
 15. In an up-and-coming town, housing prices rose an average of 115% in five years. If a home cost \$42,000 five years ago, how much would that home cost now?
- Exercises 16–22 are nonroutine and/or multistep problems. If it is possible to make an estimate, do so and briefly explain how you obtained your estimate. If you feel that an estimate

is not practical, describe what measures you took to ensure that your answer is reasonable.

16. A store reduced the price of a computer by 20% and sold it for \$1760. How much did the computer originally sell for?
17. Janice Brady borrowed \$4200 to buy a new car. If she pays the loan back in 1 year at 8.2% interest, how much will she actually pay for the car? (Assume simple interest.)
18. A store collected \$53 in sales tax alone in one day. If the sales tax is 5.5%, how much did they sell that day?
19. John paid \$330 for a new mountain bicycle to sell in his shop. He wants to price it so that he can offer a 10% discount and still make 20% profit. At what price should he mark the bike?
20. A teacher was hired at a salary of \$18,200 and is to receive a raise of 5.5% after her first year and 4% after her second year. What should her salary be after the 2-year period?
21. Your optimal exercise heart rate for cardiovascular benefits is calculated as follows: Subtract your age from 220 and then find 80% of the difference. Find the optimal heart rate for a 50-year-old and for a 20-year-old. The optimal rate for the 20-year-old is ___ % greater than the optimal rate for a 50-year-old.
22. A city budget is \$21.43 million. The city council voted to increase the budget by 4.4%. What is the new budget, rounded to the nearest \$1000?
23. In the 1980s, the company that built the Seabrook nuclear reactor in New Hampshire went bankrupt. When Northeast Utilities bought the company out, the agreement was that Northeast could raise electricity rates at least 5.5% each year for the next 7 years.
 - a. If the Jones family paid an average of \$83.00 per month for electricity last year and the rate rises 5.5% each year, what will they pay after 7 years?
 - b. If the Jones family's income is \$40,000 this year and rises at a rate of 5.5% each year, what will be their income after 7 years?
24. The owners of High Hopes Apple Farm have a crop of approximately 1000 bushels of apples. They can have the apples picked by a regular labor crew for \$2.50 per bushel and then sell the apples for \$5.00 per bushel. Or they can let the public pick the apples at \$4.00 per bushel. Which way will provide more income? One factor that they need to take into account is that when the public picks their own apples, the owners will lose about 20% of the crop from waste (eaten and dropped apples).
25. Let's say a map has a scale of 1:25,000. Two cities on the map are 3 inches apart. How many miles does this represent?
26. Arthur was not able to get his income taxes ready by April 15. The penalty for late returns is assessed at a 20% annual rate of interest. Arthur filed his income taxes on June 20 without having applied for an extension. He owed \$2000 plus the penalty. How much did he owe?
27. Selena recently bought a computer, which she paid for with her VISA card. The computer cost \$2137. When her VISA bill came the next month, the total balance was \$2534. If she paid only the minimum \$20, how much was her finance charge on the next month's bill if the APR on her VISA card is 19.6%?
28. Several years ago, newspapers reported that the average unpaid balance on VISA cards was about \$1500. Let's say that you are an average consumer, that your VISA card charges you at the rate of 18% per year, and that each month for 1 year, you have an unpaid balance of exactly \$1500.
 - a. How much will your finance charges add up to for the year?
 - b. How much money would you save if you switched to a credit card that charged only an 8% annual interest rate?
29. A *USA Today* article reported that "Senior high school principals now average \$61,768 a year. Their counterparts make \$57,504 in junior high and \$53,856 in elementary schools. The average U.S. teacher made \$33,041 in 1990–91."
 - a. Compare the average salaries in percent-greater language; that is, compare all salaries to the \$33,041 figure.
 - b. Compare the average salaries in percent-less language; that is, compare all salaries to the \$61,768 figure.
30. Express 0.5% as a ratio in simplest terms. For example, 40% can be expressed as 40 out of 100, and in simplest terms as 2 out of 5.
31. Let's say that tests on a new vaccine show that less than 0.5% of all children have a serious reaction to the drug.
 - a. Gerry is not very mathematically literate. Help her to understand mathematically what this means.
 - b. What additional information would you like to have in order to help you decide whether or not to have your child take the vaccine?
32. The figures below are disease rates for heart disease and cancer (per 100,000 people).

	1960	1988
Heart disease	286.2	166.3
Cancer	125.8	132.7
33. Let's say you read that the rate of inflation in the United States last year was 5% and the rate of inflation in

another country was 400%. Compare the two inflation rates in the way that will let the greatest number of readers relate to your statement.

34. A road sign says "7% grade." What does that mean?
35. The U.S. Bureau of the Census reported that there were 35.7 million Americans living below the poverty level in 1991, 14.2% of the population. The U.S. Bureau of the Census also gives figures for persons living "below 125% of the poverty level." They reported that 47.5 million persons, or 18.9% of the population, were in this category.
- If the poverty level for a nonfarm family of four in 1991 was \$13,924, what would be the cutoff for a family of four "below 125% of the poverty level"?
 - Why do you think they came up with the reference point of "below 125% of the poverty level"?
36. First Bank Expressline made the following claim in an advertisement. The bank was offering 9.46% interest on the monthly unpaid balance. The chart below showed how much money a consumer with a \$3500 balance would save by switching to First Bank's credit card. Is the ad valid?

	Current APR	Expressline savings
First Bank Expressline	9.46%	
Citibank	19.8%	\$361.90
Discover	19.8%	\$361.90
Sears	18.0%	\$298.90
GM Card	16.4%	\$242.90

37. Below are the infant mortality rates of 7 countries in 1993 and the predicted rates for 2000. *Infant mortality rate* is defined as the number of deaths of children under 1 year of age per 1000 live births in a calendar year.

Country	1993	2000	Change 1993-2000	Estimated % change	Actual % change
United States	8.4	6.6			
Cuba	10.5	9.2			
Egypt	78.3	65.7			
Haiti	109.5	102.2			
Japan	4.3	4.1			
Mexico	28.8	20.7			
Somalia	162.7	87.1			

- Fill in the third, fourth, and fifth columns. You need not show your work for the third column. You need to explain each of your estimates briefly. If you use a calculator for the fifth column, briefly explain the procedure you used.

- If you were writing a report on changes in infant mortality and you could choose either the "Change" or the "Percent Change" column, which would you choose? Why?
- Convert Somalia's 1993 infant mortality rate into percent language. That is, x percent of children in Somalia die before the age of 1.
- Joanne still does not understand what "infant mortality rate" means. Translate the formal description, given at the beginning of this problem, into more understandable language.

Problems 38–43 require you to make some assumptions in order to determine an answer. Describe and justify the assumptions you make in determining your answer.

38. Let's say that you read in the newspaper that last year's rate of inflation was 7.2%.
- If your grocery bill averaged \$325 per month last year, about how much would you expect your grocery bill to be this year?
 - Let's say you received a \$1200 raise, from \$23,400 to \$24,600. Did your raise keep you ahead of the game, or are you falling behind?
39. There was a proposal in New Hampshire in 1991 to reduce the definition of "drunk driving" from an alcohol blood content of 0.1 to 0.08. Explain why some might consider this a little drop and others might consider it a big drop. What do you think?
40. A recent news article stated that in 1986, women were paid only 64¢ for every dollar paid to men. Transform this statement to the following form: Men were paid ____% more than women.
41. Refer to Investigation 6.11. Jane still doesn't understand the problem. Roberto tries to help her make sense of the problem by saying that the 8% means that if we were to select 100 students at the college, 8 of them would be working full-time. What do you think?
42. Annie has just received a 5% raise from her current wage of \$9.80 per hour.
- What is her new wage?
 - What would this amount to over a year?
 - What assumptions did you make in order to answer part (b)?
 - What if the raise had been 5.4%?
43. Jack is building an office building. The local building code says that the window area in the building can be no more than 20% of the floor area.
- If the two-story building will contain 12 offices and have 3000 square feet of floor space, what is the maximum area of window area allowed?
 - About how many windows would that be?
44. The table that follows lists annual percent changes in consumer prices.

ANNUAL PERCENT CHANGE

Country	1987–88	1988–89	1989–90
United States	4.0	4.8	5.4
Brazil	682.3	1287	2938
Canada	4.0	5.0	4.8
Egypt	17.7	21.3	16.8
Japan	0.7	2.3	3.1
Mexico	114.2	20.0	26.7
Peru	667.0	3399	7482

- Anders has trouble understanding percent changes over 100. Explain Peru's percent change of 667 to Anders.
 - Approximately how much would you spend in the United States in 1990 to buy items that cost you \$100 in 1987?
 - Approximately how much would you spend in Brazil in 1990 to buy items that cost you \$100 in 1987?
 - Approximately how much would you spend in Japan in 1990 to buy items that cost you \$100 in 1987?
 - Approximately how much would you spend in Mexico in 1990 to buy items that cost you \$100 in 1987?
- If Jonah puts \$25,000 in the bank at 8% interest compounded quarterly (four times a year), how much will his investment be worth in 20 years?
 - If Liam puts \$10,000 in the bank at 5% interest compounded quarterly (four times a year), how much will his investment be worth in 5 years?
 - How long will it take \$1000 at 6% simple interest to double?
 - A company's sales increased from 1999 to 2000. It is possible to describe this increase either additively or multiplicatively. You will be asked to examine and then compare the two ways.
 - We sold 34,234 more Bender Bobbers in 2000 than we did in 1999. What does that tell you?
 - We sold 25% more Bender Bobbers in 2000 than we did in 1999. What does that tell you?
 - If you were a stockholder in the company, which sentence would be more useful to you in the annual stockholder's report, the first sentence in part (a) or the first sentence in part (b)? Explain your choice.
 - Virtually all sunscreen lotions list the SPF (sun protection factor), which is an indication of how long you will be protected from sunburn when wearing the sunscreen. The amount of time you're protected is proportional to the SPF. If wearing SPF 8 sunscreen will protect your skin for 40 minutes, how long will SPF 30 sunscreen protect you?
 - One day a newspaper reported the following information, gathered from the National Restaurant Association,

concerning the percentage of food budgets spent by the average person on eating out in different cities in the United States.

- Describe how the data might have been obtained.
- Explain why these data represent an example of pseudo-precision.

Percentage of food dollars eating out

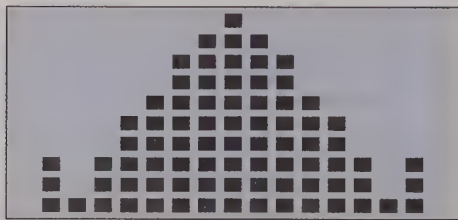
Miami	52.1%
Boston	49.5%
New York	48.9%
San Francisco	48%
Dallas/Ft. Worth	47.6%

- The *Unofficial U.S. Census* determined the following data concerning the ages of cars on the road.
 - First, estimate the percentage of cars that are 15 years old or older.
 - Determine the "exact" percentage of such cars.
 - How accurate do you think this number is? Justify your choice.

Age of car Number on road

Less than 1 year	7,812,000
1–5 years	47,569,500
5–10 years	42,687,000
10–15 years	27,760,500
15 years or older	13,671,000

- The July 20, 1996, issue of *America* had an article about the growing number of schools requiring school uniforms. The author describes a school where the amount of violence, especially between members of different gangs, was out of hand. A new principal tried various measures, "but the most effective change. . . was the one that cost the least, a school uniform policy that prescribes a white top—a white T-shirt, for instance—with black trousers or skirts. This innovation is said to have helped produce a 100 percent drop in violence at Farragut (p. 20)." Do you believe the number, or is there not enough information to determine whether the actual drop in violence is likely to be as reported? Why or why not?
- The local gym, where I work out, has a ski-type exercise machine that saves wear and tear on my knees. The machine has different programs to choose from. The one I choose simulates going up and down hills. If I program the computer for 30 minutes, the screen looks something like the image that follows. Because there are 15 vertical bars, representing the elevation (and thus the difficulty) of the machine, I spend 2 minutes at each level.



No matter how many minutes I select, there are 15 bars showing. Thus, if I select a shorter time period, the machine will move to the next level in a proportionately smaller time period.

- a. One day I decide to exercise only for 20 minutes. How long do I spend at each height now?
- b. One day I decide to exercise for 45 minutes. How long do I spend at each height now?

CHAPTER SUMMARY

1. When we compare amounts, we can do so in additive or multiplicative ways. Both have value. In many cases, one is more appropriate.
2. Ratios, proportions, and percents involve multiplicative relationships between numbers.
3. Ratios are both like and unlike fractions. Ratios can be expressed in fraction notation, but many ratios involve part-to-part or whole-to-whole relationships. Not all such ratios can be translated into fraction language. Rates never can.
4. Percent, seen only as what part of 100, is too limiting. For example, a 132% increase expresses a relationship between an original and a new amount.

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Uncertainty: Data and Chance

- ◆ 7.1 Representing and Interpreting Data
- ◆ 7.2 Distributions: Centers and Spreads
- ◆ 7.3 Concepts Related to Chance
- ◆ 7.4 Counting and Chance

We have spent a considerable amount of time examining operations with numbers and different ways to represent amounts. We can add, subtract, multiply, and divide amounts; we can represent amounts and relationships between amounts with whole numbers, negative numbers, fractions, decimals, and percents. In this chapter, we will focus on another way in which we use numbers—to understand and deal with uncertainty.

In this world, nothing can be said to be certain except death and taxes.

by BENJAMIN FRANKLIN

We expend a lot of energy trying to understand the uncertainty and chance in our lives. Let us consider a few examples of uncertainty and chance. What do the examples below have in common? Think and then read on. . . . 

- A newspaper takes an opinion poll to determine which candidate people prefer.
- The government collects information to determine the rate of inflation.
- Scientists collect samples of air to determine the extent of air pollution.
- Researchers collect data to understand the AIDS epidemic.
- Colleges collect data in order to determine how to weight admission factors such as high school GPA, SAT scores, recommendations, and so on.
- A company conducts market research to determine how much it should charge for a new product.
- A weather forecaster makes predictions based on analyses of data.
- Peter collects and analyzes information to determine which car is the best buy.

- You want to know what your chances are of getting a teaching job after graduation.

As you can see from these examples, we collect data for a variety of reasons: to make predictions, to make decisions, to answer questions, and to better understand aspects of our life.

All of the examples above have two things in common: They all involve **uncertainty**, and collecting and analyzing data increases the chances that the decision or prediction will be a good one despite this uncertainty.

We often refer to that field of mathematics that focuses on collecting and interpreting data as statistics and to that branch of mathematics that focuses on determining the chance of something happening as probability. However, as you may have discovered from the Chapter 7 explorations, much of our interpretation of data involves chance, and much of our understanding of chance involves collecting and analyzing data. This is one of the reasons why I have chosen to have you examine these ideas within one chapter rather than in two separate chapters called “Statistics” and “Probability.” Furthermore, just as you found that the words *carrying* and *borrowing* mask the essential connectedness of these two procedures, so too the terms *statistics* and *probability* tend to mask the essential connectedness of these two fields.

For example, when we say that the average annual snowfall in Buffalo, New York, is 92.2 inches,¹ that does not mean that you should expect 92.2 inches of snow in Buffalo this year (or 92 or 90 inches, for that matter). There are variations in any phenomenon; this number represents an average from data collected over 42 winters. We cannot say with certainty that we will get 92 inches of snow this year. We cannot even say with certainty that Buffalo will get between 82 and 102 inches this year. However, the average gives us an idea of what we might expect. In this sense, it tells us that when we look at the snowfall data for Buffalo, this amount is more common—it is in the middle of the data that have been collected.

Similarly, when we say that the chance of flipping a coin and getting tails is 50%, this does not mean that we will get 5 tails if we flip a coin 10 times. Even with this event, with less variation than snowfall in Buffalo, there is variation and a lack of certainty. I have flipped many coins (I was particularly fascinated by this and the dice probabilities as a youngster). I have occasionally gotten only 2 tails out of 10 throws, and it is statistically possible, though improbable, to flip a coin 10 times and have 0 tails.

In everyday life, *uncertain* has several connotations: not known or established, not determined, not having sure knowledge, and subject to change.² In one sense, our study of statistics and probability is meant to make our conclusions about phenomena that are not certain to occur more reliable and trustworthy, while at the same time recognizing the truth of Benjamin Franklin’s statement.

Data Interpretation and Chance in Society

 Collecting, organizing, describing, displaying, and interpreting data, as well as making decisions and predictions on the basis of that information, are skills that are increasingly important in a society based on technology and communication.

(Curriculum Standards, p. 54)

¹ *Information Please* (Boston: Houghton Mifflin, 1993), p. 392.

² *American Heritage Dictionary of the English Language*, 3d ed. (Boston: Houghton Mifflin, 1996), p. 1942.

Understanding uncertainty is a crucial skill that all citizens need but few have. There is much evidence that many students understand the fundamental ideas of statistics and probability even less well than they understand other ideas in the elementary school classroom. Not surprisingly, misuse of statistics—by politicians, special interest groups, entrepreneurs, and others—is pervasive. Sometimes the misuse is deliberate—for example, to manipulate decisions and opinion. However, often the misinterpretation of data occurs either out of ignorance or because the problem is very complex.

At the same time, investigating situations involving uncertainty can be some of the most exciting and informative work that you (and your students) will do in mathematics classrooms. Toward this end, I find that students need to experience statistics and probability more directly, and so the investigations have been fashioned with that in mind. The NCTM also urges more hands-on work with all aspects of what we call statistical analysis:

Teachers should have a variety of experiences in the collection, organization, representation, analysis, and interpretation of data. Key statistical concepts for all teachers include measures of central tendency, measures of variation (range, standard deviation, interquartile range, and outliers), and general distributions. Representations of data should include various types of graphs, including bar, line, circle, and pictographs as well as line plots, stem-and-leaf plots, box plots, histograms, and scatter plots.

(*Professional Standards*, p. 136)

There is another point about examining concepts in statistics and probability that is important to note: language. As you recall, the NCTM strongly emphasizes language in mathematics (Curriculum Standard 8). Many students report being intimidated by the language of mathematics. I hope you are coming to realize in this book that mathematical language can actually make mathematical ideas easier to understand and grasp. For example, the terms *digit* and *place* make it easier to discuss numeration, computation algorithms, and estimation strategies. There are a number of words used in this chapter that are used in everyday life but that have a slightly different or more precise meaning in mathematics—for example, *uncertainty*, *population*, *sampling*, *reliability*, *combination*, and *event*. To get too technical is to lose readers; to stay too informal is to cause confusion. Therefore, I have made an attempt to avoid both extremes.

SECTION

7.1

REPRESENTING AND INTERPRETING DATA

WHAT DO YOU THINK?

- How can the way a set of data is represented affect the way it is interpreted?
- How can the decisions about what data to collect affect the information learned from those data?
- Consider a representation or description of data that you have recently encountered. Was it accurate? How could you tell?

In this first section of the chapter, we will examine different methods for describing and interpreting data. In the course of this work, most of you will learn more about events outside the classroom, too.

In each of the investigations in this section, you will initially be presented with a set of data or a graph. You will be asked to record your initial impressions and conclusions. You will also be asked to note questions you have about the data or the graph. We will discuss two kinds of questions: (1) questions about aspects of the data or graph that you don't understand, and (2) questions about the reliability and validity of the data—two issues that also come up repeatedly in this chapter's explorations. Questions about *reliability* ask whether two people collecting the data would get the same numbers. Questions about *validity* ask if the methods used to collect the data are sound. Addressing all the questions that people have is beyond the scope of this book; therefore, the discussion of the data and graphs will focus on the questions that seem most

important. Another reason for emphasizing questions is to encourage you to develop a critical mind when examining any data or graph.³

INVESTIGATION 7.1 Videocassette Recorders

Although videocassette recorders (VCRs) are almost everywhere now, they are a relatively recent invention. In this first investigation, we will examine two commonly used graphs and some of the problems involved in making them. Take a minute to examine the data in Table 7.1. First, describe the growth of VCRs, as mathematically as possible, as though you were talking to someone on the phone. Please be more precise than “Wow, they sure got popular fast.” Second, describe any questions you have about the data. Then read on. . . . 

TABLE 7.1

Year	U.S. households with VCRs ('000s)
1978	200
1979	400
1980	840
1981	1440
1982	2530
1983	4580
1984	8880
1985	17,600
1986	30,920
1987	42,560
1988	51,930
1989	58,400
1990	66,940

Source: *The Universal Almanac* 1992, John W. Wright, general editor. (Andrews and McMeel [A Universal Press Syndicate Company], Kansas City), p. 253. Excerpt from *The Universal Almanac* © 1992 by John W. Wright. Reprinted with permission of Andrews McMeel Publishing. All Rights Reserved.

DISCUSSION

At the most basic level, Table 7.1 shows that the number of U.S. households with VCRs increased every year. At a slightly more sophisticated level, using our knowledge of multiplication and estimation, we can say that the number of households with VCRs just about doubled every year until 1987.

Possible questions about this table include: What does (“000s”) mean? What has happened since 1990? Because the table starts in 1978, does that mean that

³ Note that *critical* here does not refer to criticizing, but rather to being critical in the sense of “characterized by careful, exact evaluation and judgment” [*The American Heritage College Dictionary*, 3d ed. (Boston: Houghton Mifflin, 1996), p. 443]. Reproduced by permission. Copyright © 1996 by Houghton Mifflin Company. Reproduced by permission from *The American Heritage Dictionary of the English Language*, Third Edition.

that was the year in which VCRs were first sold? Who collected the data, and how did they get these numbers?

Many people wonder what ('000s) means. This is a convention that graph makers use when dealing with large numbers. There are two equivalent ways to "decode" this symbol: "Write three zeros after each number in the table to get the actual numbers" or "Each number is in the thousands; thus 200 means two hundred thousand." Recall our discussions about the choice of unit in Chapter 5. From this perspective, '000 means that the graph makers chose 1000 instead of 1 as the unit.

Graphing these data Now let us examine how a graph can help us to see the data better. What kind of graph do you think might best describe these data?⁴ Make your own graph with graph paper first, before reading on. . . . 

Look at the two graphs in Figure 7.1 and address the following questions: What do they tell us about the growth of VCRs? Are both graphs "correct," or is one "better" than the other? Summarize the pros and cons of each graph. Then read on. . . . 

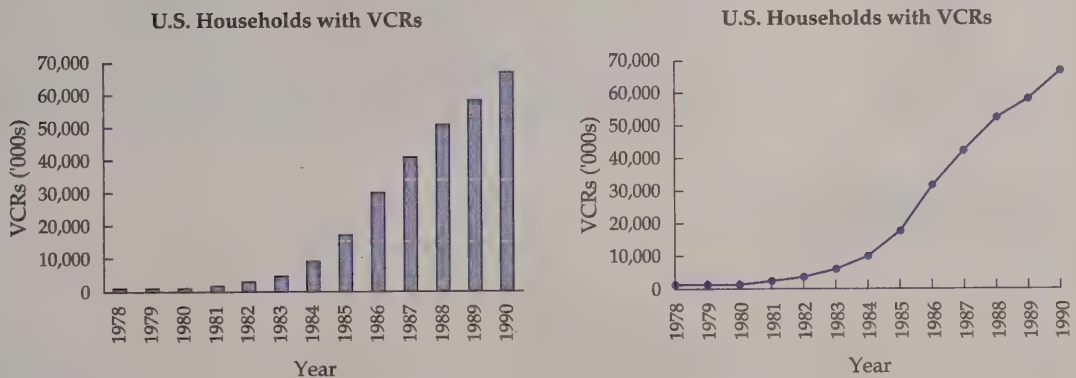


FIGURE 7.1

Both of these graphs are valid ways to represent these data. Many people prefer **bar graphs** to **line graphs**, finding the former easier to understand. The primary advantage of the line graph has to do with slope. You may recall from algebra that in a linear equation, the slope is constant, so a straight line indicates constant growth. However, when the slope keeps increasing, that indicates that the rate of growth is increasing, and we refer to such growth as exponential. The line graph more clearly shows that the rate of increase started to slow down in 1986.

Finally, let us recall the discussion at the beginning of this chapter about uses of data. Other than satisfying curiosity, what use might these data and these graphs serve? Think and read on. . . . 

An individual deciding to invest in the stock market would probably be less interested in an area whose growth was slowing down rather than picking up. Someone doing a report on the growth of the home entertainment industry might want to see how and when the VCR market grew.

⁴ The word *data* is the plural of the word *datum*. Therefore, it is correct to say that "data are used" and "the data support the following conclusion."

INVESTIGATION 7.2 World Population

Graphs can also serve an educational purpose by helping us to “see” numbers better. For example, most people know that more people live in Asia than in any other region of the world. Let us examine how graphs can help us to understand world population in a new light. Take a minute to examine the data in Table 7.2. As before, first describe what the data tell you and then describe any questions you have about the data. Then read on. . . .



TABLE 7.2

World population, by region (1990)

Africa	647,518,000
Asia	3,108,475,000
North America	275,880,000
Latin America	448,096,000
Europe	497,740,000
Former USSR	287,990,000
Oceania	26,476,000

Source: *Universal Almanac* 1992, pp. 317–318

DISCUSSION

There are many conclusions about population in 1990 that we can draw from these data. Two are that the population of Asia was much greater than that of any other region and that the population of Africa was more than double the population of North America.

There are many possible questions that people can ask about these data: Why did the people who made this table divide the world into these regions? What do they mean by Latin America (for example, does this include Central America and the Caribbean)? How did they get these numbers? How accurate are the numbers?

Changing units Many people report being overwhelmed by the size of numbers as those in Table 7.2. People who make tables take this into account by changing the unit used to report the figures. For example, look at Table 7.3, which gives virtually the same data as Table 7.2. Can you see more from this table?

TABLE 7.3

Region	Population (millions)
Africa	648
Asia	3108
North America	276
Latin America	448
Europe	498
Former USSR	288
Oceania	26

Recalling our discussions of unit in Chapter 5, we can say that the second table uses 1 million persons as the unit instead of 1 person. Many people



BEYOND THE CLASSROOM

Many of my students tell me that one of the more important lessons from the statistics unit is that the numbers they read in data tables (in newspapers, books, and so on) are not absolute. That is, they realize that many of these numbers either are approximations or are obtained from surveys. For example, the world population figures in Table 7.3 were derived from one almanac. Below are the figures from another almanac (the *1994 Information Please Almanac*, p. 133). How “close” are the two sets of numbers?

Region	Millions
Africa	661
Asia	3116
North America	278
Latin America	447
Europe	501
Former USSR	291
Oceania	27

find that relative comparisons are easier to make when the unit is larger. For example, many people can more easily see from Table 7.3 that the population of Africa was almost 400 million more than that of North America, or that there were more than twice as many people in Africa as in North America.

Graphing these data Let us now consider how we might display these data. Make your own graph(s) first, before reading on. . . 

Now look at the two graphs in Figure 7.2. What do they tell us about the population of different regions? Are both graphs “correct,” or is one “better” than the other? Summarize the pros and cons of each graph. Does your graph resemble one of these, or did you do something different?

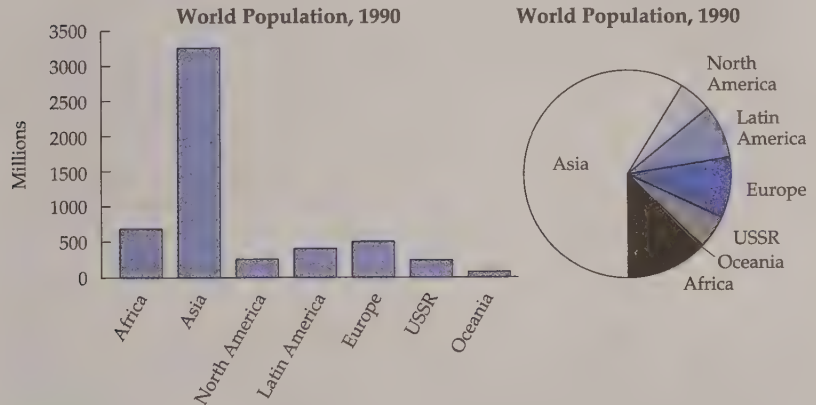


FIGURE 7.2

In this case, a line graph would not be appropriate. In general, a line graph is used when we want to look at changes in data over time. However, a bar graph is still appropriate and is commonly used when our independent variable consists of categories rather than numbers. In this case, a **circle graph** is also appropriate. Let us first compare the pros and cons of bar and circle graphs and then discuss the construction of circle graphs.

The main advantage of the bar graph is that we don't lose any data; for example, from the bar graph, we can see that the population of Asia is more than 3 billion, whereas in the circle graph, we lose the numbers. The main advantage of the circle graph has to do with fractions and parts of wholes. That is, we can quickly estimate what fraction of the entire world's people come from each region (for example, we can see that over half of the world's people live in Asia), whereas the relative proportions have to be calculated from the bar graph.

Constructing a circle graph Before reading on, first make a (rough) circle graph for these data freehand—that is, without a protractor. Then make a precise circle graph with a protractor. Then read on. . . 

In order to make a circle graph, we have to know what proportion (or percent) of the whole is represented by each group. To do that, we must first find the entire population and then determine percentages, as in Table 7.4. As always, I strongly suggest that you read this section actively—that you do the computations (either on a calculator or on a spreadsheet) and make a circle graph on your own paper.

Once we know the percent for each region, we can construct a rough circle graph. If you haven't done so yet, do so now before reading on. Note your reasoning, for there is no single "right" way. 

TABLE 7.4

	Population (millions)	Percent	Degrees
Africa	648	12%	44°
Asia	3108	59%	211°
North America	276	5%	19°
Latin America	448	8%	30°
Europe	498	9%	34°
Former USSR	288	5%	20°
Oceania	26	1%	2°
Total	5292	99%	360°

Because the percentages have been rounded to the nearest whole number, the sum here is 99% instead of 100%.

To make a rough circle graph, we make use of proportional reasoning; that is, we look at the relative amounts. For example, if we recall that 25% is equivalent to $1/4$, we can see that Africa will be about $1/2$ of a quarter circle. Similarly, Latin America and Europe will each be about $1/3$ of a quarter circle (see Figure 7.3). In one sense, we are using the quarter circle as a unit of measurement. Next, we see that the population of North America and the former USSR are both slightly over $1/2$ the population of Europe.

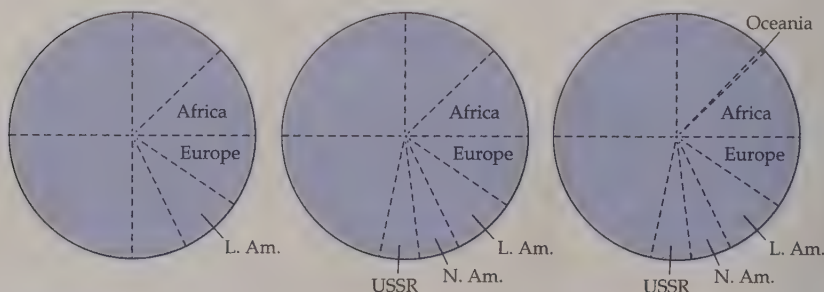


FIGURE 7.3

The remainder of the circle should be Asia. If our work is accurate, this should represent 59% of the circle. How might we check our work at this point?

We have one complete quarter-circle (25%) and one almost complete quarter-circle (23% or 24%), and we have just under half of another quarter-circle (12%). The sum of these three numbers is 60, which is pretty close to 59.

In order to make an exact circle graph, we need to know how many degrees each wedge will be. How do we determine this? Work on this yourself before reading on. . . . 

There are several ways to do this. We can make and solve a proportion. That is, if Africa is 12% of the total population, then 12 out of 100 is equal to x out of 360. Alternatively, we can see that we need to calculate 12% of 360. To be even more accurate, we would use the original population figures:

$$\frac{648}{5292} \times 360$$

In either case, we find that the Africa slice of the circle will be about 44° . We can now use a protractor to determine the size of each slice.

The placement of each slice in a circle graph is a matter of preference. Some people start by drawing a horizontal line to the right of the center of the circle, as I did in Figure 7.3. However, some people start by drawing a vertical line above the center of the circle. Some people prefer to order the slices from largest to smallest, whereas other people prefer to arrange them by geographical proximity—for example, placing North America and Latin America next to each other.

As I mentioned earlier, one of the goals of this book is to reawaken the curiosity about numbers that children naturally have. Think about these data and these graphs on world population. What kinds of questions do you have—questions asking for more information so that you can better interpret this information, or questions asking for more information so that you can better understand the graph, or questions about other aspects of world population? Note your questions before reading on. . . . 

Here are a few I have received: “I wonder if the slices for the year 2000 will be any different?” “I wonder where they put Mexico—in North America or Latin America?” “Where is the dividing line between Europe and Asia?”

INVESTIGATION 7.3 AIDS

Thus far, we have examined three fundamental kinds of graphs: line graphs, bar graphs, and circle graphs. In the next three investigations, we will examine some of the issues involved in the construction and interpretation of graphs. You may have encountered some of these issues yourself in Explorations 7.1 and 7.2.

Take a minute to think about what you know about the AIDS epidemic. Then look at the data in Table 7.5. Describe your initial conclusions from the data. In other words, what do the data tell us about the epidemic? Think and then read on. . . . 

MATHEMATICS

I attempted to update the data in Table 7.5 for the second edition of the textbook. At that time, I found that the almanac from which I had originally drawn the data in Table 7.5 did not carry an update of that table. In fact, I could not find data for the four columns in Table 7.5 in any one place. Furthermore, when I did find data—i.e. “Known Deaths to Date”—different sources gave different numbers. This bears directly on one of the themes of this chapter: uncertainty. In many cases there is no such thing as “exact” data for certain topics. You may choose to do a web search to see recent data yourself.

TABLE 7.5 AIDS in the United States

	Cases diagnosed this year	Cases diagnosed to date	Known deaths this year	Known deaths to date
1981	392	392	163	163
1982	1110	1502	449	612
1983	2974	4476	1465	2077
1984	5994	10,470	3343	5420
1985	11,263	21,733	6623	12,043
1986	18,277	40,010	11,356	23,399
1987	27,172	67,182	15,279	38,678
1988	32,669	99,851	19,602	58,280
1989	36,110	135,961	24,761	83,041
1990	30,704	166,665	22,236	105,277

DISCUSSION

From the first column, we see that the number of new cases diagnosed each year increased every year until 1990. We see that in the earlier years, the number of

new cases rose *substantially*. For example, there were almost three times as many cases diagnosed in 1982 as in 1981, and more than twice as many in 1983 as in 1982. We are looking at exponential growth rather than linear growth. When a disease is characterized by exponential growth, we use the word *epidemic*.

Graphing these data Why might we want to make a graph for these data? Think and then read on. . . . 

The graph can help us to visualize the data. For example, if we graph the “Cases diagnosed to date,” we can see the growth of the epidemic. It will be helpful if you make your own line graph for these data before reading on. . . . 

As you made the graph, what difficulties (if any) did you encounter? How did you solve them?

Figure 7.4 shows a line graph for the “Cases diagnosed to date” data. Although a bar graph would not be “wrong,” as discussed before, a line graph enables us to better see the change in the data over time, and so it is preferable for these data. One of the problems that many students struggle with is to determine the scale for these data—that is, our unit of measurement for this graph. If you are plotting these data by hand, and you want one sheet of graph paper to contain the data, each vertical interval on a sheet of typical graph paper will need to stand for about 5000 or 10,000 cases. In Figure 7.4, the scale (unit) is 5000 cases for each vertical line.

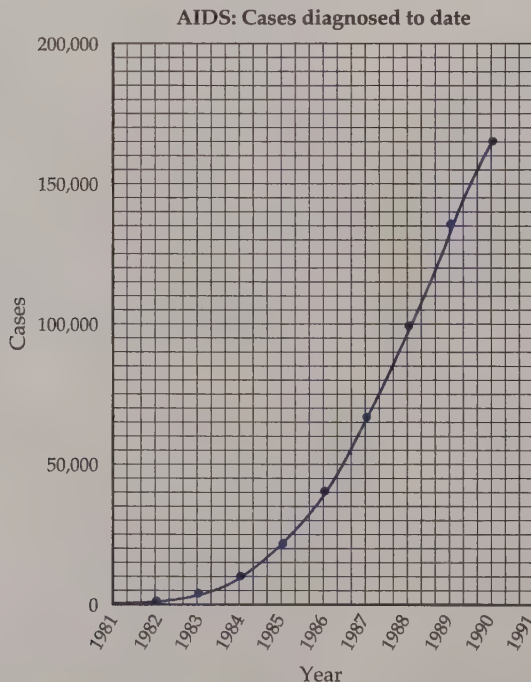


FIGURE 7.4

Suppose a classmate who was having difficulty asked you, “How do you know exactly where to put 21,733 (the 1985 figure)?” How would you explain how you made that decision? 

Once again proportional reasoning is useful. We can see that 21,733 is between 20,000 and 25,000. More specifically, we are concerned with what part 1733 is of the whole (5000). Basic estimation tells us that this point is about $1/3$ of the way from 20,000 to 25,000.

A common graphing mistake There are other issues related to scale that need to be addressed. One major mistake that people commonly make is that they choose one unit for part of the graph and then choose another unit for another part of the graph. For example, it is not uncommon for students to make graphs like the one in Figure 7.5. Do you see what the students did? Why is it wrong?

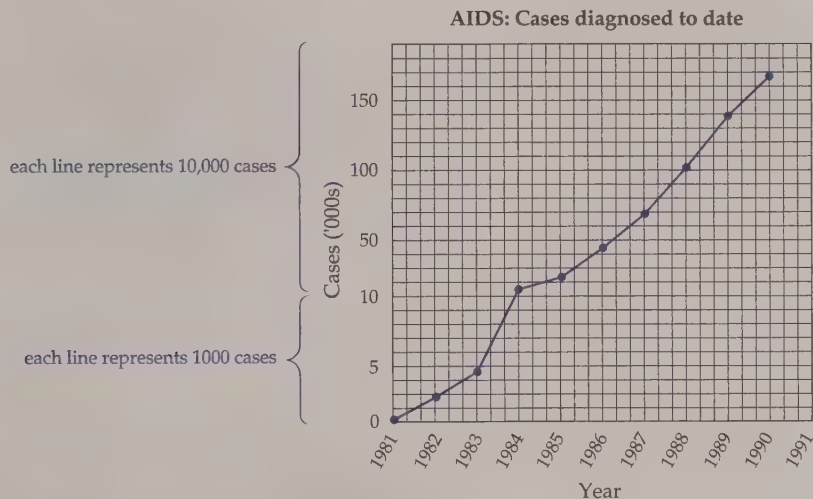


FIGURE 7.5

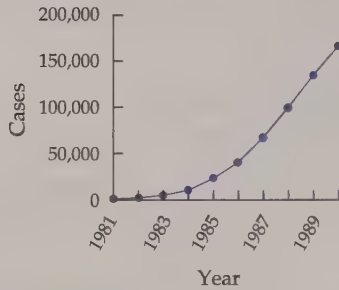
What the students did was to have two different y-axis units on the y-axis. That is, for the first ten vertical intervals, the unit is 1000 cases; thereafter, the unit is 10,000 cases. The rationale for the different scale is that by doing this, they can place the smaller numbers more accurately. However, changing the unit (or scale) on a graph is not acceptable. Why do you think this is so?

It changes the appearance of the graph dramatically. For example, Figure 7.5 implies that the rate of the epidemic slowed down after 1985, when in fact that was not true.

On the other hand, there is no single “right” answer to what is the “best” unit—for example, 5000 or 10,000 or 20,000 or 25,000. In general, the smaller the unit, the better we can see trends in the data. However, the smaller the unit, the bigger the graph.

Horizontal spacing This issue of scale and unit also applies to the horizontal spacing. For example, one can choose to have more or less horizontal space between the years, as long as the spacing is constant—that is, as long as each year is the same distance from the previous year. Although there are no rights and wrongs with respect to these decisions, the choice of units *will* alter the graph’s appearance. For example, the two graphs in Figure 7.6 represent the same data, and both use the same vertical scale. The difference is that in the graph at the left, the years are closer together. Do you see, though, why both graphs are considered to be correct?

AIDS: Cases diagnosed to date in U.S.



AIDS: Cases diagnosed to date in U.S.

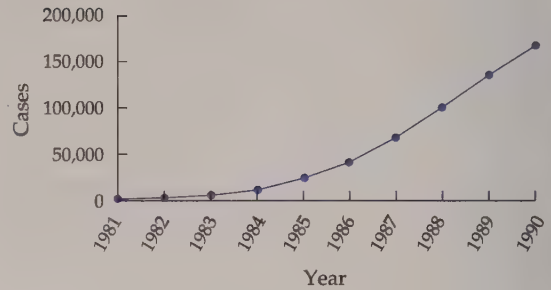


FIGURE 7.6

Although the two graphs appear to be very different, they are mathematically equivalent. For example, the number of cases in 1985 was double the number of cases in 1984. In both graphs, the point representing 1985 is twice as high as the point representing 1984.

Think about the AIDS epidemic, and record any further questions you have about the data we have examined or other questions about AIDS on which there might be data. Note your questions before reading on. . . . 

Here are a few I have received: "I wonder if the graph is still climbing as fast now." "I wonder what the numbers are for different groups of people, such as males and females, people of different ages, etc." "I wonder how accurate these numbers really are."

INVESTIGATION 7.4 Life Expectancy Over the Past 70 Years

We frequently use graphs not simply to "see" or analyze one set of data, but to compare two sets of data. Table 7.6 shows average life expectancy in the United States since 1920. What are your first impressions from examining the data? What questions do you have about the data? Think and then read on. . . . 

TABLE 7.6 Life Expectancy in the United States

	Males	Females
1920	53.6	54.6
1930	58.1	61.6
1940	60.8	65.2
1950	65.6	71.1
1960	66.6	73.1
1970	67.1	74.7
1980	70.0	77.4
1990	71.5	78.3
2000	73.0	79.7

DISCUSSION

A quick look shows that the life expectancy for both males and females has increased every decade and that during this time period, the female life expectancy has always been greater than the male life expectancy.

Before graphing these data and examining what a graph can show us, let us stop and think about what the data actually mean. For example, what does 73.0 years (the 2000 number for males) actually mean?

These numbers are actually **predictions** (which represent **probabilities**). The 73.0 means that the authors of the table are *predicting* that if we were to take all the males born in the United States in 2000 and go forward in time to a point when they had all died, the “average” lifetime of these males born in 2000 would be 73.0 years. Some would have lived fewer years—for example, died in childhood—and some would have lived more years, but the average lifetime of all these males would have been 73.0 years. That is, the authors of the table are predicting that 73.0 years is the most probable lifetime.

Graphing these data From the data, we have a general sense that the life expectancies are getting longer, but a graph would give us a visual sense of the data. What kind of graph do you think would best help us to see both the male and female life expectancies over this time period? Think and then read on. . . . 

Although a bar graph would not be wrong, a line graph enables us to see changes over time more clearly. What do you “see” from the graph in Figure 7.7? Describe your reactions before reading on. . . . 

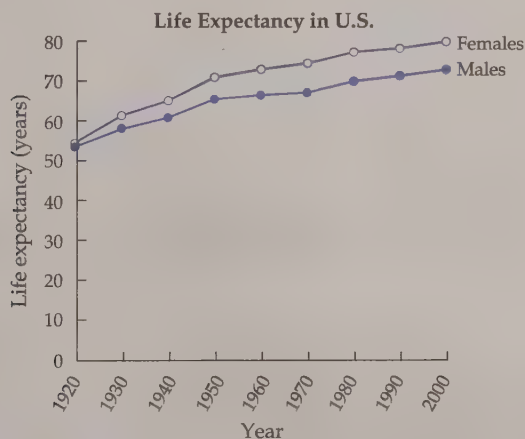


FIGURE 7.7

We can see that the life expectancies of both males and females were increasing at a faster rate between 1920 and 1950 than they were between 1950 and 2000. However, the data are all crammed into the top part of the graph. To better see trends in the data, we can “zoom” in on the data. In Figure 7.8, the vertical scale begins at 50 years, because all of the data are above 50. What additional conclusions can you draw from this graph?

We can see that in some decades, the life expectancy of women grew faster than that of men (that is, the line is steeper). We can see that since 1970, the two graphs have been more nearly parallel than before, indicating similar growth rates. From this graph, we can also more accurately estimate the actual numbers. For example, if you did not have the data but only the graph and wanted to estimate the life expectancies in 2010, you could make a more accurate estimate from Figure 7.8 than from Figure 7.7.

What further questions do you have about life expectancy?

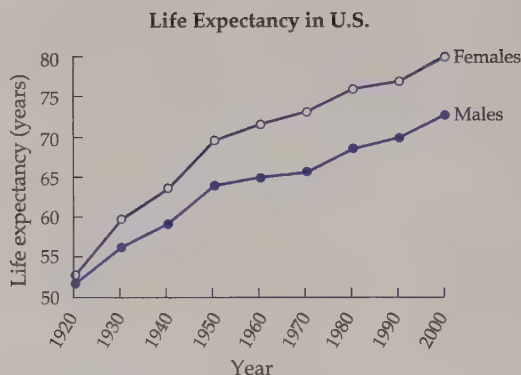


FIGURE 7.8

Interpreting Graphs

In everyday life, we generally don't collect data and graph data as much as we interpret other people's graphs of data that they or yet *other* people have collected. The ability to interpret and to critique graphs is important. As we will find, graphs often distort the data. Sometimes this is intentional on the part of the people making the graph; other times it is unintentional, the result of carelessness or ignorance.

In Explorations 7.1 and 7.2, we critically examined data collected by other people and then constructed graphs. In the next three investigations, we will critically examine graphs made by other people. As you read each graph, I encourage you to think about three kinds of questions before you read the discussion. As always, if you write down your responses, you are likely to retain more from your work.

Conclusions

What conclusion(s) can I draw from the graph?

Do the conclusion(s) that I read seem reasonable?

Construction of the graph

Are the scales and the units clear or are they misleading?

Would another graph be more appropriate? Why or why not?

Reliability/validity

Do I have questions about how the data were obtained that could affect the accuracy of the data?

Further questions

Questions to help you better interpret the data and graph

Questions to help you better understand the data and graph

Questions that this data set and graph provoke in you

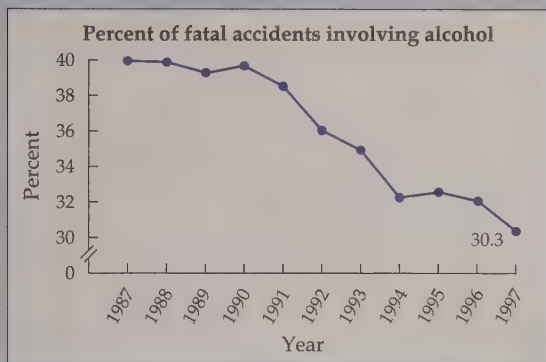
Let us begin with a graph that indicates hopeful news.

INVESTIGATION 7.5 Fatal Crashes

Examine the graph in Figure 7.9 and answer each of the four kinds of questions before reading on. . . . 

BEYOND THE CLASSROOM

Interestingly, in the first edition of this textbook, I had data from a different source. The caption on the graph was "Percent of fatal accidents involving drunken drivers," and the data were from 1982 through 1990. When writing the second edition of the textbook, I could not find an updated set of these data, which had come from an article in *USA Today*. I searched the Web to see whether the Centers for Disease Control had an updated version of these data. No luck. Finally, I got the current data from the National Highway Traffic Safety Administration. However, their heading was "Percentage of traffic fatalities" and they had three columns: one in which no one involved in the crash had any alcohol in their bloodstream, one in which any person in the crash had a blood alcohol concentration ≥ 0 , and the third column (which I used) in which the highest blood alcohol concentration of any person in the crash was ≥ 0.10 . In the discussion of these data, we are presuming that in virtually all of these cases, at least one of the people was drunk.



Source: NHTSA Fatality Analysis Reporting System (FARS) 1998

FIGURE 7.9

DISCUSSION

One student wrote the following: "The percent of fatal car crashes in which the driver was drunk fell dramatically between 1987 and 1997." What do you think of her summary? Think and then read on. . . .

Actually, there are several problems with the student's summary. First, the data don't seem to be restricted to car crashes. What kinds of "traffic fatalities" do you think count in these data? Think and then read on. . . .

Other kinds of traffic fatalities include crashes between two motor vehicles (trucks, cars, motorcycles) in which one or both drivers were drunk, single-motor-vehicle accidents in which the driver was drunk, and possibly accidents in which a motor vehicle hit a pedestrian or a person on a bicycle.

A second problem with the student's summary has to do with what "drunk" means. What do the graph makers mean by "drunk"? How did the people who recorded the data know that a driver was drunk? How were the data gathered? Think and then read on. . . .

It seems reasonable to expect that there are data for every motor vehicle accident in which there was a fatality. However, how did the people who recorded the data determine the number of such accidents in which at least one driver was drunk? Did a sobriety test or a blood test show that the person was drunk? Furthermore, the definition of *drunk* varies from state to state: In some states, a person with a blood alcohol level of 0.08% is considered drunk, whereas in other states the blood alcohol level has to be 0.10%.

Now let us examine the student's use of the word *dramatically* to describe the change in fatalities. Look back at the graph—what does the jagged line just below 30% mean? Think and then read on. . . .

It means that this is a **truncated graph**—that is, the authors of this graph deleted the 0%–30% interval. In the previous investigation, we truncated the life expectancy graph so that we could zoom in on the actual data. Why else might someone want to truncate a graph? Think and then read on. . . .

Sometimes graphs are truncated to save space. However, sometimes they are truncated to distort the data. Let us see how the graph would look if it had not been truncated (see Figure 7.10). How does the decline in the percentage of drunk drivers in fatal accidents look now? Think and then read on. . . .

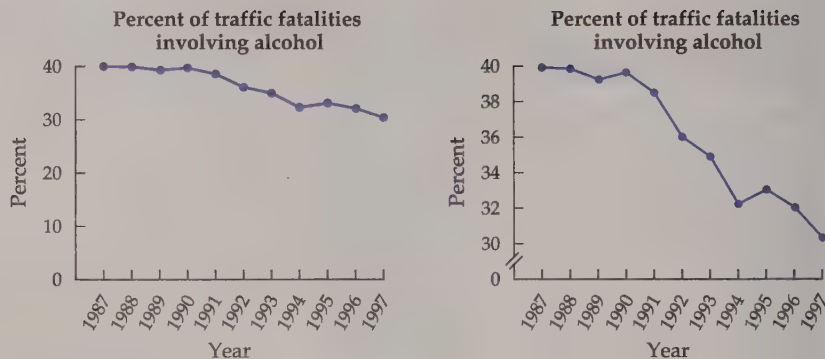


FIGURE 7.10

As you can see, the decline does not seem so great in the untruncated graph. The actual percent decrease from 39.9 to 30.3 is about a 24% decrease (that is, $\frac{39.9 - 30.3}{39.9} = 0.24 = 24\%$).

Which graph would you have picked if you were working for a beer company preparing an advertisement showing that drunk driving is on the decline? What if you were a member of SADD (Students Against Drunk Driving)?

What do the numbers mean? Let us examine two more aspects of this graph. In 1997, the percentage of traffic fatalities involving drunk drivers was about 30%. Write a sentence describing what that 30 means, as though you were talking to someone who didn't understand. Then read on. . . 

Below are two of many possible responses.

- In 1997, 30% of all fatal motor vehicle accidents involved a drunk driver.
- In 1997, of every 100 fatal motor vehicle accidents, 30 involved a drunk driver.

Pseudo-precision Another aspect of the graph worth examining is the 1997 figure of 30.3%. This figure looks very precise—that is, not 30.4% or 30.2%, but 30.3%. Are the actual data really that precise? Think and then read on. . . 

Let's say there were 254 fatal accidents in a state and that state reported that drunken drivers were involved in 77 of them. That state could report that 30.3% of the fatal accidents involved drunk drivers. Given the discussion above, how much room for error do you think there is? Think and then read on. . . 

Let's say we determine that this number might be off by as much as 5. That is, it is possible that the actual number of accidents involving a drunk driver was as low as 72 or as high as 82. What would the percentages have been in those cases? Think and then read on. . . 

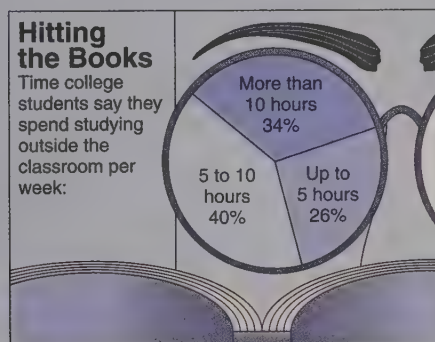
If the actual number had been 72, the percentage would have been 28.3%; if the actual number had been 82, the percentage would have been 32.3%. Given the possible sources of error in the data—definition of *drunk*, definition of *fatal* (does it mean that an injured person dies within 24 hours, 1 week, 1 month?), and likely inconsistency among different states—the 30.3% is another example of pseudo-precision.

Further questions Finally, let us examine additional questions that we might ask. What do you predict has happened since 1997? Do you think the percent-

age has leveled off or has continued to decline? Where could you go to find out? What other data would you like to see?

INVESTIGATION 7.6 Hitting the Books

Take a few moments to examine Figure 7.11. As in Investigation 7.5 and Explorations 7.1 and 7.2, a critical aspect of this examination is asking yourself questions about conclusions, construction of the graph, and reliability/validity. Write your responses to these questions before reading on. . . .



Source: "Hitting the Books," from *USA Today*. Copyright 1992, *USA Today*. Reprinted with permission.

FIGURE 7.11

DISCUSSION

Describing the graph Critique the following two statements, which represent conclusions that people have made from this graph.

- Barely one-third of college students study more than 10 hours per week.
- Most college students average less than 2 hours a day on homework.

The first conclusion is simply taken straight from the graph: "Barely $1/3$ " is consistent with 34%. The second statement represents a valid interpretation of the graph, because less than half of the students spend more than 10 hours a week.

What do the data mean? Now let us look beyond the first impressions to examine what the data mean. If you could meet with the people who collected these data, what questions would you ask? Think and then read on. . . .

Below are three of many possible questions that we might ask. As you read these questions and consider your own questions, ask yourself whether the answers to these questions would make a difference. Then read on. . . .

- How did you get the data (take a survey outside the dining commons, have questionnaires filled out in some classes)?
- What kind of students are included (full-time, a "representative" sample)?

- What did you ask? For example, did you ask, "About how many hours did you study last week?" or, "Last week, did you spend less than 5 hours studying, between 5 and 10 hours, or more than 10 hours?"

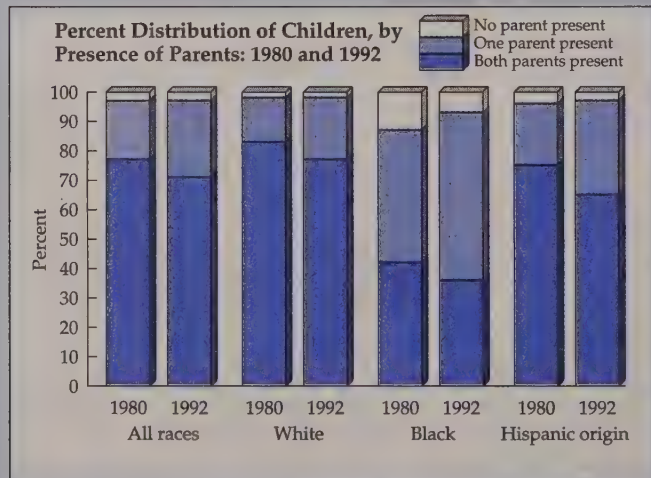
As you may have seen in Exploration 7.3 (Typical Person), the way in which we gather data can make a big difference in how the graph appears. For example, if the questions were asked of students coming out of the dining commons, that means that off-campus students were not asked. And if off-campus students tend to study a lot more or less than on-campus students, this could change the data quite a bit. Similarly, if the data included both part-time and full-time students, the number of hours reported would naturally go down. (Why?) Finally, how we ask the question has an influence on our data. You might want to replicate this study on your own campus and compare the results of asking the question in two different ways.

Choice of graph The makers of this graph chose a circle graph. What other choices would be appropriate, or is the circle graph the "best" choice? What do you think? . . . 

Here it is a matter of personal preference. Circle graphs work well when we are looking at parts of wholes. In this case, the whole represents all students, and the makers of the graph have divided the whole into three subsets. However, a bar graph would not be wrong.

INVESTIGATION 7.7 Presence of Parents

Figure 7.12 presents some very sobering data about the status of American families. However, it is not a simple graph. Take a minute to look at it. Do you understand what it means? If not, focus on just one bar—for example, 1992 "All races"—and think about what you know about percent. Then read on. . . . 



Source: The American Almanac: Statistical Abstract of the United States 1993–94

FIGURE 7.12

DISCUSSION

Let us look at the 1992 “All races” bar to make sure you are interpreting the graph correctly. According to the data, in 1992, about 70% of all children lived with both parents, about 26% or 27% lived with one parent, and about 3% or 4% lived with neither of their parents.

Why did the authors choose this kind of graph, which is called a stacked bar graph? Think and then read on. . . . 

Because all the data are in percentages, the total heights of the bars in all columns are equal. This makes it much easier to compare (among white, black, and Hispanic) the percentages of children that live with both parents; in other words, we can make proportional (multiplicative) comparisons. For example, we can say that a black child in 1992 is more than twice as likely to live with neither parent than is a white child, because the “No parent present” segment of the bar for blacks is more than twice as long as the “No parent present” segment for whites.

After examining these bar graphs, one of my students suddenly stopped and said, “Wait a minute, the stacked bar graph combines the advantages of a bar graph and a circle graph.” Do you see that? If not, go back and look at the earlier discussion of the advantages of bar and circle graphs. Do you see how this kind of graph retains the advantages of bar graphs and incorporates the advantages of circle graphs?

Summary

Record your thoughts about issues that arise in describing data and interpreting graphs. Then read on. . . . 

We have examined three basic kinds of graphs: line graphs, bar graphs, and circle graphs.

We have learned that in some cases, which graph is made is a matter of preference and that each graph has certain characteristics that we might want to use:

- Line graphs enable us to see change over time.
- Bar graphs enable us to compare the quantity of each category.
- Circle graphs enable us to see what part each category is of the whole.

We have learned that we need to take care that our graph does not distort the data. For example, we need to be careful when choosing the units for our spacing, both horizontally and vertically, in line graphs and bar graphs.

We have also learned that when interpreting others’ graphs, we need to think about what the data mean and how the data were gathered, and we need to realize that some graphs are distorted, either intentionally or out of ignorance or carelessness. At the same time, we have learned that although a truncated graph does indeed distort the data, it sometimes offers advantages. We have learned that whether we are making graphs from data collected by other people or interpreting graphs made by other people, it is important to examine the data and the graphs with respect to fairness, appropriateness, reasonableness, reliability, and validity. “Students should begin to develop a critical attitude toward information presented in the media and learn to ask relevant questions before making judgments based on that information.”⁵

My hope is that the graphs and discussions in this section have inspired many of you to investigate some of these topics further on your own.

⁵ Center for Statistical Education and the American Statistical Association, *Teaching Statistics: Guidelines for Elementary Through High School* (Palo Alto: Ca. Dale Seymour Publications), p. 23.

EXERCISES 7.1

1. a. Make two line graphs for the data shown in the table below. Make the first graph with the "Total graduates" data and the second graph with the "Graduates per 100 17-year-olds" data.
- b. What did you learn or see from each graph that you didn't see just from the data?
- c. Write for each graph a 2 to 3-sentence summary that might appear in a newspaper article describing the increase in numbers of people earning a high school diploma.
- d. How is it possible that the first graph increases between 1969–70 and 1979–80, whereas the second graph decreases here?

HIGH SCHOOL GRADUATION BECOMES MORE COMMON

School year	Total graduates	Graduates per 100 17-year-olds	School year	Total graduates	Graduates per 100 17-year-olds
1899–1900	95,000	6.4	1949–50	1,200,000	59.0
1909–10	156,000	8.8	1959–60	1,858,000	69.5
1919–20	311,000	16.8	1969–70	2,889,000	76.9
1929–30	667,000	29.0	1979–80	3,043,000	71.4
1939–40	1,221,000	50.8	1989–90	2,587,000	74.2

Source: Data from *The World Almanac and Book of Facts*, Primedia Reference Inc., © 1999, p. 242.

2. a. Make a line graph for the data shown in the table below.
- b. What did you learn or see from the graph that you didn't see just from the data?
- c. Write a 2 to 3-sentence summary for this graph that might appear in a newspaper article describing the changes in production of motor vehicles.
- d. Describe how the world production of motor vehicles changed during this time period, beyond saying simply that it has increased.

World Motor Vehicle Production, 1950-97

Source: American Automobile Manufacturers Assn.
(in thousands)

Year	United States	Canada	Europe	Japan	Other	World total	U.S. % of World total
1997	12,119	2,571	17,773	10,975	10,024	53,463	22.7
1996	11,799	2,397	17,550	10,346	9,241	51,332	23.0
1995	11,985	2,408	17,045	10,196	8,349	49,983	24.0
1994	12,263	2,321	16,195	10,554	8,167	49,500	24.8
1993	10,898	2,246	15,208	11,228	7,205	46,785	23.3
1992	9,729	1,961	17,628	12,499	6,269	48,088	20.2
1991	8,811	1,888	17,804	13,245	5,180	46,928	18.8
1990	9,783	1,928	18,866	13,487	4,496	48,554	20.1
1985	11,653	1,933	16,113	12,271	2,939	44,909	25.9
1980	8,010	1,324	15,496	11,043	2,692	38,565	20.8
1970	8,284	1,160	13,049	5,289	1,637	29,419	28.2
1960	7,905	398	6,837	482	866	16,488	47.9
1950	8,006	388	1,991	32	160	10,577	75.7

Note: As far as can be determined, production refers to vehicles locally manufactured.

Source: *World Almanac*, 1999, p. 708. Reprinted with permission from *The World Almanac and Book of Facts* 1999. Copyright © 1998 World Almanac Education Group. All Rights Reserved.

3. a. Make a line graph for the data shown in the table at the right.
- b. What did you learn or see from the graph that you didn't see just from the data?
- c. Write a 2 to 3-sentence summary for this graph that might appear in a newspaper article describing the increasing numbers of women in these four professions.

No. 327. First Professional Degrees Earned in Selected Professions: 1970 to 1995

(First professional degrees include degrees which require at least 6 years of college work for completion (including at least 2 years of preprofessional training).)

TYPE OF DEGREE AND SEX OF RECIPIENT	1970	1975	1980	1985	1990	1991	1992	1993	1994	1995
Medicine (M.D.):										
Degrees conferred, total	8,314	12,447	14,902	16,041	15,075	15,043	15,243	15,531	15,368	15,537
Percent to women	8.4	13.1	23.4	30.4	34.2	36.0	35.7	37.7	37.9	38.8
Dentistry (D.D.S. or D.M.D.):										
Degrees conferred, total	3,718	4,773	5,258	5,339	4,100	3,699	3,593	3,605	3,787	3,897
Percent to women	0.9	3.1	13.3	20.7	30.9	32.1	32.3	33.9	38.5	36.4
Law (LL.B. or J.D.):										
Degrees conferred, total	14,916	29,296	35,647	37,491	36,485	37,945	38,848	40,302	40,044	39,349
Percent to women	5.4	15.1	30.2	38.5	42.2	43.0	42.7	42.5	43.0	42.6
Theological (B.D., M.Div., M.H.L.):										
Degrees conferred, total	5,298	5,095	7,115	7,221	5,851	5,695	5,251	5,447	5,967	5,978
Percent to women	2.3	6.8	13.8	18.5	24.8	23.4	23.3	24.8	24.8	25.7

Source: U.S. National Center for Education Statistics, *Digest of Education Statistics*, annual, *Statistical Abstract of the U.S.*, 1998, p.202.

4. a. Make a line graph for the data shown in the table below.
- b. What did you learn or see from the graph that you didn't see just from the data?
- c. Write a 2 to 3-sentence summary for this graph that might appear in a newspaper article on the growing numbers of mothers in the workplace.

Percent of married women with children under six who work outside the home

1960	18.6
1970	30.3
1980	45.1
1990	58.9
1996	62.7

Source: US. Bureau of Labor Statistics.

5. Look at the Participants data in the table below.
 - a. Make a line graph for these data.
 - b. What did you learn or see from the graph that you didn't see just from the data?
 - c. Write a 2 to 3-sentence summary for this graph that might appear in a newspaper article describing the participation in high school sports.
6. Look at the Most Popular Sports data in the table below.
 - a. Make two graphs, one for males and one for females.
 - b. Justify your choice of graphs.
 - c. Write a 2 to 3-sentence summary for these graphs that might appear in a newspaper article describing the participation in high school sports.

No. 441. Participation in High School Athletic Programs: 1971 to 1997

[Data based on number of state associations reporting and may underrepresent the number of schools with and participants in athletic programs]

YEAR	PARTICIPANTS ¹		SEX AND SPORT	MOST POPULAR SPORTS 1996-97 ²	
	Males	Females		Schools	Participants
1971	3,666,917	294,105	MALE		
1972-73	3,770,621	817,073			
1973-74	4,070,125	1,300,169			
1975-76	4,109,021	1,645,039			
1977-78	4,367,442	2,083,040			
1978-79	3,709,512	1,854,400			
1979-80	3,517,829	1,750,264			
1980-81	3,503,124	1,853,789			
1981-82	3,409,081	1,810,671			
1982-83	3,355,558	1,779,972			
1983-84	3,303,599	1,747,946			
1984-85	3,354,284	1,757,884			
1985-86	3,344,275	1,807,121	FEMALE		
1986-87	3,364,082	1,836,356			
1987-88	3,425,777	1,849,684			
1988-89	3,416,844	1,839,352			
1989-90	3,398,192	1,858,659			
1990-91	3,406,355	1,892,316			
1991-92	3,429,853	1,940,801			
1992-93	3,416,389	1,997,489			
1993-94	3,472,967	2,130,315			
1994-95	3,536,359	2,240,461			
1995-96	3,634,052	2,367,936			
1996-97	3,706,225	2,472,043			
			Basketball	16,704	544,025
			Track & field (outdoor)	14,580	468,061
			Baseball	14,212	444,248
			Football	13,119	957,507
			Golf	11,573	150,578
			Cross country	11,485	174,599
			Tennis	9,322	136,451
			Wrestling	8,798	227,596
			Soccer	8,432	296,587
			Swimming & diving	5,028	93,523
			Basketball	16,325	447,687
			Track & field (outdoor)	14,525	393,946
			Volleyball	12,988	370,957
			Softball (fast pitch)	11,895	313,607
			Soccer	6,971	226,636
			Tennis	9,228	150,346
			Cross country	10,934	145,624
			Swimming & diving	5,270	123,886
			Field hockey	1,462	56,502
			Track & field (indoor)	1,874	41,024

¹ A participant is counted in the number of sports participated in. ² Ten most popular sports for each sex in terms of number of participants.

Source: *The 1997 High School Athletics Participation Survey* (copyright), Kansas City, MO: National Federation of State High School Associations. *Statistical Abstract of the United States*, 1998, p. 268.

7. The table below shows the U.S. immigration rate by decade, from 1820 to 1990.

Period	Total number ('000s)	Rate per 1,000 U.S. pop.
1820–30	152	1.2
1831–40	599	3.9
1841–50	1,713	8.4
1851–60	2,598	9.3
1861–70	2,315	6.4
1871–80	2,812	6.2
1881–90	5,247	9.2
1891–1900	3,688	5.3
1901–10	8,795	10.4
1911–20	5,736	5.7
1921–30	4,107	3.5
1931–40	528	0.4
1941–50	1,035	0.7
1951–60	2,515	1.5
1961–70	3,322	1.7
1971–80	4,493	2.1
1981–90	7,338	2.9

Source: U.S. Immigration and Naturalization Service, 1990 *Statistical Yearbook* (1991).

8. Use the data in the table below.
- Make a graph to compare computer use in public schools in 1984–1985 and 1997–1998.
 - Justify your choice of graph.
 - What did you learn or see from the graph that you didn't see just from the data?
 - Write a 2 to 3-sentence summary for this graph that might appear in a newspaper article describing changes in computer use in schools.
 - If you could use either "Number of computers" or "Students per computer" to show the rise in the numbers of computers in schools, which would you choose? Why?

Computer Use in Schools, 1984–1985 and 1997–1998

Level	1984–1985			1997–1998		
	Total enrollment	Number of computers ¹	Students per computer ²	Total enrollment	Number of computers ¹	Students per computer ²
Public schools	39,186,000	569,825	63.5	43,769,320	7,415,007	6.3
Elementary	19,373,000	215,393	79.3	23,499,195	3,401,082	6.9
Middle/junior high	6,662,000	100,331	61.2	8,549,137	1,348,058	6.3
Senior high	11,191,000	228,726	51.5	12,287,093	2,180,520	5.6
K–12/other	1,959,000	25,375	45.8	2,433,895	485,345	5.0

1. Includes estimates for school not reporting number of computers. 2. Excludes schools with no computers.

Source: Market Data Retrieval, Shelton, CT, unpublished data (copyright). *Time Almanac*, 1999, p. 882. "Computer Use in Schools," from *Time Almanac* 1999. Reprinted by permission of Market Data Retrieval, Shelton, CT (1998).

9. The table at the right shows the 1990 population and the predicted population in 2025 (in millions) for the following regions. Make graphs to compare these two sets of figures. Justify your choice of graphs.

Region	1990	2025
Africa	648	1581
Asia	3108	4889
North America	276	333
Latin America	448	760
Europe	498	512
Oceania	26	39

Source: *Universal Almanac* 1992, pp. 317–18.

- Describe your first impressions from examining these data. Are there any surprises?
- Make two separate line graphs, one using the total number and one using the rate per 1000 population. Describe the different impressions conveyed by the two graphs. Which graph do you think gives a "better" comparison of immigration to the United States? Why?
- What did you learn about mathematics from this exercise?
- What did you learn about immigration from this exercise?

10. The National Education Association reported the highest education attainment levels for adults in the United States in 1990. Select and make an appropriate graph for the data below. Justify your choice.

Category	Percent
Eighth grade or less	10.4
Some high school, no diploma	14.4
High school diploma	30.0
Some college, no degree	18.7
Associate degree	6.2
Bachelor's degree	13.1
Graduate degree	7.2

Source: *NEA Today*, 12, No. 6 (February 1994), 32.

11. It is estimated that 36 million Americans were victims of crimes in 1990 but that 22 million of these victims did not report the crime. The following table gives the estimated numbers for the eight most common reasons people gave for not reporting a crime.

Reason for not reporting crime	Number
The crime was not successfully completed	5,676,000
Perceived lack of proof	2,354,000
Private matter	1,606,000
No ID number on lost property	1,518,000
Police would not want to be bothered	1,430,000
Too inconvenient	748,000
Police would not be able to help	594,000
Fear of reprisal	286,000

Source: Tom Heymann, *The Unofficial U.S. Census: What the U.S. Census Doesn't Tell You* (New York: Ballantine Books, 1991), p. 118.

- Select and make an appropriate graph for these data. Justify your choice.
- Write a brief summary to accompany the graph.
- How do you think the author obtained these data?
- How accurate do you think the data are?
- What other data would you like to see?

For problems 12 and 13 fill out a table like the one shown. State whether you think each type of graph would be a valid or an invalid representation of the given data. If invalid, explain why. If valid, briefly summarize the advantages/strengths and disadvantages/limitations of that type of graph for these data.

	Valid?	Advantages/Strengths	Disadvantages/Limitations
Line			
Bar			
Circle			

12.

Blood type	Number having
O +	91,131,000
A +	88,668,000
B +	22,167,000
O -	17,241,000
A -	14,778,000
AB +	7,389,000
B -	4,926,000
AB -	2,463,000

Source: *The Unofficial U.S. Census, What the U.S. Census Doesn't Tell You* (New York: Ballantine Books, 1991), p. 17.

13.

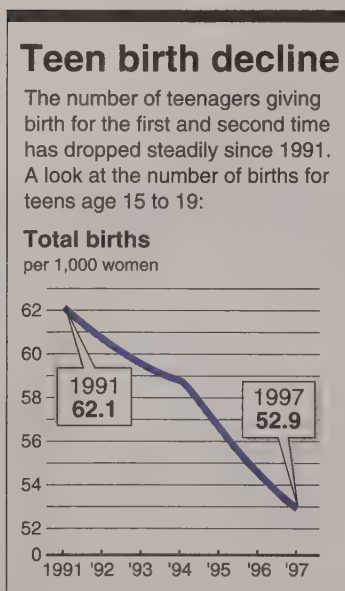
WORLD'S 10 MOST POPULOUS CITIES		
Rank	City and country	Population
1.	Seoul, South Korea	10,776,201
2.	Bombay (Mumbai), India	9,925,891
3.	Mexico City, Mexico	9,815,795
4.	São Paulo, Brazil	9,393,753
5.	Jakarta, Indonesia	9,160,500
6.	Shanghai, China	8,930,000
7.	Moscow, Russia	8,436,447
8.	Tokyo, Japan	7,966,195
9.	Istanbul, Turkey	7,774,169
10.	New York City, U.S.	7,380,906

Source: Data from *Time Almanac* 1998, p. 153.

For problems 14–25, answer questions a through e and any additional questions that may be asked about that specific graph.

- Describe in full sentences and in everyday English one major quantitative conclusion that you can make from the graph and data. In some cases, one sentence is sufficient; in other cases, two or more sentences will be required. The key word is *major*.
- Critique the data from the perspective of reliability and validity. That is, state and explain at least one question you have that is about how the data were obtained and could affect the accuracy of the data. This question has two parts: (1) state your question and (2) explain how this question has to do with accuracy of the data.
- Critique the choice of the graph (circle, bar, histogram, or line). If you think the choice is fine, explain why. Be specific. (Don't just say, "I think a bar graph is fine because it is easy to read.") If you think a different kind of graph would have been more appropriate, explain why.
- Critique the construction of the graph—its scales, categories, and labels. For example, if you think some of the labels on the graph are unclear or poor or missing, or if you think the scale is not accurate or useful, describe your criticism specifically.
- What other questions do you have? These may be requests for more information to help you interpret the information better or requests for more information to help you better understand the graph. (This answer will not be graded; the question is included because it is a natural next step.)

14.



Source: National Center for Health Statistics. "Teen Birth Decline" graph © AP/Wide World Photos. Reprinted by permission.

Reference: *Keene Sentinel* 12/18/98

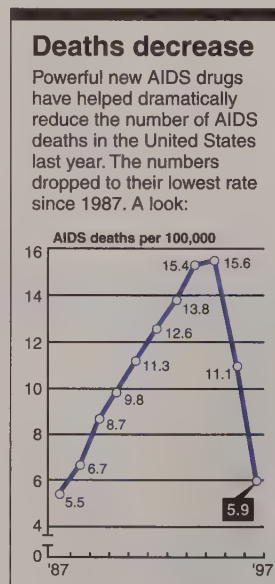
15.



Source: "Food" Data Source: Federal Reserve Bank of Dallas © 1998. Reprinted by permission.

Reference: *Minneapolis Star Tribune* 7/19/98

16.



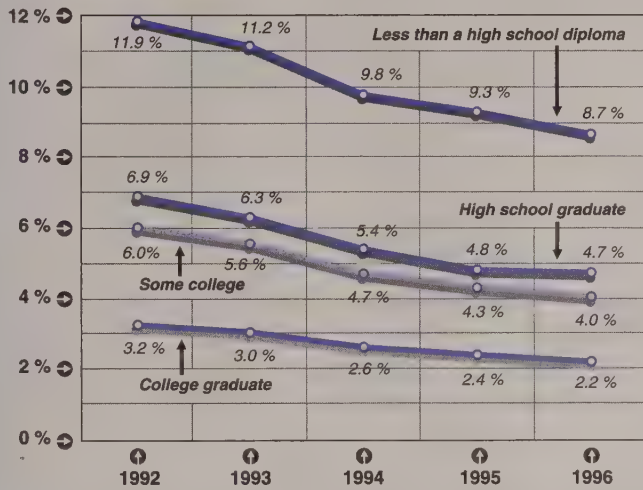
Source: Department of Health and Human Services AP/Justin Gilbert "Deaths Decrease" graph © AP/Wide World Photos. Reprinted by permission.

- Why do you think the makers of this graph chose to use rates instead of actual AIDS deaths?

17.

Reworking welfare

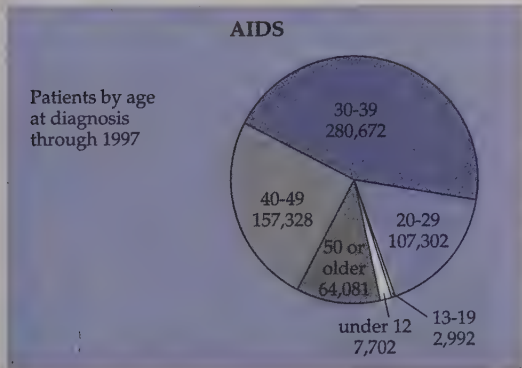
Unemployment rate by education level



Source: "Reworking Welfare" graph © AP/Wide World Photos. Reprinted by permission.

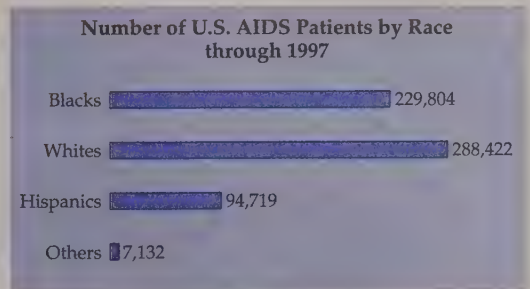
Reference: *Keene Sentinel* 2/18/97

18. The graph below shows who is getting AIDS and at what age.



Source: Data from the *Statistical Abstract of the United States* 1998, p. 147.

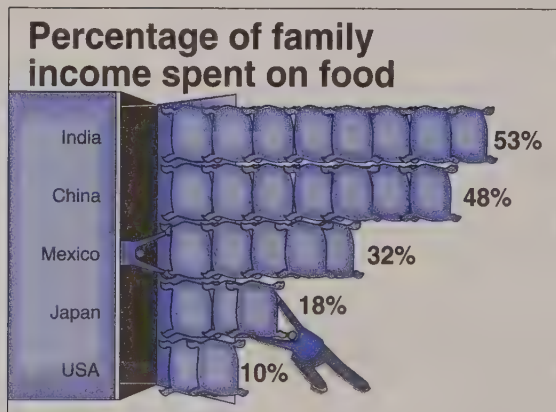
19. The graph below shows U.S. AIDS patients by race.



Source: Data from the *Statistical Abstract of the United States* 1998, p. 147.

f. The numbers of blacks and Hispanics are disproportionate. Explain what this statement means.

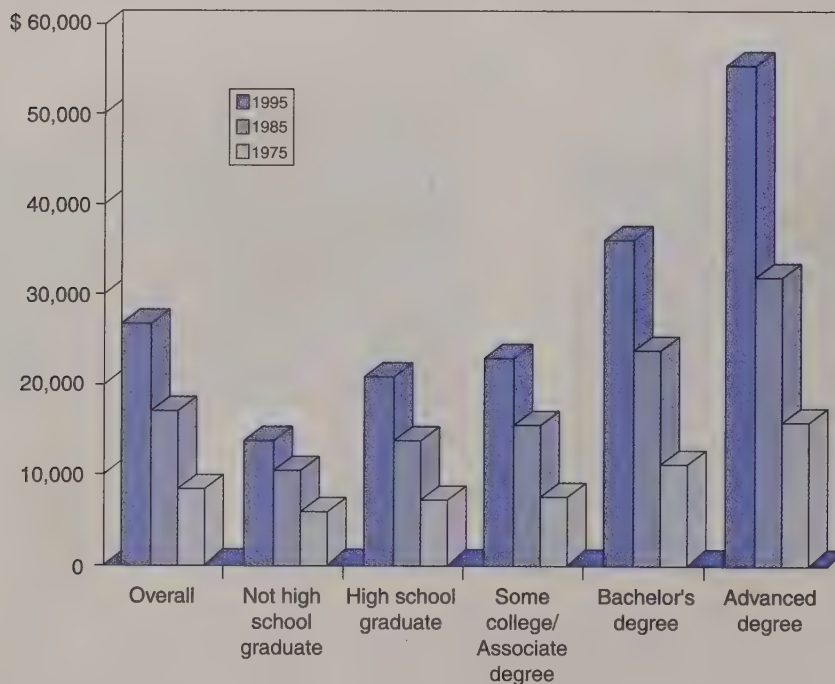
20.



Source: "Percentage of family income spent on food," from *USA Today*. Copyright 1995, *USA Today*. Reprinted with permission.

Reference: *USA Today* Calendar, April 30, 1995.

21. **Mean Earnings of Workers 18 Years and Over by Educational Attainment, 1975 to 1995**

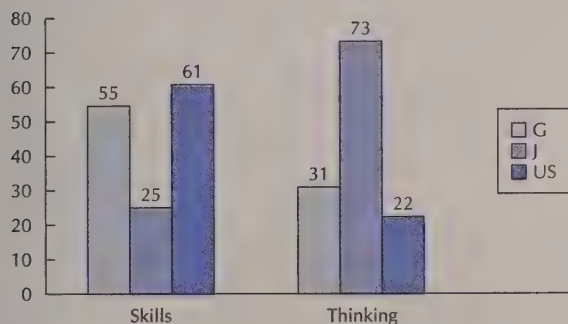


Source: "Mean Earnings of Workers 18 Years and Over by Educational Attainment, 1975 to 1995" from *Time Almanac* 1998. Reprinted by permission of Information Please LLC.

The graphs in Exercises 22–24 are from the TIMSS (Third International Mathematics and Science Study). TIMSS is the third international study of students' mathematics and science achievement. Students were assessed at fourth grade, at eighth grade, and at the end of high school. The study included about a half-million students from 41 countries. The National Center for Education Statistics (NCES) also sponsored a study of teaching methods in eighth-grade mathematics in Japan, Germany, and the United States. In this study, classes were observed and videotaped. The three graphs below are from that study.

22. Teachers were asked to tell the researchers what they wanted students to learn from the lessons that were videotaped. The researchers found that most of the answers fell into one of two categories: *skills*, where the answers focused on students being able to do something, and *thinking*, where the answers focused on students being able to understand something about a mathematical concept or idea.

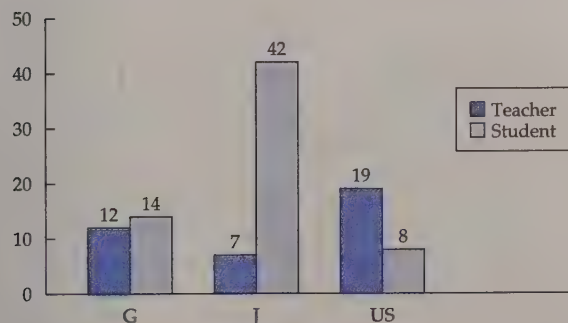
Percentage of teachers who describe the goal of the video-taped lesson as skills vs. thinking



Source: <http://nces.ed.gov/pubs99/timssvid/chap3.htm#12>

23. In this case, the researchers compared the frequency with which students were presented with the teacher's solution versus a student-generated solution.

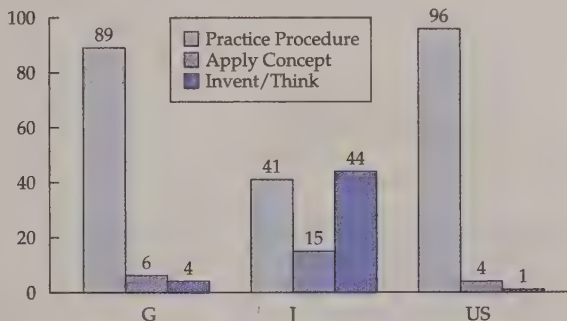
Percentage of lessons that included teacher-presented and student-generated alternative solution methods



Source: <http://nces.ed.gov/pubs99/timssvid/chap3.htm#22>

24. In this case, the researchers examined the kind of work that students did during the lesson. They coded three types of student work: practicing routine procedures, applying concepts to new situations, and inventing new solution methods or thinking.

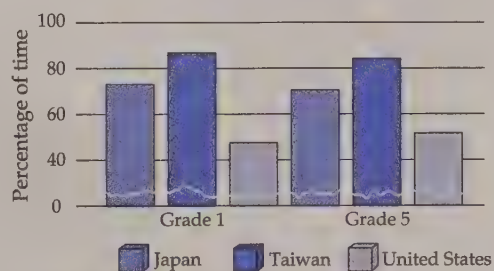
Average percentage of seatwork time in each country spent working on three kinds of tasks



Source: <http://nces.ed.gov/pubs99/timssvid/chap5.htm#65>

25. The graph below is taken from the book *Making the Grade in Mathematics*.

Percentage of time teachers led mathematic classes



Source: Harold W. Stevenson, Max Lummis, Shinying Lee, and James W. Stigler, *Making the Grade in Mathematics: Elementary School Mathematics in the United States, Taiwan, and Japan* (Reston, VA: NCTM, 1990), p. 16. Reprinted with permission from *Making the Grade in Mathematics*, copyright 1990 by the National Council of Teachers of Mathematics. All rights reserved.

26. Why is a bar graph less appropriate than a line graph for showing data that have changed over a period of time?
27. Describe how the valid changing of the vertical and/or horizontal scales of a graph can change the appearance of a graph.
28. When data are presented, either in tables or in graphs, sometimes we see raw numbers and sometimes the raw numbers have been translated into rates. Unfortunately, many people do not fully understand rates. Explain why we have rates, as though to someone who asks, "Why not just use the raw numbers?"

29. Find or make up a set of data for which:
- A bar graph would be an appropriate representation
 - A circle graph would be an appropriate representation
 - A line graph would be an appropriate representation
 - A box plot would be an appropriate representation
- e. A stem plot would be an appropriate representation
30. Thelma says that she assumes a circle graph is appropriate when the data are percentages. What do you think of Thelma's rule of thumb?

SECTION

7.2

DISTRIBUTIONS: CENTERS AND SPREADS

WHAT DO YOU THINK?

- What do we mean by average?
- How can there be different averages for the same set of data, depending on which center of a distribution you use?
- Why do mathematicians discourage reporting just the average when describing a population?

In the previous section, we examined changes in populations over time (AIDS, fatal crashes), and we examined specific subsets of other populations (the regions of the world, students studying more than 10 hours). In this section, we will focus on analyzing populations in order to understand the distribution of the data within a population. This focus is consistent with what has been traditionally seen as statistical analysis.

Tools We Will Use

When we are presented with a set of data describing some characteristics of a population (income, height, age, and so on), there are three kinds of mathematical tools that we commonly use to better understand the population: (1) tools to help us determine the center of the population, sometimes called the typical or representative or average; (2) tools to tell us how the data are dispersed or spread—for example, the range of values, the quartiles, and the standard deviation; and (3) words to describe the overall distribution, such as *uniform*, *normal*, *skewed*, and *random*. These are not esoteric tools that only some people can understand. Rather, they are tools that are based on common sense. It is my hope that this section helps to demystify the word *statistics*. This section is critical because one of the most common uses of statistics is to gather information on a subset (called a sample) of a population in order to make generalizations about the whole population. Many of the examples of data collection and analysis cited at the beginning of this chapter involve sampling—for example, opinion polls, market research, and studies to determine whether a vaccine is effective.

Lynn Steen edited a book called *On the Shoulders of Giants: New Approaches to Numeracy*, a book written by mathematicians to help mathematics educators and others realize the vision of the NCTM standards. As we have seen, a key aspect of this vision is to get people to realize that mathematics is more than just numbers, formulas, and computations. The six chapter titles are especially relevant to this section: Pattern, Dimension, Quantity, Uncertainty, Shape, and Change. Much of the reason why people collect and analyze data is to deal with *uncertainty* and to understand and predict *changes* in populations; we look for *patterns* and trends in the data; the *shape* of the data tells us much, and certain *quantities* (mean, standard deviation) also tell us much about the data. These ideas are what this chapter deals with: uncertainty and change, shapes, patterns, and quantities.

HISTORY

The name of the book edited by Steen comes from a quotation from Isaac Newton, who most people think of as the guy who discovered gravity when the apple fell on his head. However, he is also one of the creators of calculus, which he developed to explain his observations in physics. Newton remarked, "If I have seen farther than others, it is because I have stood on the shoulders of giants."

What Is Average?

Let us begin with the concept of average. When you hear the word *average*, what comes to mind? Think and then read on. . . .





LANGUAGE

A number of students told me that the use of the word *population* in this section is confusing. Because the mathematical use of *population* is similar but not identical to the everyday use, let me try to help those readers for whom the term is not clear. A connection to the set language of Chapter 2 is useful—the “*population*” is like the universal set. Let’s say someone gathers data on the study habits of college students and finds that the average full-time college student studies 20 hours per week, and someone else gathers data on traffic accidents and finds that 30% of all traffic accidents in which someone dies involve the use of alcohol. In the first case, the population is the set of all full-time college students, and in the second case, the population is the set of all traffic accidents in which someone dies. In the first case, we do not ask every full-time college student, but rather a sample (subset). In the second case, the gatherer of the data probably does gather data on all traffic accidents in which someone dies, because every state maintains such records. In one case the population consists of people; in the other it consists of traffic accidents involving a fatality.

Most of my students tell me *how to find* the mean; that is, “add all the values and divide by how many you have.” However, *average* is not a formula; rather, it is one of many concepts that can help us to understand a population.

We see and use references to average all the time. For example, what is your grade point average, and what is the average starting salary of a teacher these days? Some of the investigations in Section 7.1 dealt with averages—for example, average life expectancy. Can you think of other examples where you have seen or used the concept of average?

When we stop and think about it, we find that this concept is used frequently. Let me now ask you to reflect on another question: Why do we want to know the average this or the average that? Think about this, and then read on. . . . 

Mean, Median, Mode

The idea of average was created (by somebody, somewhere) so that one number or one word would give us a sense of what is typical or representative of the population in question. Depending on the population we are studying and the question we are asking, we may find the “average” by determining the *mean*, the *median*, or the *mode*. Statisticians refer to the mean, median, and mode as **measures of central tendency** of a distribution, although this long term is being replaced simply by **measures of the center**.

In some cases, we want to know the **mean**, which we find by taking the sum of the data and dividing by the number of data. If you took five tests in a course, your instructor would generally determine your average score by using the mean—adding up the scores and dividing by 5. The 1994 *Information Please Almanac*⁶ reports that the mean number of people in U.S. households in 1992 was 2.62.

In some cases, we want to know the **median**, which we find by determining the numerical middle of the data. For example, if you determined the height of all the students in your class and ordered the numbers from smallest to greatest, the number in the middle would be the median. If the number of students in the class is an even number, then there is no middle number. In that case, you “split the difference” of the two numbers closest to the middle; technically, you take the mean of those two numbers. The 1993–1994 *American Almanac*⁷ reports that the median age in the United States in 1991 was 33.1 years.

In some cases, we want to know the **mode**, which is the datum that appears most often. The mode is often used when the characteristic we are studying is not a number. For example, if you were to collect data on what state the students in your class were born in, the mode would be the state that occurred most frequently. We often use the word *typical* with mode. For example, the typical professional basketball player is over 6 feet. If you did Exploration 7.3, you defined *typical* for a specific population: you and your classmates in this course. When a newspaper reports that more women in the U.S. work force were employed in administrative support than in any other category, it is saying that when we examine the population of working women and look at the characteristic called “type of job,” “administrative support” is the mode.

One of the reasons for determining the average of a set of data is that one number or one phrase can give a quick summary of the data. The mean, median, and mode are all candidates to be considered as a representative of the data. In some cases, the mean, median, and mode are very close, but sometimes

⁶ 1994 *Information Please* (Boston: Houghton Mifflin, 1993), p. 827.

⁷ *The American Almanac: Statistical Abstract of the United States 1993–1994* (Austin, TX: The Reference Press, 1994), p. 15.

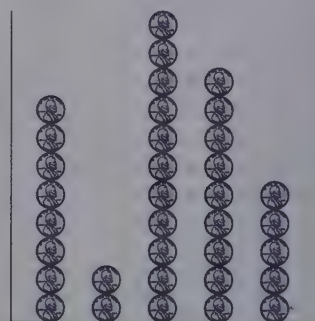
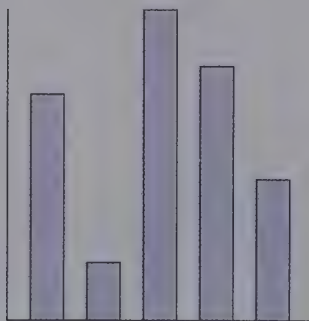
they are not. Let me connect this notion of representative of a population to a real-life situation. When I was in the Peace Corps in Nepal, the country director told new volunteers that we needed to think about our behavior because we were representatives of the United States. In one sense, the director was referring to average. Because the people we worked with had had little or no previous contact with Americans, they were likely seeing whatever we did as typical. If we wore blue jeans or drank alcohol, many people would assume that all (or at least most) Americans wear blue jeans and drink alcohol. This is related directly to the notion of sampling—that is, making generalizations about a whole population on the basis of data from a sample of the population.

The concept of “mean” is frustrating for a college teacher, because so many students enter the course believing that *mean* and *average* are the same thing. This concept falls in the “rubber band family of learnings.” Imagine the student as a rubber band (hey, creativity and humor are important parts of teaching). The professor teaching the new idea is stretching the student’s understanding. However researchers have found that, in all too many cases, several months after the course the student’s understanding is like the rubber band—it snaps back to its initial state. I have discussed this earlier as the difference between rented and owned knowledge. The following investigation is one that I first discovered in a methods textbook for elementary teachers; I have since seen variations of it in many places. My students enjoy it because it is fun and because they can quickly see that they can use it with their students also.

INVESTIGATION 7.8 Going Beyond a Computational Sense of Average

Imagine that five elementary school children are asked how many movies they saw in the past year, and they responded: 7, 2, 9, 8, and 4. First, write down what you think the mean tells you about a set of data. . . .

Below is a standard bar graph and a slightly unconventional bar graph representing these data. *Do not compute the mean.* Rather, draw a horizontal line across the standard bar graph where, on the basis of your current sense of what the mean is, you “feel” the mean will be. Now, I want you to get some pennies and make the bar graph at the right. If you don’t have pennies, other coins will do. If you don’t have coins, find a bunch of other objects (such as paper clips), or tear a sheet of paper into a number of small pieces of roughly the same size. Now move the pennies so that all the bars are the same. What did you just learn about the mean? Now read on . . .



DISCUSSION

The mean can be viewed as the number you get when all the values are leveled off. In this case, if we “give” values from the larger numbers to the smaller numbers until all the numbers are the same, then the heights of the bars (on the standard bar graph) and the number of pennies in each column (in the penny bar graph) are all the same. Before this course, most students’ conception of the mean is purely procedural—you just do it. However, in order to have mathematical power, you also need conceptual understanding. If you have only procedural knowledge, you can do only problems “just like the ones in the book.” The catch is that most real-life classrooms are never “just like the ones in the book.” This one investigation does not give you a complete understanding of the mean. However, it can enhance your understanding of average and other data on populations—an understanding that will enable you to interpret intelligently the graphs and data you find in newspapers and elsewhere and that will equip you to teach in such a way that your children will own what they learn.

Average is only one of several tools we use to better understand a population. Rather than explain the others now, let us move on to some investigations of average and see what we find. Again, this is a modeling of the NCTM problem-solving standard, in which teachers are encouraged to use problem-solving approaches to investigate and understand content. We will examine data for which it makes sense to ask about averages. In the following five investigations, we will examine graphs that can help us to “see” the data, and will discover some tools that help us to describe the center of the data and how the data are dispersed (spread). As in Section 7.1, we will also look critically at the data—what they mean, where they came from, and the like.

INVESTIGATION 7.9 Years of Experience

LEARNING

In Explorations, you have several opportunities to gather your own data. Most of my students find it more interesting to gather real data than to analyze some made-up set of data. So too with children. *Teaching Children Mathematics*, the NCTM monthly journal for elementary teachers, and other publications have had many articles written by elementary teachers about students gathering their own data. On one occasion, fourth graders had complained that the playground was inadequate and unsafe. They gathered and analyzed data, presented it to the School Board, and experienced great satisfaction when a new playground was built!

Let us begin with a relatively simple case and work toward more complex cases. Maplewood Elementary School is preparing its accreditation report. One of the questions on the report asks for the average years of experience of the staff. What do the people who prepare the report need to do to answer this question? That is, what data do they need, and what do they do with the data? Think and then read on. . . .

First, the people who prepare the report need to decide what “years of experience” means. Let’s say they decide to interpret this question as the number of years of experience a teacher had at the beginning of the school year. However, they still have to decide what to do about teachers who taught part-time (job sharing) or taught for part of a year (started in January), among other things. Let’s say they decide to use whole numbers.

Here are the numbers of years of experience that Maplewood’s teachers have:

0, 2, 2, 3, 3, 4, 20, 2, 4, 3, 2

How might you analyze these data to answer the question on the accreditation report? Take some time to explore this set of data, using knowledge you already have. State your conclusions and then read on. . . .

DISCUSSION

The mean number of years is 4.09 years, which is very close to 4. The median is 3 years. Some people mistakenly conclude that the median is 4. If you are one of these people, remember that the median is the middle number only when the numbers have been arranged from smallest to greatest. It is not the middle number in the original list; do you see why? The mode is 2 years.

In one sense, a case can be made for each of these three numbers as representing the average (typical) teacher. Do you see why?

The mean is the term most people think of when they hear *average*. The median is literally in the middle. One could argue for the mode, because more teachers have 2 years of experience than any other number of years. So what do they write on the report?

They decide to make a bar graph to see whether that will help. One teacher makes the graph in Figure 7.13(a), and another makes the graph in Figure 7.13(b). Before we discuss the graphs, a note on interpretation. The numbers on the x axis represent years of experience. The numbers on the y axis represent frequency—that is, how many teachers have that amount of experience. Thus, the first bar tells us that 1 teacher has 0 years of experience, the second bar tells us that 4 teachers have 2 years of experience, and so on.

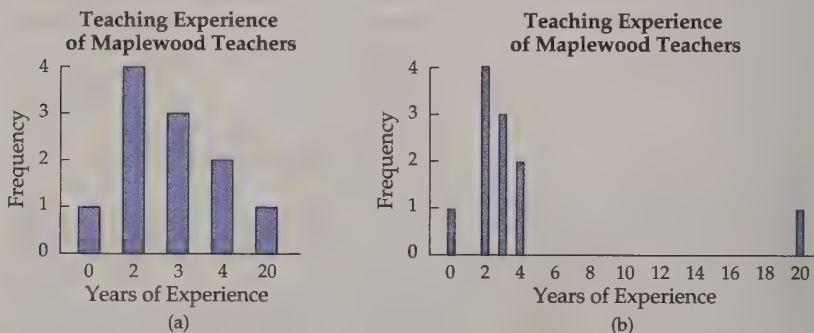


FIGURE 7.13

Are both graphs right, or is one better? What do you think?

When we make bar graphs for numerical data, graphs like that in Figure 7.13(a) are frowned upon. Do you see why? In Figure 7.13(a), the numbers are all clumped together. Like the AIDS graph in Investigation 7.3, this distorts the fact that 20 is much farther from 4 than 4 is from 3.

As you can see from Figure 7.13(b), the number 20 stands out dramatically. Data whose values are significantly greater or less than those of other data are referred to as **outliers**. We will explore them more in the next investigation.

The bar graph shows that the overwhelming majority of teachers (9 out of 11) have 2, 3, or 4 years of experience. Because 3 is in the middle of this cluster, the teachers decide that the median (3 years) makes the most sense to report as the average years of experience.

A more efficient way to determine the mean In this investigation, we could determine the mean of the data by adding all 11 numbers and then dividing by 11. We will use Table 7.7 to examine two easier ways to determine the mean; this is especially useful as the size of the sample becomes larger. The first column shows all the years for which we have people; that is, there are no teachers with 1 year of experience, none with 5 to 19 years of experience, and none with over 20 years of experience. The second column shows the frequency of each of these numbers. For example, there are 1 teacher with 0 years of experience, 4 teachers with 2 years of experience, and so on. The third column is

obtained by multiplying the numbers in the first two columns. Why will multiplying these pairs of numbers help us to determine the mean?

TABLE 7.7

Number of years of experience	Frequency	Product
0	1	0
2	4	8
3	3	9
4	2	8
20	1	20
Sum	11	45

Basically, we are simply applying the repeated addition context of multiplication. Thus, rather than adding all 11 numbers, like this:

$$0 + 2 + 2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 20$$

we are doing the following:

$$0 + 4(2) + 3(3) + 2(4) + 20$$

In cases of large sets of data, we can reduce the tedium of calculations further by entering the first two columns of data into a spreadsheet program and then “telling” the computer to multiply the rows together and then find the sum of the third column.

These numbers are measurements This investigation brings out another important aspect of statistical analysis. When we are representing a characteristic of a population with a number, we are actually measuring. In this case, the measurement is rather straightforward, although the teachers still had to make some decisions—for example, whether to count partial years and whether to round experience to the nearest year. In many cases, the measurement is not so straightforward and obvious. For example, if we were to graph the SAT scores of the freshman class at your college, what does SAT measure? What does IQ measure? How do we measure productivity?

Let us now focus on a situation in which we have more data.

INVESTIGATION 7.10 Class Sizes in a College

 Cost of education is always an issue in the minds of taxpayers. One of the ways to reduce cost is to increase class size. Let’s say you are the principal of a small high school and these are the sizes of classes in your school. What would you say is the average class size? 

7, 28, 29, 31, 31, 16, 18, 29, 29, 19, 24, 30, 34, 30, 40, 25, 27, 29, 7,
28, 30, 31, 30, 25, 26, 31, 30, 30, 32, 26, 27, 32, 17, 33, 16

DISCUSSION

The mean is 26, the median is 29, and the mode is 30. The school board has decided to use the mean as the average. However, the principal argues that it is not fair to use the mean as the average for these data. Why do you think the principal objected to the mean as the best representative of “average class

size?" As before, take some time to consider what analyses you might do of these data. Summarize your observations and conclusions, and then read on. . . .



Stem plot This is a good time to demonstrate three other graphs: stem plots, line plots, and boxplots, which are also good ways to organize data. The **stem plot**, also referred to as a stem-and-leaf plot, uses the idea of place value. In this case, we look at the tens digits of our data and make a row for each, as shown on the left-hand side of Table 7.8. Now we simply insert each datum into the proper row. For example, 34 is entered into the 3 row as 4, and 28 is entered into the 2 row as 8.

Once this is done, it is a simple task to put the (ones) digits into numerical order. Now examine the stem plot. What does it tell you?

CONNECTION

From one perspective, a stem plot is similar to a bar graph that has been laid on its side.

TABLE 7.8

4	0
3	1 1 0 4 0 0 1 0 1 0 0 2 2 3
2	8 9 9 9 4 5 7 9 8 5 6 6 7
1	6 8 9 7 6
0	7 7

4	0
3	0 0 0 0 0 0 1 1 1 1 2 2 3 4
2	4 5 5 6 6 7 7 8 8 9 9 9 9
1	6 6 7 8 9
0	7 7

From the stem plot we can easily see that there are more classes in the 30s than in any other "decade" and that only two classes have fewer than 16 students.

Line plot We can also construct a **line plot**. First, we make a scale. Next we place an $\times$ (or other mark) at the appropriate place for each piece of data. What can you tell about class sizes in the school from the line plot in Figure 7.14? Think and then read on. . . .

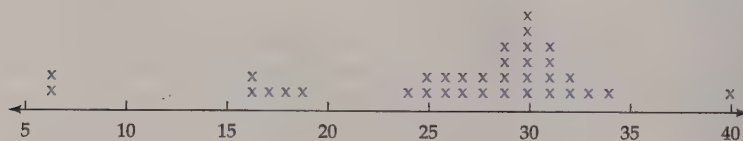


FIGURE 7.14

We can use ordinary English to describe some of our observations:

We can see that the class sizes vary from a low of 7 to a high of 40. The difference between the extreme values is called the *range*. Thus, the range for these data is 33.

We can see that there is a **cluster** of data around 30.

There are also **gaps in the data** between 7 and 16, between 19 and 24, and between 34 and 40.

One of the primary advantages of such a graph is that not only can we see the range of the data but we also get a much richer sense of how the data are **distributed**, or spread out, and we don't lose any information about individual values.

Thus, the line plot helps us to interpret how the numbers are actually distributed. We see that the mean (26) does not seem as fair a representative of the center of this set of data as the median (29) or the mode (30).

Boxplot Finally, let us consider one more graph, which builds on the line plot. The box-and-whisker graph, or **boxplot** (invented by J. W. Tuckey in the twentieth century), packs a lot of information into one graph. It is often difficult for students to understand at first. Therefore, I recommend that you read through the description first, just to get an idea, and the second time through actually make the boxplot along with me.

1. Copy the class sizes onto your paper horizontally and circle the median of the data; in this case, there are 17 values to the right and to the left of the circled 29.

7 7 16 16 17 18 19 24 25 25 26 26 27 27 28 28 29 (29) 29 29 30 30 30 30 30 30 31 31 31 31 32 32 33 34 40

2. Find and circle the median of the bottom half of the data, and then find and circle the median of the upper half of the data.

7 7 16 16 17 18 19 24 (25) 25 26 26 27 27 28 28 29 (29) 29 29 30 30 30 30 30 30 (31) 31 31 31 32 32 33 34 40

We call 25 the **first quartile** and 31 the **third quartile** because these two numbers and the median essentially partition the data into four quarters that are (roughly) equal in size. The first quartile lets us know the top of the first quarter, and the third quartile lets us know the top of the third quarter. We use these three numbers plus the low and high data (7 and 40) to make the boxplot.

3. Make a scale and mark a dot for each of these five numbers at the appropriate spot on the scale, as in Figure 7.15.

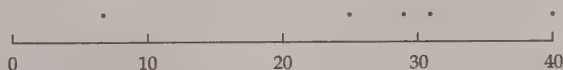


FIGURE 7.15

4. Make the boxes and whiskers, as in Figure 7.16!

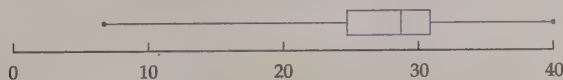


FIGURE 7.16

Stop for a moment and look only at the boxplot. What does it tell you about the sizes of classes in the school? Think and then read on. . . . 

- The largest class (the end of the right whisker) has 40 students.
- The smallest class (the end of the left whisker) has 7 students.
- The median class size (the line in the middle of the boxes) is 29.
- Half of the classes (the two boxes) have between 25 and 31 students.

Let us consider the boxplot from a more general perspective. That is, in general, what do the boxes represent? What do the whiskers represent? Think and then read on. . . . 

The left whisker tells us the range of the bottom quarter of the data, and the right whisker tells us the range of the top quarter of the data. In this case, about $1/4$ of the classes are between 7 and 25, and about $1/4$ of the classes are between 31 and 35. The two boxes tell us where the middle half of the data lie.

Outliers and boxplots Recall the observation in Investigation 7.9 that the number 20, representing the teacher with 20 years of experience, was an outlier. Let us now quantify that term. To determine whether a datum is an outlier, we first find the difference between the third and first quartiles. We call this amount the **interquartile range (IQR)**. In one sense, the IQR is like a mini-range—it is the range of the middle half of the data. We multiply the interquartile range by 1.5. If a datum is more than 1.5 IQRs from the first or third quartile, we call it an outlier. (It is important to note that not all statisticians use 1.5; even statisticians do not agree on everything!) Apply this procedure to this example to determine whether there are any outliers, and then read on. . . . 

The difference between the third quartile and the first quartile is 6; $31 - 25 = 6$, and $6(1.5) = 9$. Thus, any number below 16 or above 40 is an outlier, because $25 - 9 = 16$ and $31 + 9 = 40$.

In this case, we see that the class size of 7 is an outlier because it is less than 16, and the class size of 40 is on the verge of being an outlier.

Figure 7.17 illustrates the notion of outliers in this instance. If a value is outside the region shown, then it's an outlier.

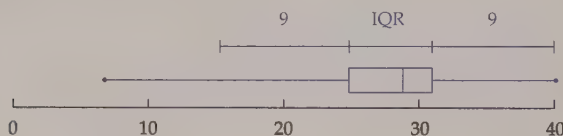


FIGURE 7.17

Because the boxplot is such a new kind of graph, most students need to do several to get a feel for it. Some readers notice that the line plot and the boxplot look “similar.” Explaining their similarities will be left as an exercise. The interested reader is encouraged to make boxplots of some data from the exercises.

In the following investigation, we examine a situation in which we still have quite a few data and they are much more spread out.

INVESTIGATION 7.11 Average Time to Finish the Exam

One year, I recorded the time (in minutes) it took each student to do the final examination in my course. Technically, it was a two-hour test, but I told the students that I would stay for three hours. I recorded the time each student took to finish the exam to determine whether or not I should make the exam shorter. The data below are the times, in minutes, that the students took on the exam:

62, 76, 88, 89, 95, 96, 98, 99, 101, 102, 104, 106, 111, 112, 114, 114,
115, 115, 123, 123, 125, 125, 132, 134, 134, 136, 138, 139, 139, 144,
146, 149, 149, 154, 154, 155, 160

What analyses might you do of these data to advise me? Take some time to explore the data, using knowledge that you already have. Summarize what you learned and state your conclusions, and then read on. . . . 

DISCUSSION

Examining the spread of the data A line plot (Figure 7.18) gives us a sense of the distribution without losing any data. In this case, the line plot doesn't tell us much beyond what we knew already. The range is so great that patterns in the data are not apparent.

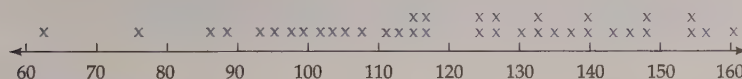


FIGURE 7.18

TABLE 7.9

16	0
15	4 4 5
14	4 6 9 9
13	2 4 4 6 8 9 9
12	3 3 5 5
11	1 2 4 4 5 5
10	1 2 4 6
9	5 6 8 9
8	8 9
7	6
6	2

A stem plot helps us to organize the data (see Table 7.9). As with the line plot, we don't lose any data (for example, we still know the minimum and maximum). In this case, the stem plot shows us that as we get closer to the middle of the data, the number of students is greater.

The stem plot suggests another way of organizing and representing the data. We can group the data using an interval size suggested by the stem plot, 10 minutes. By selecting an interval size, we can make a **grouped frequency table** for the data. The intervals are called **classes**. It is important to note that there is no one "right" interval size. For example, we could choose an interval size of 10 minutes, in which case we have Table 7.10. However, this choice produces 11 classes. Alternatively, we could choose an interval size of 20 minutes, in which case we have Table 7.11, which gives us 6 classes.

TABLE 7.10

Interval	Frequency
60–69	1
70–79	1
80–89	2
90–99	4
100–109	4
110–119	6
120–129	4
130–139	7
140–149	4
150–159	3
160–169	1

TABLE 7.11

Interval	Frequency
60–79	2
80–99	6
100–119	10
120–139	11
140–159	7
160–179	1

From these data, we can make a new kind of graph, a **histogram**, which is a bar graph in which there is no space between the bars. Histograms are commonly used when the data are numbers as opposed to categories. Examine the two histograms in Figure 7.19. What do you see now that you didn't see before? What do they tell you about the data that is similar? What do they tell you about the data that is different? Then read on. . . . 

The histogram in Figure 7.19(a) indicates that the majority of the times are between 90 and 150 minutes, and it shows two peak intervals: 110–119 and 130–139. The histogram in Figure 7.19(b) indicates that the majority of the times lie between 80 and 159 minutes.

Note that with the second set of grouped frequencies, we could also make a circle graph for the data. This would give a quicker sense of what proportion

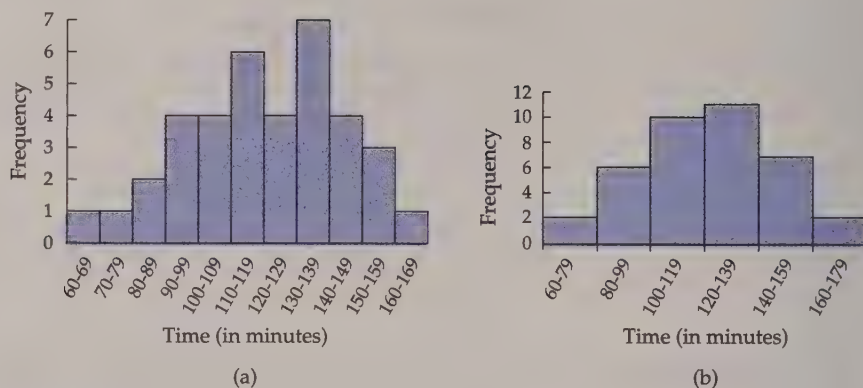


FIGURE 7.19

of the class finished in each of the time intervals. Technically, we could do this with the first set, but a circle graph with 11 slices is a bit much.

Finding the center If you haven't already done so, estimate the median from the line plot or one of the histograms. Then read on. . . 

From the line plot (Figure 7.18), we can see that there are about as many test lengths above 120 minutes as below. From Figure 7.19(a), we can see that there are roughly as many test lengths above the 120–129 group as there are below. In fact, the median is 123 minutes. In this case, the mean is close, 120 minutes.

The strict interpretation of the mode is relatively meaningless in this case. There are several test lengths that occurred twice, but a frequency of only 2 in a set of 37 hardly makes a number a candidate for typical. Thus, when we make grouped frequency tables, we speak of a **modal class**—that is, the class that occurs most frequently. A look at Figure 7.19(a) reveals two intervals that stand out: the 110–119-minute interval and the 130–139-minute interval. Even though the 130–139 interval has one more time than the other, both of them stand out from the other intervals, and so we can say that this set of data has two modal classes. In such a case, we say that the distribution is **bimodal**; that is, it has two modes.

We can draw several conclusions from the data and the graphs. When given more than two hours, over half the class took the extra time. The “average” time for completion was about two hours, but it is ironic that although the mean and median are both very close to two hours, this is where the actual distribution dipped. I wondered whether that was just a coincidence or whether it was related to the traditional time limit. In any case, because over half the class took longer than two hours, I decided to make a shorter exam.

Let us now see how our knowledge can be used by a teacher to analyze the test scores of a whole class.

INVESTIGATION 7.12 Scores on a Test

Let's say you have a friend, also a teacher, who has just finished grading his students' midterms. The scores are given below.

77, 96, 58, 100, 66, 76, 88, 73, 94, 75, 76, 84, 91, 74, 87, 92, 67

You ask how his students did, but the teacher does not know much mathematics. All he can say is that this is an unusual class. He can see that the scores range from 58 to 100. Take some time to analyze these data. What can you do with the data to help your friend better understand the overall picture of his students' performance? State your conclusions and then read on. . . . 

DISCUSSION

Simply from examining the numbers as listed, we can quickly determine that the median score is 77. Either by observation or from a stem plot, we can also see that grades of C and A dominate the data. Thus, grades of C and A are the modal classes for these data. (Recall that when we group the data, we speak of modal classes instead of modes. However, in both cases, the idea is the same: the value or interval that occurs most frequently.)

Look at the histogram and circle graph in Figure 7.20. What do they add to our understanding of the scores? Consider this, and then read on. . . . 

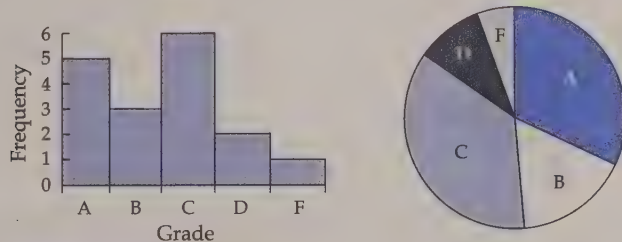


FIGURE 7.20

From the histogram, we can see that technically the modal class is a grade of C. However, because there is only one more C than A, and because these two intervals stand out, we call this distribution of scores bimodal. From the circle graph, we could frame our analysis optimistically (almost half the class got As and Bs) or pessimistically (more than half the class got C or lower).

What do the line plot and boxplot (Figure 7.21) add to our understanding of the distribution of the scores? Consider this, and then read on. . . . 

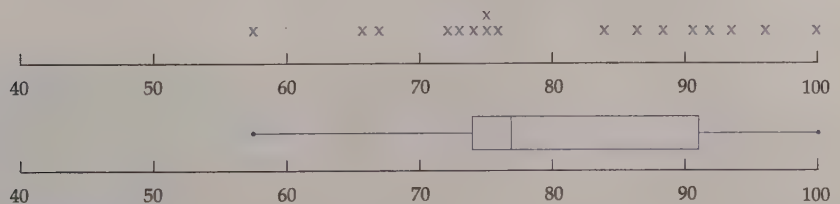


FIGURE 7.21

The line plot shows two major gaps, one from 58 to 66 and the other from 77 to 84. The line plot also shows a sense of a "top" group and a "bottom" group of the class. From your friend's perspective, this might help him understand why this is an unusual class. If he teaches to the top group, he is likely to lose the bottom group; on the other hand, if he teaches to the bottom group, he is likely to bore the top group.

In this case, because the number of data is small, the boxplot has no advantages over the line plot. From a teaching perspective, an examination of the boxplot is useful, so that you can use it and interpret it when working with larger sets of data. As stated earlier, the boxplot shows the range. Using the idea that the boxplot partitions the data into four roughly equal groups, we can focus on the two boxes and see that about $1/2$ of the students scored between 75 and 90. We can also say that roughly $3/4$ of the class scored above 75. Note that the left-hand box is narrower than the right-hand box. The narrowness of the left-hand box indicates a cluster of scores in that range—that is, the mid-70s.

A geometric view of median and mean Thus far we have determined the median (77) and the modal classes (C and A). What about the mean? Let us take this opportunity to develop another perspective for understanding the mean. From a computational perspective, we can compute the mean. From a geometric perspective, the mean is the center of gravity of the data. Imagine the data as weights placed on a seesaw. Where would the point be at which the seesaw would balance? Take a few moments to look at the line plot and estimate the balance point of this set of data. Do you think it will be greater than, equal to, or less than the median of 77? Why? Then read on. . . . 



FIGURE 7.22

If you imagine a seesaw, then you realize that numbers that are farther from the center will tip the scale, like a seesaw (see Figure 7.22).

In this set of data, the data won't balance at the median (77). Only one score in the lower half is far from 77, but several scores in the top half are far from 77. Thus, we can see that the mean will be higher than the median. In fact, the mean is 81.

With respect to the center of the data, we can tell your friend that the score of 77 is the median, that the mean score is 81, and that grades of C and A are the modal classes.

With respect to the spread of the data, we can report that the scores range from 58 to 100. Both the line plot and the boxplot can help give the teacher a visual sense of how the scores are spread out.

Thus far, we have used our graphs and our measures of the center and spread to describe and analyze one set of data. These tools can also be used to compare two sets of data.

INVESTIGATION 7.13 Which Battery Do You Buy?

Let's say you were going to buy an expensive, long-life battery for your videocamera. Two companies claim that they have developed batteries that will last an average of 40 hours instead of lasting only a few hours. The cost of the batteries is the same. What other information would you want to know before making a decision?

Do you think it doesn't matter which you buy? Let's run some numbers. Say you had these data:

Battery A: 37, 42, 40, 38, 42, 40, 39, 38, 41, 40, 39, 41, 43, 39, 40, 40, 41

Battery B: 30, 48, 44, 36, 42, 36, 40, 34, 46, 38, 40, 40, 42, 44, 38, 40, 32, 50

What analyses of these data might you do to help you make your decision? Take some time to explore these two sets of data, using knowledge

you already have. Summarize what you learned and state your conclusions, and then read on. . . 

DISCUSSION

If you did explore the data, you found that for each set, the mean, median, and mode were 40. In this case, the measures of the center tell us nothing about differences between the two sets of data. This occasionally happens with real-life data, and it illustrates a point that statisticians make: Measures of the center of a distribution give an incomplete picture of the data. Let us now discuss some graphs that help us to see how the data are spread out. In this case, because we are comparing two sets of data, we can make a back-to-back stem plot (Table 7.12).

TABLE 7.12

Battery A		Battery B
	5	0
3 2 2 1 1 1 0 0 0 0	4	0 0 0 0 2 2 4 4 6 8
9 9 9 8 8 7	3	0 2 4 6 6 8 8

The stem plot enables us to determine quickly the ranges of the two sets of data. We see that the data for Battery B are much more spread out (dispersed) than the data for Battery A:

- The range of the data for Battery A is 6 (37 to 43).
- The range of the data for Battery B is 20 (30 to 50).

If we make line plots (Figure 7.23), what do we see?

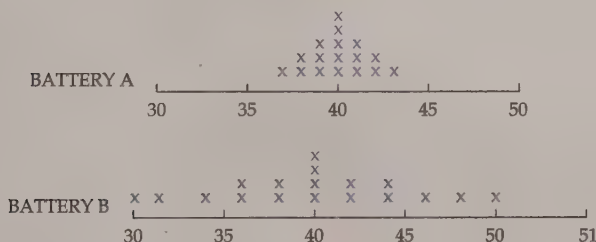


FIGURE 7.23

Finally, we can make boxplots, which we can stack on top of each other as in Figure 7.24.

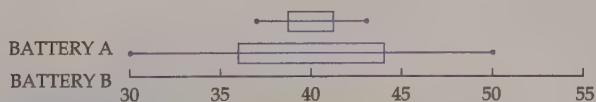


FIGURE 7.24

Both the line plots and the boxplots indicate that the data for Battery A are more clumped together, whereas the data for Battery B are more spread out. We will return to our analysis of the battery data after we examine a new concept

for measuring how spread out a set of data is. However, before we do that, we need to spend some time looking at some generalized kinds of statements we can make about distributions of data.

Different distributions There are many ways in which a set of data can be distributed. Another way of saying this is that graphs of populations can have many different kinds of shapes. In this course, we will focus on five distributions: **uniform**, **skewed to the right**, **skewed to the left**, **bimodal**, and **normal**. The graphs in Figure 7.25 represent idealized (smoothed) versions of these distributions. In real life the data are seldom so smooth. Can you think of a scenario for each of these graphs? Try to do so before reading on. . . .



HISTORY

As you have discovered in this book, much of what we call elementary mathematics is relatively recent knowledge. So, too, is the idea of looking at distributions of data. In the 1830s a Belgian mathematician, L. A. J. Quetelet, collected measurements of many kinds of data on people, including height, weight, length of arms, and intelligence. He did this for many people. What he found was that in most of these cases, the graphs were similar. What do you think all the graphs looked like? Think and read on. . . .



All of the distributions were similar to the normal curve.

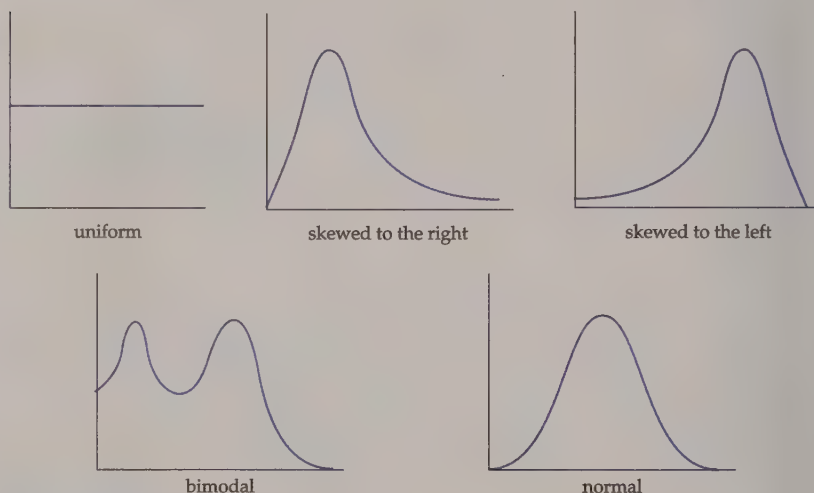


FIGURE 7.25

Some students report having difficulty making sense of the preceding discussion. If this applies to you, the following may be helpful. The distributions shown in Figure 7.25 represent five (of many) ways in which a set of data may be distributed. The line graphs shown can be thought of as evolving from bar graphs (with which most students report being more comfortable). For example, if we collected data on the number of siblings, we would have the bar graph shown at the left in Figure 7.26. If we made a line graph from those data, we would have the line graph shown at the right in Figure 7.26. If you look at the idealized graph, you can see that we can say these data are skewed to the right. That is, this graph has the characteristics of a “skewed to the right” distribution.

Table 7.13 gives one example for each of the five distributions.

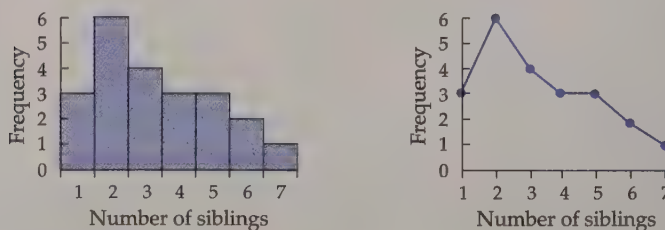


FIGURE 7.26

TABLE 7.13

Distribution	x axis (independent variable)	y axis (dependent variable)
Uniform	Months of the year	Temperature in Hawaii
Skewed (right)	Salaries in a factory	Frequency
Skewed (left)	Class sizes in a high school	Frequency
Bimodal	Number of people with cars made in a certain year	Frequency
Normal	Hours that Battery A lasted	Frequency

Take a minute or two to make sure you understand each of these examples. Otherwise, it's the classic "in one ear and out the other" story. I mention in the Instructor's Manual for this course that the NCTM process standards should permeate every class; it is my responsibility to pose, through the Explorations and Investigations, questions that cause you to see the value of these standards—that is, owning rather than renting. Making sense of these distributions involves the standards Connections and Communication. In order to make sure that you understand these distributions, I am giving you examples to help you make connections. In order to make a better (as opposed to a weaker) connection, you need to talk to yourself—to describe, in words of your own choosing, why each of these examples matches the distribution shown in the table. This is also what we call active learning, as opposed to passively reading the text and moving on.

For example, consider the row where "Salaries in a factory" is used to illustrate the distribution "Skewed (right)." Do you see that the graphical representation of this situation will be skewed to the right? Visualize a factory, who works there, what kinds of jobs are involved, and the like before reading on. . . . 

If the shape of the graph of "Salaries in a factory" is skewed to the right, that means that the frequency of salaries will peak somewhere near the middle but that the graph will slope more sharply to the left than to the right. In other words, there will be people much farther to the right of the center (making much higher salaries) than to the left of the center. From another perspective, the peak of this graph is not in the exact middle of the highest and lowest salaries but is closer to the lowest. In Exercise 4 at the end of this section, you will examine the problems of skewed distribution when different groups determine average salary.

What Measures of Central Tendency and Dispersion Tell Us

Investigations 7.9 through 7.13 have focused on three tools that enable us to make statements about a set of data: graphs, measures of the center (mean, median, and mode), and **measures of dispersion** (range, clusters, gaps, outliers, and interquartile range).

Before moving on, let us take some time to reflect on our understanding of measures of the center and measures of dispersion. We will focus first on measures of the center. For each of the three terms for average, describe the reasons for using it and its disadvantages. Then describe a real-life scenario in which that term would not be appropriate and justify your choice. Then read on. . . . 

Table 7.14 summarizes the main reasons for using each measure and some of the drawbacks of each.

TABLE 7.14

	Reasons for using	Disadvantage
Mean	It is often easier to compute with a large set of data. It is the center of gravity of the data.	It can give a distorted sense of the middle if there are extreme data (outliers). It might not make sense. For example, what does a mean shoe size of 6.3 mean?
Median	It is not affected by outliers. It is the physical center of the data.	It is not always appropriate—for example, for a baseball player's batting average.
Mode	It shows the most common datum. It is easy to determine from a graph.	It might be far from the center. There might be no modes or more than one mode.

We can make some generalizations about using these terms to describe the center of a set of data:

- If the distribution of the data is skewed, the median will often be more representative than the mean.
- If the data are categories rather than numbers (for example, favorite TV show versus age), there is no mean or median, and so the mode is used to convey the center of the data—that is, the most commonly occurring datum. We might say, for example, that the typical American family eats hot dogs on the Fourth of July; this statement indicates that it has been determined that more families eat hot dogs on the Fourth of July than any other food.
- If the distribution is symmetric (for instance, normal) the mean, median, and mode will be close to one another.

I hope that you have also seen that the average alone yields an incomplete picture of the data. One of the things that graphs do is to give us a sense of the spread of the data. The stem plot, boxplot, line plot, and histogram all enable us to gain a sense of how the data are spread out. Some show clusters, outliers, and gaps in the data better than others.

Thus far, we have found that one measure of the center (the median) is often used together with the boxplot, with the median telling us the middle of the data and the boxplot telling us about the spread of the data. That is, shorter boxes and whiskers indicate that there is a cluster of data, and longer boxes and whiskers indicate that this portion (quarter) of the data is more spread out.

Not surprisingly, there is a measure of spread that works well with the mean. It is called the *standard deviation*. However, most students do not feel comfortable computing it, and few students are able to use it confidently when analyzing data. In order to understand the standard deviation, however, we first need to examine more closely the normal distribution (often referred to as

the normal curve) because the standard deviation has most power with a set of data that are normally distributed.

Normal Distribution

A line graph depicting a set of data that are normally distributed is often referred to as a bell-shaped curve; that is, the frequency values are highest in the middle, and the graph is symmetrical. The discussion below is what is referred to as a “thought experiment”; we are not going to use data from an actual situation, but rather we are going to imagine data from an ideal situation. Suppose we have collected data on the heights of all students at a small college. As you have seen from Section 7.1, we could make a histogram from these data. The graph at the left in Figure 7.27 shows how the actual histogram might look in this ideal case; that is, the graph is symmetrical. If we trace a smooth line over the tops of the bars, we have the line graph at the right in Figure 7.27. When normally distributed data are visually depicted, the writers generally show a line graph. For example, in the case of the heights at the college, the figure on the right is a smoothed-out line graph for the heights. However, in actuality the heights have been measured in whole inches, and therefore the histogram on the left is a truer depiction of the data. Thus, whenever you see a “normal” curve, it will help if you realize that this curve is an idealization of a histogram. For example, we say that SAT scores and IQ scores are normally distributed. In each case, the x axis represents the score and the y axis represents the frequency of each score.

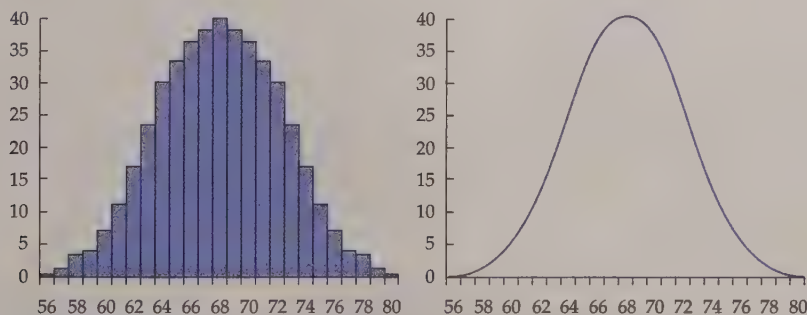


FIGURE 7.27

There are, of course, variations within kinds of distributions. For example, we would say that both of the graphs in Figure 7.28 represent data that are normally distributed. Do you see why?

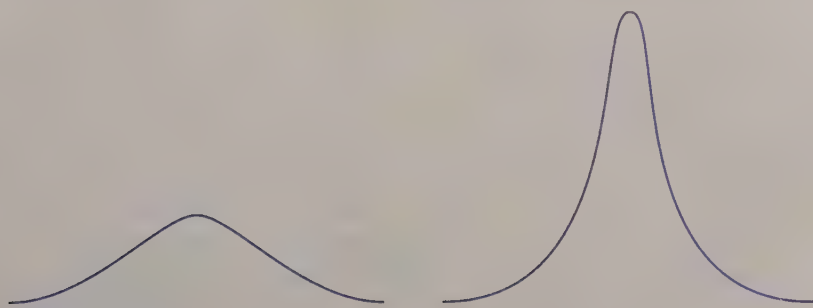


FIGURE 7.28

HISTORY

The beginning of modern statistics can be traced to the work of the Englishman John Graunt (1620–1674), who collected data on births and deaths. In his book *Natural and Political Observations of Mortality*, he presented such findings as the following: More boys are born than girls, more men than women die violent deaths, and the death rate in urban areas is greater than the death rate in rural areas.

Now let us examine what *standard deviation* means and how it helps us to make generalizations from sets of data that are normally distributed. Many of my students have told me that this investigation actually makes standard deviation make sense! For the time being, if you have had negative experiences with standard deviation, please suspend your judgment and pretend you have never heard of this idea before.

INVESTIGATION 7.14 Understanding Standard Deviation

Up to now, we have relied on graphs to help us get a sense of how spread out the data are. For example, returning to the batteries from Investigation 7.13, if we make histograms to represent the battery data (see Figure 7.29), we see that the data for Battery B are more spread out than the data for Battery A.

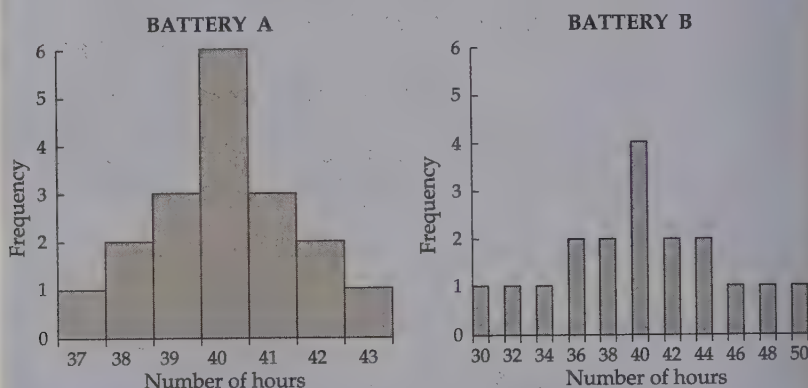


FIGURE 7.29

Up to now, the only number associated with spread has been the range. However, just as we have found that the center of some sets of data can be characterized by a single number (mean, median, or mode), we will develop the idea of one number that can give us a sense of the “spread” of a set of data by revisiting our data on the two batteries. As stated above, we can see that the data for Battery B are more spread out than the data for Battery A. How else might we describe how spread out the data in these two sets are? Think and then read on. . . .

DISCUSSION

One way to do so would be to note how far each datum is from the mean and then take the average (mean) of those numbers. This would give us an “average distance from the mean.” Recall that in both cases, the mean life of the batteries was 40 hours. Determine the “average distance from the mean” for each of the two classes and then read on. . . .

Table 7.15 shows the computations. If we look at how far each number is from the mean and find the mean of *those* numbers, we can say that for Battery A, the average distance from the mean is 1.2 hours, whereas for Battery B, the average distance is 4.0 hours. Just as the mean, median, and mode are concepts that enable us to give a sense of the middle of the data with a single number, “average distance from the mean” enables us to compare the relative spreads

TABLE 7.15

BATTERY A		BATTERY B	
Number of hours	Distance from the mean	Number of hours	Distance from the mean
37	3	30	10
38	2	32	8
38	2	34	6
39	1	36	4
39	1	36	4
39	1	38	2
40	0	38	2
40	0	40	0
40	0	40	0
40	0	40	0
40	0	40	0
41	1	42	2
41	1	42	2
41	1	44	4
42	2	44	4
42	2	46	6
43	3	48	8
		50	10
Total distance from the mean			72
Average distance		$20/17 = 1.2$ hours	
		$72/18 = 4.0$ hours	

of the data with a single number. Because 4.0 is more than 3 times 1.2, if someone were to tell you that the means from both sets of data were equal but that the average distance from the mean for Battery B was 4.0 hours compared to 1.2 hours for Battery A, this number would instantly tell you that the former set of data is more spread out.

Now the skeptical reader might be wondering, “Is it possible that real mathematicians do something this simple?” or “Why have I never heard the term ‘average distance from the mean’ before?” The answer to the first question is yes, and we will defer the second question temporarily. What we call the standard deviation is actually quite closely related to the average distance from the mean. For reasons that will be explained very shortly, the computation is slightly more complicated.

Finding the standard deviation To determine the **standard deviation**, we can use much of the work we did for average distance from the mean. There are two more steps (see Table 7.16).

To determine average distance from the mean:

Find the distance from the mean of each datum.

Add this column of numbers.

Divide this sum by the number of data.

To determine standard deviation:

Find the distance from the mean of each datum.

Square each of these numbers.

Add this column of numbers.

Divide this sum by the number of data.

Take the square root of this number.

TABLE 7.16

BATTERY A			BATTERY B		
Score	Distance from the mean	Square of the distance	Score	Distance from the mean	Square of the distance
37	3	9	30	10	100
38	2	4	32	8	64
38	2	4	34	6	36
39	1	1	36	4	16
39	1	1	36	4	16
39	1	1	38	2	4
40	0	0	38	2	4
40	0	0	40	0	0
40	0	0	40	0	0
40	0	0	40	0	0
40	0	0	40	0	0
41	1	1	42	2	4
41	1	1	42	2	4
41	1	1	44	4	16
42	2	4	44	4	16
42	2	4	46	6	36
43	3	9	48	8	64
			50	10	100
Sum	20	40		72	480
Sum/ <i>n</i>	Average distance = 1.2	2.35	Average distance = 4.0		26.7
Square root		Standard deviation = 1.53			Standard deviation = 5.16

Now that we have examined the meaning of the standard deviation and developed the procedure for determining it, you are more likely to be able to make sense of the mathematical definition of the standard deviation.

If we have a set of data whose values are denoted $x_1, x_2, x_3, \dots, x_n$, and the mean of this data is represented as $\bar{x}$, then the standard deviation is

$$\text{Standard deviation}^8 = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \dots + (x_n - \bar{x})^2}{n}}$$

In this case, the standard deviation for the data for Battery A is 1.53 compared to the standard deviation of 5.16 for Battery B. Like the average distances from the mean, the two standard deviations tell us that the data for Battery B are much more spread out. Once again, the skeptical reader will ask, "If they both give us a sense of spread, why not use the average distance from the mean, because it is easier to understand and compute?"

Using standard deviation The reason why we use the standard deviation has to do with some rather amazing generalizations that we can make about

⁸ Technically, because we are finding the standard deviation of a sample, as opposed to the whole population, we should determine the standard deviation by dividing by $n - 1$ instead of n , as we would do in an ordinary average. A full explanation of why one should divide by $n - 1$ is beyond the scope of this book. Our focus is on understanding the concept, so I have chosen to divide by n , as have authors of most elementary mathematics textbooks.

a set of data that is normally distributed. Let us look at those generalizations, make sure you understand them, and then see how we can apply these generalizations.

When a set of data is normally distributed, we can make the following conclusions (see Figure 7.30):

- 68% of the data lie within 1 standard deviation of the mean.
- 95% of the data lie within 2 standard deviations of the mean.
- 99.8% of the data lie within 3 standard deviations of the mean.

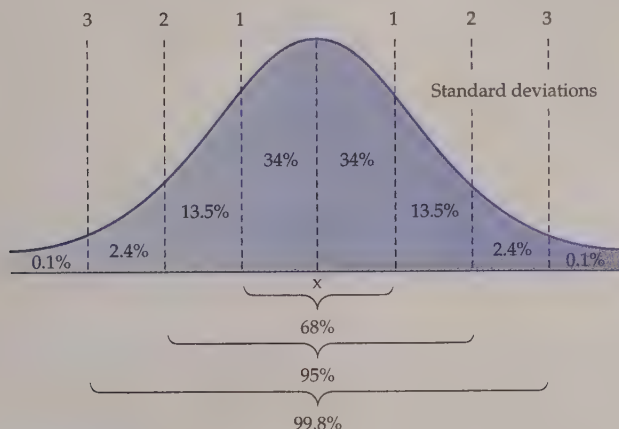


FIGURE 7.30

Let us apply these conclusions to our data for Battery A. Determine what percent of the data lie within 1 standard deviation of the mean. Then read on. . . 

In this case, the mean was 40 hours and the standard deviation was 1.5 hours. Thus, we are looking for how many data points lie between $40 - 1.5$ and $40 + 1.5$ —that is, between 38.5 and 41.5. We find that 10 of the 17 data points lie within this range, and $10/17$ is equivalent to 65%. If we look at the data for Battery B and find how many data points lie within 1 standard deviation of the mean, we are looking for data points that lie between $40 - 5.2$ and $40 + 5.2$ —that is, between 34.8 and 45.2. We find that 12 of the 18 data points lie within this range, and $12/18$ is 67%.

What about other distributions? When data are not normally distributed, then these generalizations are not as useful. We can summarize our knowledge of how to use standard deviation by saying the following:

- When data are not normally distributed, the standard deviation is a tool that enables us to describe how spread out the data are or to compare the spreads of two sets of data.
- When data are normally distributed, the standard deviation is a unit for measuring dispersion; that is, approximately 68% of the data will lie within 1 standard deviation of the mean, and so on.



HISTORY

There was a time when many (if not most) educators thought that grades should be normally distributed. Thus the grades were curved so that the “appropriate” proportion of students received grades of A, B, C, D, and F. In this course, I have found that the distribution of grades is often bimodal. That is, I often have a large proportion of students who do well (A), a large proportion of students who struggle to master the basic ideas (C), and a smaller proportion of students earning the other grades.

It is important to note that these numbers (68%, 95%, and 99.8%) are theoretical figures that are realized when the number of data is very large and when the data are “perfectly” normally distributed. However, even when the number of data is smaller and the distribution is not perfectly normal, we will generally find close to 68% of the data lying within 1 standard deviation of the mean.

Now that we have examined standard deviation, we can summarize the uses of measures of the center and of spread for different kinds of distributions. In general, the mean and standard deviation are used together to describe the center and spread of a set of data that is normally distributed. In general, the median and quartiles are used together to describe the center and spread of a set of data that is skewed.

Let us now consider some examples from real life that use the concept of standard deviation.

INVESTIGATION 7.15 Analyzing Standardized Test Scores

Let's say that a standardized test has a mean of 250 and a standard deviation of 50. Suppose 2000 students took the test and their scores were normally distributed. Think about the following questions, and then read on. . .

- A. How many students would you expect to score over 350?
- B. How many students would you expect to score at least 200?

DISCUSSION

A. Because a score of 350 is 2 standard deviations above the mean, we conclude that 2.5% of the scores will be over 350 [Figure 7.31(a)]. Then we simply need to compute 2.5% of 2000. We can use a calculator, or we can just as quickly do it mentally: 10% of 2000 is 200, so 5% is 100, and 2.5% is 50.

B. This question is slightly more complex. The answer is that about 1680 students will score at least 200. If you got this wrong, it is possible that you may understand the procedures but not the concepts related to standard deviation. Figure 7.31(b) illustrates one solution path. If we think of the results of the test in terms of a histogram, we can see that all the shaded bars represent scores over 200. However, when we superimpose this histogram on the graph showing the percent of scores with respect to standard deviations, we find that this means that 84% of the students scored at least 200.

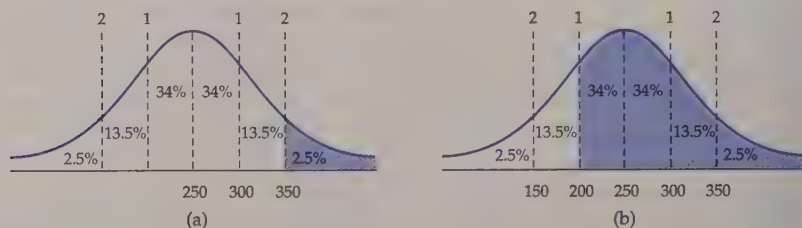


FIGURE 7.31



MATHEMATICS

When the SAT was first introduced, it was calibrated so that it had a mean of 500 and a standard deviation of 100. Over the past 20 or so years, the average score has declined considerably. In 1994, the Educational Testing Service proposed adjusting the test so that the mean would once again be 500. However, this proposal provoked quite a controversy. Why do you think the ETS made the proposal? Why do you think some people objected? Think about these questions before reading on. . . .



The rationale given by the ETS was that if 500 were the “average” score, then the scores would be more meaningful. That is, if a student’s score was above 500, that student would know that the score was above average. Many people objected because such a move would make it appear that scores had not gone down.

The concept of standard deviation is used regularly by industries. Let us examine one such example.

INVESTIGATION 7.16 How Long Should the Tire Be Guaranteed?

A tire company has developed a new tire and has tested it extensively. The results of the tests showed that the “average” tire lasted 44,000 miles, that the distribution of wearing was normal, and that the standard deviation was 2500 miles. If the company guarantees that the tire will last 39,000 miles, what percent of the tires are likely to wear out before 39,000 miles and thus be subject to refund?



DISCUSSION

Again, a graph makes the solution to this problem much easier (see Figure 7.32). From the graph, we can conclude that 2.5% of the tires are likely to wear out before 39,000 miles.

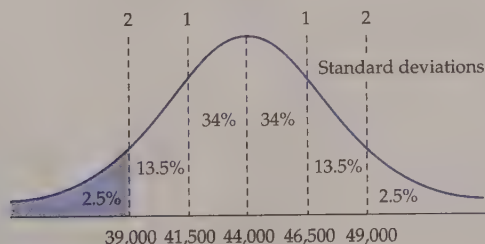


FIGURE 7.32

What if the company wanted to guarantee 40,000 miles? What percent of the tires are likely to wear out before 40,000 miles? In this case, you cannot get an “exact” answer. What could you do to increase the accuracy of your estimate? This question will be left as an exercise.



HISTORY

A classic example of a sampling error occurred in the 1936 presidential election. The *Literary Digest* sent questionnaires to 10 million voters. Of the 2,266,566 responses, 1,293,669 were for Landon and 972,897 were for Roosevelt. That is, more than 57% were for Landon. In electoral politics, if a candidate receives more than 55% of the vote, this is considered to be a landslide. History shows that, in fact, Roosevelt received 62% of the votes, himself winning by a landslide! What happened?

Analysis of the sampling process showed two flaws. First, the editors did not send the survey to a random sample of potential voters. They had gotten their sample (that is, the names and addresses) from telephone directories and lists of automobile owners. Second, less than one-fourth of the people responded to the survey. These two factors (the sample not being randomly determined and the response rate being too low) caused their error. Let us focus on the idea of obtaining a random sample. Why wasn't the *Literary Digest* sample a random sample? How could that have biased the data?

In those days, a much smaller fraction of the population owned telephones and/or cars. Therefore, the sample was not representative but, rather, was biased toward upper-income voters.

Sampling

In this section, we have developed tools that enable us to describe sets of data: measures of the center, measures of the spread, and graphs that can be useful. One of the applications of these tools is in sampling. As stated at the beginning of the chapter, people collect data on samples (opinion polls, product research, samples of a vaccine) and then make generalizations about the whole population. For example, if 55% of our sample favor Candidate X, then we believe that about 55% of the whole population will favor Candidate X. If 90% of the tires in the sample lasted more than 40,000 miles, then we believe that about 90% of all tires of this model will last more than 40,000 miles.

Engaging in opinion sampling is beyond the scope of this course. However, it is important that you have a basic understanding of this enterprise.

A **sample** is a subset of a population. If our sampling of the population is to give us accurate information, we must try to ensure that the sample is as representative of the total population as possible. This involves developing a method for selecting subjects (whether they be people, fish, or tires) that gives every member of the population an equal chance of being selected. That is, we want our sample to be a **random sample**.

Let's say that a large university wants to survey freshmen to see whether they are planning to transfer. There are currently 8000 freshmen, and the administrators decide to survey 250 of them randomly. How might they do so?

They could put all the students' names in a hat and draw out 250, but this would be very time-consuming. Computers are often used to accomplish the same purpose. For example, we could assign each student a number between 1 and 8000. We could next have the computer select the numbers 1 through 8000 in a random fashion. We would then match the first 250 numbers selected by the computer with our code, and these would be the students we would interview.

Sometimes we need a **stratified random sample**, a sample in which there are distinct subsets of the population that need to be proportionally represented. For example, if we are surveying students on campus and 25% of the students are married, we may want to make sure that 25% of our sample are married students. Consider one more example: Let's say that 60% of the voters in a district are white, 30% are black, and 10% are Hispanic. In this case, we will want to make sure that our sample contains these proportions also.

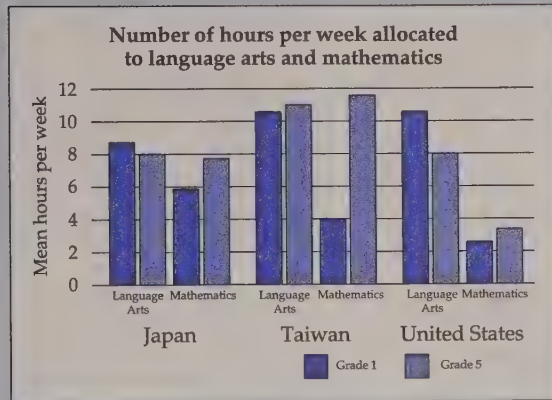
Let us consider a real-life situation involving sampling and apply our knowledge of statistics to better understand the authors' graphs and report.

INVESTIGATION 7.17 Comparing Students in Three Countries



Figure 7.33 is taken from a book entitled *Making the Grade in Mathematics: Elementary School Mathematics in the United States, Taiwan, and Japan*. Before reading on, look at the graph. Take some time to understand what information the authors are trying to present. What questions would you ask the authors in order to better understand the data? Then write down what you think they are saying. Then read on. . .





Source: Harold W. Stevenson, Max Lummis, Shinying Lee, and James W. Stigler, *Making the Grade in Mathematics: Elementary School Mathematics in the United States, Taiwan, and Japan* (Reston, VA: NCTM, 1990), p. 16. Reprinted with permission from *Making the Grade in Mathematics*, copyright 1990 by the National Council of Teachers of Mathematics. All rights reserved.

FIGURE 7.33

DISCUSSION

In their report, the authors state that three cities were selected: Minneapolis (U.S.), Taipei (Taiwan), and Sendai (Japan). They chose “representative samples of ten schools in each city. Within each school, we randomly selected two first- and two fifth-grade classrooms. . . . Thus, our report deals with data from 1440 children (240 first graders and 240 fifth graders in each of the three cities).”⁹ The data were collected in the early 1980s.

Note that there are several cases in which the authors needed to ensure that the samples were representative. All of these choices could affect the results: the choice of cities, the choice of schools, the choice of classrooms, and the choice of students.

For this investigation, we will focus on the differences among the countries in grade 5 mathematics. How would you summarize the differences? To give this question more context, assume that you are a writer for a national television network. The nightly newscast is going to have a feature story on the differences among the educational systems in these three countries. You are to write the story. At one point, this graph will appear in the background. What do you want the anchor to say? Think and then read on. . . . 

First, let us make sure we are interpreting the information correctly. The numbers refer to the numbers of hours per week allocated to mathematics. The graph indicates that these are means (as opposed to medians or modes). If we assume that the data are normally distributed, then all three measures will be close. Given the use of computers, it is easier to determine the mean than the median.

Now let us analyze the wording. Does “allocate” mean how much time is scheduled for mathematics, or does it mean how much time is actually spent

⁹Harold W. Stevenson, Max Lummis, Shinying Lee, and James W. Stigler, *Making the Grade in Mathematics: Elementary School Mathematics in the United States, Taiwan, and Japan* (Reston, VA: NCTM, 1990), p. 2.

on mathematics? I was not able to infer from the book exactly how the data for this graph were gathered. My interpretation (from reading the book and consulting the dictionary) is that the hours allocated to mathematics were determined by adding up the periods of time scheduled for mathematics in the teacher's plan book.

Now let us examine the hours allocated to mathematics each week in grade 5 and determine how we might convey that information to other people. Before we can make comparisons, we need to determine the actual times. Before reading on, determine the number of hours per week spent on math in grade 5 in each of the three countries. 

Looking at the graph, we can conclude that in Japan, the number of hours allocated is about 8; in Taiwan, the number of hours is about 12; and in the United States, the number of hours is slightly over 3, or about $3\frac{1}{2}$ hours.

You have several options for how you write the news item:

1. *Simply report the numbers.* "In the ten Japanese schools, almost 8 hours per week was allocated to math in the fifth grade, compared to almost 12 hours per week in the ten Taiwanese schools and about $3\frac{1}{2}$ hours per week in the ten U.S. schools."
2. *Make additive comparisons.* "The Japanese schools allocate about $4\frac{1}{2}$ hours more per week than the U.S. schools, and the Taiwanese schools allocate about $8\frac{1}{2}$ hours more per week than the U.S. schools."
3. *Make multiplicative comparisons.* "The Japanese schools allocate more than twice as many hours per week as the U.S. schools, and the Taiwanese schools allocate more than three times as many hours per week as the U.S. schools."
4. *Make multiplicative comparisons using percent language.* "The Japanese schools allocate about 140% more time than the U.S. schools, and the Taiwanese schools allocate about 225% more time than the U.S. schools."

I hope you have seen that even when examining a reputable study, we need to examine critically and question the data and graphs that we read. Given their understanding of statistical analysis, the authors took pains to make the data as representative as possible. They are therefore confident that if they had had more money and had been able to sample more students in more places, those results would have been close to the results that they found.

Weighted Average

We will consider one more concept in this section: *weighted averages*. In our work thus far with the mean, we have encountered situations in which we computed the simple mean. However, there are many situations in which we use what we call **weighted means**. This concept is a bit more complex and requires a thorough understanding of the concept of *mean*. Let us investigate some examples.

INVESTIGATION 7.18 Grade Point Average

 Table 7.17 gives Ed's semester grades. First estimate his grade point average (GPA), and then compute his actual GPA. Recall that A = 4 points, B = 3 points, C = 2 points, and so on. How can you explain and justify your estimation and computations? 

TABLE 7.17

Course	Grade	Credits
Mathematics 151	B	3
Philosophy 205	A	3
Computer Science 100	C	3
Biology 102	A	3

DISCUSSION

One way of estimating is to use the concept of balance; recall the seesaw analogy. Where would the balance point be in this case?

C	B	A
x	x	x
		x

Clearly it would be between the B and the A. If you place the wedge under the B, the seesaw will not balance there because the two As “weigh” more than the one C. Numerically, we can determine Ed’s average by taking the mean of the four grades:

$$\frac{(3 + 4 + 2 + 4)}{4} = 3.25$$

What if the computer course had been only a 1-credit course? What would Ed’s GPA be then? Think and read on. . . . 

Intuitively, many students realize that his GPA will go up, since the low grade is now worth only 1 credit—that is, it doesn’t “count as much” as the others. We can capitalize on this intuitive sense by connecting it to the idea of unit, which we have seen is one of the big ideas of mathematics. That is, we can count the B in mathematics (worth 3 points) three times, the A in philosophy 3 times, the C in computer science only once, and the A in biology three times: $(3 + 3 + 3) + (4 + 4 + 4) + 2 + (4 + 4 + 4) = 35$. We divide the sum by 10 (that is, the number of credits Ed is taking), and so we find that his GPA is 3.5 (see Table 7.18).

TABLE 7.18

Course	Grade	Numerical equivalent	Credits	Grade points
Mathematics 151	B	3	3	9
Philosophy 205	A	4	3	12
Computer Science 100	C	2	1	2
Biology 102	A	4	3	12
			10	35

$$\frac{\text{Total grade points}}{\text{Total credits}} = \text{GPA}$$

$$\frac{35}{10} = 3.5$$

We can understand the most efficient procedure for determining GPA by *connecting* our understanding that multiplication can be seen as repeated addition. We now multiply the numerical equivalent for each grade by the number of credits to get the grade points for each course. Then we add the grade points to get the total number of grade points for the whole semester and divide by the number of credits.

INVESTIGATION 7.19 What Does Amy Need to Bring Her GPA Up to 2.5?

Now let us use this knowledge to solve a problem many college students face: determining how their grades in the present semester will affect their overall GPA. At Keene State College, education majors need to have a 2.5 GPA in order to be able to student-teach. (You might want to check the policies at your college if you haven't already done so.)

Let's say Amy didn't do so well her first three semesters in college (Table 7.19). What GPA will she need this semester (she is taking 16 credits) in order to bring her overall GPA up to 2.5? Once again, estimate first. Then determine the exact answer. 

TABLE 7.19

	Credits	GPA
Beginning of the semester	50	2.35
Spring semester	16	?
Total	66	2.50

DISCUSSION

We can use the balance idea again. The 50 credits Amy has already taken will count more than the 16 credits she is taking this semester. Because the ratio of 50 to 16 is approximately 3:1, we can represent the problem by placing three weights at 2.35 and the balance point at 2.5 (see Figure 7.34). How far away must this semester's GPA be in order to balance the 2.35?

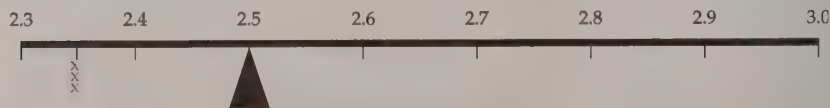


FIGURE 7.34

If we combine the idea of ratios and number lines with our "balance sense," we realize that the one weight (representing this semester) will have to be three times as far from 2.50 as is 2.35. Do you see why? If not, go out and play on a seesaw with some friends. Seriously! Therefore, we can estimate that she will need a GPA of about 2.95. Actually, because 16 is less than $1/3$ of 50, it will have to be a bit farther to the right in order to balance the 50 credits at 2.35.

One way to determine the exact answer is to determine how many total grade points she has now (2.35×50) and how many total grade points she needs if she is to have a GPA of 2.5 at the end of the semester (2.5×66). The difference of these numbers will tell us how many grade points her 16 credits this semester have to provide.

That is, $2.35 \times 50 = 117.5$ and $2.5 \times 66 = 165$. We now divide the difference of 47.5 by 16 (the number of credits she is taking this semester), and we

find that she must get a GPA of at least 2.97 to bring her GPA up to at least 2.5. As you can see, the estimate was very close.

Some students understand the solution better with a slightly different phrasing. At this point (50 credits), she has accumulated $50 \times 2.35 = 117.5$ grade points. At the end of the semester (66 credits), if she is to have a GPA of 2.5, she will need to have accumulated a total of $66 \times 2.50 = 165$ grade points. What does she need to accumulate this semester? She needs $165 - 117.5 = 47.5$ grade points. Now we compute: 47.5 grade points divided by 16 credits = 2.97.

INVESTIGATION 7.20 Course Grades Are Weighted Averages

Another area in which weighted averages are encountered is figuring one's grade in a course. Table 7.20 gives Amy's grades for a course, the weight of each component, and the grading scale for the course. What is her average for the course? Please work on this before reading on. . . . 

There are several different ways to determine her average.

TABLE 7.20

Component of course	Weight	Amy's grades
Homework	30%	72, 84, 50, 72, 85, 80
Exams	30%	75, 83, 97, 78
Quizzes	15%	60, 0, 100, 90, 80
Final exam	25%	81

DISCUSSION

We first need to determine her average (mean) for each component. We find that her homework average is 74, her exam average is 83, and her quiz average is 66.

STRATEGY 1: Use accumulated grade points

Using the same idea that we used in determining GPA, we see that the homework counts 30 times, the exams count 30 times, the quizzes count 15 times, and the final exam counts 25 times.

$$\begin{array}{ccccccc}
 \text{hw} & & \text{exam} & & \text{quiz} & & \text{final} \\
 (30 \times 74) + (30 \times 83) + (15 \times 66) + (25 \times 81) = \\
 2220 & + & 2490 & + & 990 & + & 2025 = 7725
 \end{array}$$

What does this 7725 mean?

It means the total accumulated points. Because the entire course is 100 percent, we have to divide 7725 by 100, and we get her average of 77.25.

STRATEGY 2: Apply proportional reasoning

Some students realize that what is crucial in the computation is the *proportion* of each component. The ratios of the four components of the course are 30:30:15:25, and we can simplify these ratios to 6:6:3:5. Even though the weight of each component is smaller, the ratios are the same; for example, $30:15 = 6:3$. Can you use this idea to compute her average now? Do so before reading on. . . . 

$$(6 \times 74) + (6 \times 83) + (3 \times 66) + (5 \times 81) = 1545$$

What must we divide 1545 by to determine her average? Think and read on. . . . 

The sum of the weights is 20, and $1545/20$ gives us 77.25.

The exercises will examine some of the many nuances of determining students' averages.

Summary

In this section, we have investigated a variety of tools that enable us to analyze different sets of data. We have added stem plots, grouped frequency tables, histograms, line plots, and boxplots to our graphing toolbox. We have seen similarities between stem plots and histograms and similarities between line plots and boxplots. We have seen that when the range of a set of data is great, it is often helpful to select intervals and make a grouped frequency table, from which we can then make a histogram to give us a visual sense of the data.

We have learned about the three candidates for the “center” of a set of data: mean, median, and mode. We have discussed when each of these is an appropriate representation of the center and when it is not. We have found that these numbers, just like people, are not always good representatives! You encountered dilemmas of this nature if you did Exploration 7.3 (Typical Person).

We have also examined different ways to quantify the dispersion (or spread) of a set of data, from simple ideas like range, clusters, gaps, and outliers to more complex ideas like interquartile range and standard deviation. We have seen that the mean and standard deviation can be used together to describe the center and spread of a set of data that is normally distributed, and that the median and quartiles can be used together to describe the center and spread of a set of data that is skewed.

One chapter in a book called *Benchmarks for Scientific Literacy* focuses on the kind of mathematical knowledge especially useful in scientific inquiry. Below is an excerpt from the description of the students' need to be able to interpret data:

One common misunderstanding is the belief that averages are always highly representative of a population; little or no attention is given to the range of variation around averages. . . . Because there is a persistent misconception, even in adults, that means are good representations of whole groups, it is especially important to draw students' attention to the additional questions, “what are the largest and smallest values?” and “how much do the data spread on both sides of the middle?”¹⁰

We have touched upon the notion of sampling. The actual practice of gathering data on samples of a population in order to make generalizations about the entire population is rather complex. However, we have explored some of the issues that are encountered in this practice. Taking measures to ensure that the sample is as representative of the total population as possible involves taking measures to ensure that the samples that are obtained are random samples. In cases where there are distinct subgroups within the population, this often involves stratified random samples.

Finally, we examined the notion of weighted means. We connected the computation of weighted means and simple means, and we connected the concept of both kinds of means—that is, the notion of the balance point of the set of data.

¹⁰ American Association for the Advancement of Science, *Benchmarks for Scientific Literacy* (New York: Oxford University Press, 1993), pp. 226–227.

EXERCISES 7.2

1. Suppose the instructor in Investigation 7.12 gave the same test to another class whose scores are given below. Tell the instructor how the students did in each class and how the classes compare.

Class 2: 96, 94, 93, 92, 89, 85, 85, 85, 83, 80, 79, 77, 77, 77, 75, 74, 74, 68, 67, 65, 65

2. Below are the heights of adults in a college class.
- Make a histogram for these data.
 - Determine the mean and standard deviation.
 - What percent of the values lie within 1 standard deviation of the mean?

Height (inches)	Frequency
59	1
60	1
61	3
62	8
63	11
64	13
65	17
66	14
67	11
68	7
69	4
70	2
71	1
72	1

3. Below are the ages of students in a class in mathematics for elementary teachers:

18, 18, 18, 19, 19, 19, 19, 19, 19, 20, 20, 20, 21, 21, 21, 21, 21, 23, 24, 24, 26, 27, 37, 42, 43, 46

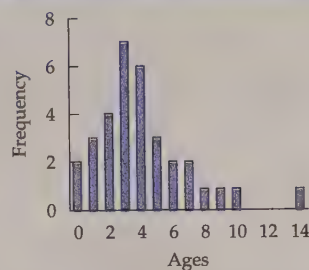
- Use one or more graphs to show the data.
- What is the average age? Justify your answer.
- Describe the spread of the data.

4. Below are the annual salaries (in thousands of dollars) of workers in Bates Ball Bearing factory:

12, 12, 13, 13, 13, 16, 16, 16, 16, 18, 18, 18, 21, 21, 21, 26, 26, 28, 31, 51, 51, 75

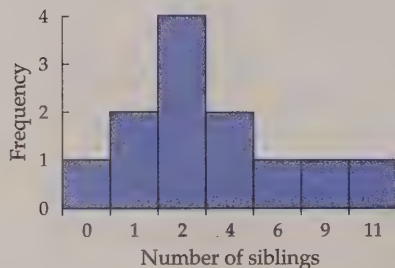
- Use one or more graphs to show the data.
- What is the average salary? Justify your answer.
- Describe the spread of the data.

5. The teachers at Gates Memorial High School got to talking about cars and how long they keep them. Someone wondered about the average age of the teachers' cars. Following is a graph representing the ages, rounded to the nearest year.

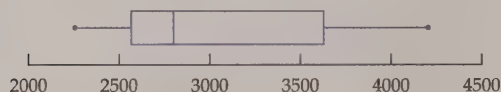


- Before computing, predict the mean and the median. Explain your reasoning.
 - Determine the mean and the median.
6. When doing Exploration 7.3 (Typical Person), you found that you have to think about how to word a question. For example, if we simply ask, "How many brothers and sisters do you have?" there might be problems with half-siblings and step-siblings, adopted family members, and so on. Assume that you are in a group that has decided to ask a question about how much time people study. Describe the actual question you would ask and explain the thinking behind your phrasing. That is, convince the reader that the respondent will answer the question you think you are asking and that the data you get will be meaningful and analyzable (as opposed to problematic, as in the example cited above).
7. Another group in Exploration 7.3 asked, "How many drinks did you have in the past week?" Here are the data: 0, 0, 0, 0, 0, 0, 1, 2, 2, 3, 5, 5, 8, 12, 12, 15, 10–15. Imagine you are in that group and have been asked to present your conclusions to the class. In other words, what did you learn about the drinking habits? This includes making a graph—it can be a sketch, as opposed to a polished, precise graph. *Note:* You will need to decide what to do with the 10–15 datum and to justify your decision.
8. In Exploration 7.3, a group in a previous class asked the following question: "What is your favorite sport?" Here are their data:
- | Sport | Frequency |
|--------------|-----------|
| Basketball | 5 |
| Volleyball | 4 |
| Soccer | 3 |
| Tennis | 3 |
| Softball | 3 |
| None | 2 |
| Skiing | 2 |
| Rugby | 2 |
| Horseback | 1 |
| Gymnastics | 1 |
| Swimming | 1 |
| Cheerleading | 1 |
- Select and make an appropriate graph.
 - Justify your choice of graph.
 - Critique the question. If you feel it is fine, explain why; if you feel it has problems, explain why and then suggest a rewording.
 - Determine the mean, median, and mode. If any of these three is inappropriate, explain why.
9. Mr. Arnold asked his students how many brothers and sisters they had. The graph following summarizes what he found.

- Determine the mean and the median number of siblings.
- There is something wrong with the graph. Explain not only what is wrong with the graph but also why it is wrong.



- The graph below is a box-and-whisker graph for the lengths (in miles) of the 18 longest rivers in the world. What information does the graph give us?



- Willie has an average (mean) of 83.4 on the first five exams. What is the least he can score in order to have an average of 85 for all six exams?
- John's bowling average is 152. The other night, John's scores on the first two games were 132 and 138. What does he need to bowl on the third game in order to maintain his average?
- In driving to a nearby town, it took Maurice 27 minutes to go 18 miles. What was his average speed?
- In a special triathlon, Betty swam 1 mile in 25 minutes, biked 25 miles in 45 minutes, and then ran 5 miles in 40 minutes. What was her "average" speed for the race?
- Lucy took three tests. If her median score was 82, her mean score was 87, and the range was 17, what were her three test scores?
- The mean of five numbers is 6. If one of the five numbers is removed, the mean becomes 7. What is the value of the number that was removed?
- On a certain exam, Tony corrected 10 papers and found the mean for his group to be 70. Alice corrected the remaining 20 papers and found that the mean for her group was 80. What is the mean of the combined group of students?
- Create a set of 10 test scores for which the mean is higher than the median.
- A student scores 16 points on a 20-point quiz and 35 points on a 50-point quiz. The professor determines her average in the following unconventional way. What do you think of this system?

$$\frac{16}{20} + \frac{35}{50} = \frac{51}{70}$$

- I recently read an article in which the following sentence appeared: "The average cost of a car is \$16,000." How do you think that number was determined? In your answer, you need to state whether you think that figure represents the mean, median, or mode.
- How far above or below the average temperature do the numbers need to be in order for people to refer to a month as a hot month or a cold month?
- We found that in a normally distributed set of data, the mean, median, and mode are virtually identical. However, this is not necessarily true for other distributions. What about data whose distribution is skewed to the right? Can you predict the numerical order of the three averages? For example, will the mean be largest, followed by the median, followed by the mode? Or do you think "it depends"? Justify your answer.
- Explain the similarities between a line plot and a box-plot. You are encouraged but not required to use data from a set of data such as that in Investigation 7.10 or 7.11.
- There is another way to figure the mean of a set of data. Here's how it works. Let's say we want to find the mean weight of these boxes of fruit: 26, 27, 38, 35, 34, 29. I estimate the average as 30 and then add or subtract the distance of each datum from my estimate. In this case, the arithmetic would be

$$-4 + -3 + 8 + 5 + 4 + -1 = +9$$
 Next, $9/6 = 1.5$, and so the mean is 31.5.
 - Explain why this procedure works.
 - Make up another set of data and use this strategy to determine the mean.
- Make up three questions such that in each case one of the centers (mean, median, or mode) would be most appropriate. In each example, all of the three must be able to be determined. ("What is the hair color of children at Riverside Elementary School" is not acceptable because the mean and median cannot be determined.) Explain why the center you picked is, indeed, the most appropriate.
- A student organization at Keene State College had an essay contest for Hunger and Homelessness Week one year. Students were asked to share their views on the following question: "The average age of a person experiencing homelessness is 9 years old. How does that make you feel and what would you do about it?"
 - Which of the centers (mean, median, mode) do you think were used here?
 - Explain how this statement could be true or argue that it is impossible.
 - Research homelessness at the library or on the Web. Present the data that you find.
- Suppose the distribution of heights of a group of 400 children is normal, with a mean of 150 centimeters and a standard deviation of 12 centimeters. About how many of these children are taller than 162 centimeters?

28. The heights of 1500 boys at West High School were measured, and the mean was found to be 66 inches, with a standard deviation of 2.5 inches. If the heights are approximately normally distributed, 95% of the boys are between ____ and ____ inches tall. About how many of the boys are less than 5 feet tall? *Note:* You cannot get an exact answer here. Any answer will involve some estimating.
29. Let's say a student scored 45 on a math achievement test for which the mean was 36 and the standard deviation was 6. Based on this performance, what score would you predict for her on a standardized test for which the mean is 100 and the standard deviation 15?
30. What happens to the mean and the standard deviation of a set of data when the value of each datum is increased by the same amount?
31. A tire company tested a particular model of super radial tire and found the tires to be normally distributed with respect to wear. The "average" (mean) tire wore out at 59,000 miles, and the standard deviation was 2500 miles.
- If 2000 tires are tested, about how many are likely to wear out before 54,000 miles?
 - What if the company wanted to guarantee 55,000 miles? What percent of the tires are likely to wear out before 55,000 miles? In this case, you cannot get an "exact" answer. What could you do to increase the accuracy of your estimate?
32. A tire company tested another model of tire and found the tires to be normally distributed with respect to wear. The "average" (mean) tire wore out at 46,000 miles, and the standard deviation was 2400 miles. If 2000 tires are tested, about how many are likely to wear out before 45,000 miles? You cannot get an exact answer. I want your best estimate. Please use graph paper to help you estimate.
33. Jack and Jill go to different schools. Jack got an 82 on a test with a mean of 60 and a standard deviation of 15. Jill got a 78 on a test with a mean of 60 and a standard deviation of 9. If both tests are graded on a curve, who did better?
34. Wendy took three standardized aptitude tests: English, mathematics, and general information. The table below gives her score, the mean, and the standard deviation for each test. On which test did she do the "best"? Justify your answer.

Test	Wendy's score	Mean	Standard deviation
English	85	75	7.5
Math	63	55	6
General information	104	94	8

35. Compute Ed's GPA, to two decimal places, for the semester ($A = 4$, $B = 3$, $C = 2$).

Course	Grade	Credits
Mathematics	B	3
Elementary Methods	A	6
Computers in School	C	1
Biology	B	4

36. Perry has a 2.53 GPA after 112 credits. This is his last semester, he is taking 15 credits, and he has senioritis.
- What is the minimum GPA Perry can get and still graduate with at least a 2.5 GPA?
 - He is taking five 3-credit courses. Give one scenario that will let him "squeak by."

Grade

Course 1
Course 2
Course 3
Course 4
Course 5

37. Joe has a 3.22 GPA after 46 credits. He says that he has a total of 148.12 grade points. Julie says that's impossible. What do you think?
38. Below are Amy's grades for each component of the course, the weight of each component, and the grading scale for the course.

Component of course	Weight	Amy's grades
Attendance/class participation	15%	78
Homework	30%	70, 84, 40, 82, 85, 60
Exams	25%	75, 83, 87, 78
Quizzes (10-point quizzes)	5%	5, 0, 10, 9
Final exam	25%	81

- What is Amy's overall average for the course?
 - What would her overall average have been if her grade on the homework on which she got a 40 had been an 82?
 - How much did the missed quiz hurt her average? Assume she would have gotten an 8 instead of the zero.
39. Refer to Investigation 7.20, in which we determined Amy's average for the course. One student says that I should not have rounded. For example, her homework average is really 73.83. What do you think? Could rounding, rather than using the exact scores, actually affect the student's grade?

40. The following scenarios have been taken from the *Instructor's Course Planner* for the textbook *Children*.¹¹ In each case, do you think the sample is a random sample or a biased sample? Justify your response.
- The researchers asked teenage boys about their driving records and habits by going to a movie drive-in, a local bar, the beach, and a baseball park.
 - Parents at a PTA meeting were interviewed about the quality of the local public school system.
 - A telephone survey assessed a community's attitudes toward welfare recipients.
 - Children were asked their opinion of Santa Claus on December 28.
41. Why do you think many statisticians recommend that when describing a population, one should give at least three pieces of information: some sense of the "average," some sense of the spread, and some sense of how the data are distributed? For example, they would recommend not simply reporting that the average age of teachers in New Hampshire is 43 or that the average adult American goes to church 1.3 times a month (both figures are made up).
42. Several secretaries in an academic building on a college campus share a copy machine. They believe the copy machine is too slow and want the college to buy a newer high-speed copy machine. Describe what kind of data they should collect and how they should present their data so as to best support their case.

In problems 43–48, you are given data from various surveys that I have read about. In each case:

- State at least two questions you would ask the people who did the survey to satisfy yourself of the validity and reliability of the survey. Briefly explain the reasoning behind your questions.
- Write the actual question that you would ask if you were to conduct a similar survey.
- Make a graph for these data.
- Justify your choice of graph.
- What does the graph add to your understanding of the question? If you feel the graph adds nothing, explain why not.

43.

PERCENTAGE OF TELEVISION HOUSEHOLDS WITH:	
Number of TVs/VCRs	Percent
Two or more TVs	74
One TV	26
Two or more VCRs	26
One VCR	74

Source: Roper Organization survey

44.

HOW PARENTS WAKE UP THEIR CHILDREN	
Response	Percent
Call to them	43
Alarm Clock	22
Kids wake on their own	16
Other	19

Source: Aunt Jemima survey poll of 400 parents of children under the age of 18

45.

GENDER OF CHILD MOST PEOPLE THINK IS EASIER TO RAISE	
Response	Percent
Boy	43
Girl	27
No difference	23

Source: Gallup survey of 1239 adults

46.

HOW OFTEN FAMILIES EAT DINNER TOGETHER	
Nights per week	Percent
0	5
1 to 2	17
3 to 4	26
5 to 6	30
7	22

Source: BKG Youth survey for Kodak

47.

HOW MANY FRIENDS PEOPLE SAY THEY HAVE	
Number of friends	Percent
0	1
1	2
2 to 5	36
6 to 10	25
11–20	18
More than 20	18

Source: MCI/Louis Harris survey

¹¹ Allen Keniston and Blaine Peden, *Instructor's Course Planner for Children*, by John Santrock, (Dubuque, IA: Brown & Benchmark, 1995), p. 36.

48.

AVERAGE NUMBER OF COLDS PER YEAR	
Age group	Number
0–4	5
5–19	3
20–39	2
40 plus	2

Source: University of Michigan

SECTION

7.3

CONCEPTS RELATED TO CHANCE

WHAT DO YOU THINK?

- How can we figure out the probability that something will happen before it actually happens?
- How can we express that probability?

The study of probability . . . should not focus on developing formulas or computing the likelihood of events pictured in texts. Students should actively explore situations by experimenting and simulating probability models. . . . Students should talk about their ideas and use the results of their experiments to model situations or predict events.

(Curriculum Standards, p. 109)

If you did one or more of Explorations 7.7 through 7.11, you have conducted experiments and simulations and grappled with the fundamental probability ideas that we will now examine more formally.

Stop for a moment and think of situations in which probability enters into our lives. Then read on. . . . 

We make and are affected by probability decisions every day. For example,

- If you decide to have two children, what is the probability that you will have a boy and a girl?
- What is the probability that you will get a teaching job after you graduate?
- If you are independently employed, should you buy health insurance? People's decisions are influenced by their estimate of the probability that they will have a catastrophic illness.
- If you drive over the speed limit or decide not to put money in the parking meter, what is the probability of your getting a ticket?
- What is the probability that everyone who buys an airline ticket will actually take the trip? Airlines regularly sell more tickets than they have seats because they know that the probability of this is very low.

These situations all have in common that their results are unknown. Much of our understanding of probability is based on collecting and analyzing data on occurrences of situations that, individually, are random. Analyzing the data helps us to see patterns in these occurrences and to make predictions. For example, because the probability that a 21-year-old male will have an automobile accident is much greater than the probability that a 50-year-old male will have an accident, car insurance rates for 21-year-old males are higher than those for 50-year-old males.

Our work with data that are normally distributed connects to probability. For example, if a tire company's sample of tires lasted an average of 42,000 miles with a standard deviation of 2000, the company can say that there is an 84% probability that a specific tire will last at least 40,000 miles.

Random phenomena It is important to note the way in which mathematicians use the word *random*. When mathematicians use the word *random*, we are referring to phenomena that are unpredictable individually but that have regular patterns when considered as a group or when considered over the long run. *Random* is not a synonym for *haphazard*; rather, it refers to phenomena for which we cannot make individual predictions. The following example nicely illustrates this point. If we drop a coin from a certain height, we can predict with a great deal of precision the time it will take to reach the ground. However, if we flip that coin, we can give only the probability that it will land heads or tails. We do not say that flipping a coin is haphazard but rather that it is random.¹² When we flip many coins, we find that there are many patterns, as you will see in this section.

In this section, we will investigate both well-defined questions (for example, the probability of tossing 4 heads in a row) and questions that are not as well defined (for example, insurance premiums).

Preliminary Terms and Concepts

We hear probability statements all the time. Let's say you turn on the television and hear the weather forecaster say, "There is a 25% chance of rain tomorrow." What does that statement mean? Write your thoughts before reading on. . . . 

When we make a probability statement, we are giving a numerical value that represents the degree to which we believe that event will or will not happen.

Probabilities can be represented by fractions, decimals, percents, ratios, and odds. The following five statements are equivalent:

- There is a 25% chance of rain tomorrow.
- The probability of rain tomorrow is $1/4$.
- The probability of rain tomorrow is 0.25.
- There is 1 chance in 4 of rain tomorrow.
- The odds against rain tomorrow are 3 to 1.

Outcomes and events There are two terms whose definition will make our discussion about probabilities much clearer: *outcome* and *event*. Answer the following two questions and then read on. . . . 

- If you select a card from a regular deck of playing cards, what is the probability of drawing the queen of spades?
- If you select a card from a regular deck of playing cards, what is the probability of drawing any queen?

The probability of drawing the queen of spades is $1/52$, and the probability of drawing any queen is $4/52$, or $1/13$.

We often use notation as a shorthand to express probabilities:

$$P(\text{Q of spades}) = \frac{1}{52} \quad P(\text{Q}) = \frac{1}{13}$$

We refer to "queen of spades" as an outcome and "any queen" as an event. Let us examine the difference between the two. When we define a situation (in this case, drawing a card from a deck), each possibility is called an **outcome**.

¹² David Moore, "Uncertainty" in *On the Shoulders of Giants: New Approaches to Numeracy*, ed. Lynn Steen (Washington, D.C.: National Academy of Sciences, 1990), p. 98. Reprinted with permission from *On the Shoulders of Giants: New Approaches to Numeracy*. Copyright © 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington D.C.

The set of all possible outcomes is called the **sample space**. Thus, each outcome is an element of the sample space. Within the sample space, there are many possible subsets—for example the subset E :

$$E = \{\text{queen of diamonds, queen of spades, queen of hearts, and queen of clubs}\}$$

Any subset of a sample space is called an **event**.

When All Outcomes Are Equally Likely

In this case, we are examining situations in which each outcome is **equally likely**. When we are examining a situation in which all outcomes are equally likely, we can determine the probability of an event by computing the following ratio, expressed here both in probability language and in set language. If all outcomes in a sample space are equally likely, then the probability of event E is given by

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{number of total outcomes}} \quad P(E) = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

When All Outcomes Are Not Equally Likely

On many occasions, all outcomes are not equally likely. Consider the spinner in Figure 7.35. In this case, there are four outcomes: 1, 2, 3, 4. However, they are not equally likely. In this case, what is the probability of spinning an odd number?

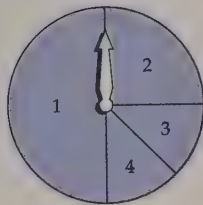


FIGURE 7.35

$$P(\text{Odd}) = P(1) + P(3) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

We can generalize from this example to state the probability of an event when all outcomes are not equally likely:

The probability of an event is equal to the sum of the probabilities of all outcomes in that event.

That is, if event E consists of n different outcomes $O_1, O_2, O_3, \dots, O_n$, then $P(E) = P(O_1) + P(O_2) + P(O_3) + \dots + P(O_n)$.

Note that this more general definition of probability works with both kinds of situations.

When we speak of the probability of an event happening, at one extreme we have **impossible events**—for example, the probability of a student having a GPA of 4.4. At the other extreme, we have **certain events**—for example, the probability that it will rain this year in Jacksonville, Florida. If an event is impossible, we say that the probability is 0. If it is certain, we say that the probability is 1. Thus, the probability of any event is between 0 and 1.

Theoretical and Experimental Probabilities

When working with probability situations, we distinguish between theoretical probabilities and experimental probabilities. In some situations involving probabilities, we can determine the theoretical probability—for example, the probability of a couple having 5 boys in a row or of rolling doubles 3 times in a row when playing Monopoly. In situations where we can determine the total number of outcomes and we can count the number of outcomes in a specific event, we refer to the probability as the **theoretical probability**. That is, the theoretical probability refers to our expectation, assuming that things turn out ideally. For example, we can say that when we flip a coin, theoretically the probability of its being heads is $1/2$.

There are many real-life situations in which it is either impossible or very expensive to determine theoretical probabilities. In those situations, we

determine the experimental probability by collecting and analyzing data. For example, manufacturers determine the probabilities that their products will last a certain time by testing some of the models. For example, if 90% of the sampled tires last at least 40,000 miles, the manufacturer will assume that close to 90% of all the tires of that model will last at least 40,000 miles. In this case, we would say that the experimental probability of a tire's lasting at least 40,000 miles is 90%. In situations in which the probability of an outcome or event has been determined by collecting data and determining the fraction of the time in which the outcome or event actually occurred, we refer to that fraction as the **experimental probability**. Some books and authors use the term **empirical probability** instead.

Let us now investigate some situations to deepen our understanding.

INVESTIGATION 7.21 Probability of Having 2 Boys and 2 Girls

Let's say a couple is planning to begin their family and would like to have 4 children. What is the probability that they will have 2 boys and 2 girls? What do you think? What problem-solving tools might help? Take some time to work on this question before reading on. . . . 

DISCUSSION

Three common initial hypotheses are 50%, 20%, and 37.5%. Let us investigate each of these.

Those who answer 50% reason something like this: Each time, there is a 50% chance of getting a boy or a girl, so it makes sense that the "balanced" possibility (same number of boys and girls) will take place 50% of the time.

Those who answer 20% reason something like this: There are 5 possibilities (events)—4G0B (that is, 4 girls and 0 boys), 3G1B, 2G2B, 1G3B, 0G4B. Thus, 2G2B is 1 out of 5, which is 20%.

Those who answer 37.5% reason something like this: There are 16 possible outcomes (BBBB, BBBG, BGBB, and so on), and 6 of the 16 involve 2 boys and 2 girls. Therefore, the probability of having 2 boys and 2 girls is $6/16 = 3/8 = 37.5\%$.

What do you think now?

Rather than present the answer just yet (presenting the answer too quickly means that more students are more likely to rent the solution rather than own it), let us do a **simulation**. This is what scientists, economists, and mathematicians often do when they are beginning an investigation or when they have different hypotheses: They construct a model of the situation and then run a simulation. How might we run a simulation of this question? Please think for a minute yourself and then read on. . . . 

There are many ways to simulate this scenario:

1. We could write a computer program using a random number generator. That is, we could ask the computer to select randomly between two numbers, such as 0 and 1, and then translate the 0 as a boy and the 1 as a girl. We could program the computer to generate and translate four numbers at a time and then translate that outcome as 4 boys, as 3 boys and 1 girl, as 2 boys and 2 girls, and so on.
2. We could roll dice: If we roll a 1, 2, or 3, it's a girl; if we roll a 4, 5, or 6, it's a boy. We could also tape a B on three sides of a die and tape a G on the other three sides. We could then roll one die 4 times, or we could roll 4 dice and arrange them in a line.

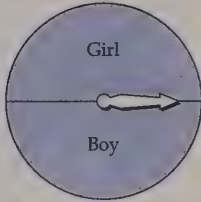


FIGURE 7.36

3. We could write B and G on the same number of slips of paper and put them in a hat and then take slips out. This would be slower and has one other potential problem. What is it?

We would need to put the paper back after each draw. For example, we could put a B on 5 pieces and a G on 5 pieces. If we picked a G and didn't put it back, the probability of getting a B would then be 5/9, and the probability of getting a G would be 4/9.

4. We could label a spinner (see Figure 7.36). Every four spins would represent one random outcome, such as BBGG.

Your instructor may have you do this simulation in class. If not, do it yourself. As you may have learned from Explorations 7.7 through 7.11, we can use simulations not only to help us determine the experimental probability but also to understand better how to develop the theoretical probability.

With the knowledge gleaned from the simulation, let us now explore how the theoretical probability is determined.

STRATEGY 1: Be systematic and look for patterns

There are many ways to be systematic. Let us examine one way.

First, the couple could have all girls or all boys: GGGG and BBBB.

Note that we use letters to represent boy and girl rather than spelling them out. This saves time and makes patterns in the data easier to see.

Next, we could determine the different ways in which the couple could have 3 girls and 1 boy. If you did not do the problem this way, stop now and see whether you can determine the answer to this particular question. . . . 

One way to do this systematically is to see that the boy could be in four different positions: oldest, second, third, youngest. Symbolically, this looks like

B G G G
G B G G
G G B G
G G G B

Do you see a pattern here? How would you describe it?

If you look only at the B, you can see it move diagonally. Do you see that this systematic approach also practically gives us the ways to have 3 boys and a girl?

If you stop and think for a minute, you might realize that because there are four different ways to have 3 girls, there must also be four different ways to have 3 boys. Do you see why? There are many symmetries in probability situations.

Finally, we need to determine how many different ways there are to have 2 boys and 2 girls. Do this first yourself before reading on. . . . 

There are a variety of methods you might use to come up with the 6 different ways in which one could have 2 boys and 2 girls. (In probability language, we are saying that the event 2B, 2G consists of 6 outcomes.)

B G B G
G B G B
B G G B
G B B G
B B G G
G G B B



CHILDREN

When my daughter and son were small, they would come into our bed when they woke up. One morning, my daughter noted that there was a pattern. Our position was Emily, Dad, Mom, Josh. She said, "Girl, boy, girl, boy." She then jumped over me and said, "This is another pattern." I was expecting her to say, "Boys on the outside, girls on the inside." What she said was, "Oldest and youngest on the outside." I am older than my wife (by three months, which was amusing to our children when they were young: Sometimes I was "older" than my wife and sometimes we were the "same" age), and Emily is older than Josh.

Because 6 of the 16 outcomes have 2 Bs and 2 Gs, the probability we sought is $6/16$, or 37.5%.

STRATEGY 2: Make a tree diagram

Figure 7.37 is a tree diagram for this problem; it is a different way to represent and solve the problem. Can you interpret the tree diagram? Do you see why it is called a tree diagram? How is the tree diagram made? How is this similar to other diagrams used in the book? 

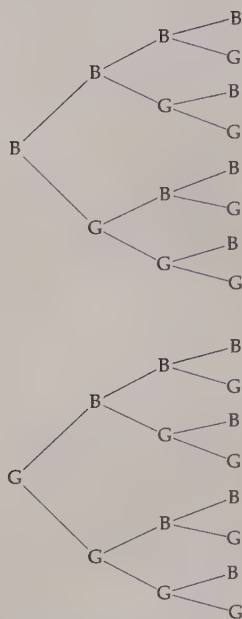


FIGURE 7.37

Now make a table of the 16 outcomes by using the tree diagram. (The table is not shown—you have to make it!) What patterns do you see in the table? Write the patterns you see and then read on. . . . 

There is a pattern in each of the columns. The right column simply alternates between B and G. The next column alternates between 2 Bs and 2 Gs. The third column alternates between 4 Bs and 4 Gs. The last column consists of 8 Bs and 8 Gs.

Some students see the symmetrical nature of the table. Pair each row in the table in the following way: row 1 and row 16, row 2 and row 15, and so on. Another way of seeing this is to fold the table in half and look at the rows that lie on top of each other. Each half is like a mirror image of the other: BBBB and GGGG, BBBG and GGGB, and so on.

Note that this table is not identical to the table that comes from Strategy 1, although they contain the same 16 outcomes. The other table exhibited different patterns. What is important here is that both tables represent a systematic approach to the problem. If a student were to generate the 16 outcomes in a more random fashion, there would probably be fewer patterns and hence a greater probability of missing one or more outcomes.



MATHEMATICS

Blaise Pascal, whom we have encountered before, once said, "Chance favors the prepared mind." A powerful illustration of this occurred one year while my students were investigating this question. Some students noted that since the letters G and B are similar, it was hard to see patterns. One student replaced the Gs with 0 and the Bs with 1. The contrast between 0 and 1 is much greater than that between B and G. The student then listed all 16 outcomes on the board, noting that 0 represented B and 1 represented G. This notation made it easier to discuss patterns and also provoked another discovery. Does Table 7.21 remind you of something you have studied before?

TABLE 7.21

0000
0001
0010
0011
0100
0101
0110
0111
1000
...
1111

This table is simply the first 16 numbers in base 2!

Let us now examine two related questions for this couple, from which new probability concepts will emerge.

INVESTIGATION 7.22 Probability of Having 3 Boys and 2 Girls

What if the couple were planning to have 5 children? What is the probability of their having 3 boys and 2 girls? Work on this and then read on. . . . 

DISCUSSION

There are several different ways to answer this question. You can make a new table, extend the tree diagram, or extend the base 2 counting table and look for outcomes that show two 0s and three 1s or three 0s and two 1s. However, in any of these cases, obtaining the solution is relatively cumbersome. It becomes downright tedious if we extend the question further. For example, what is the probability of having 4 girls and 4 boys?

What is happening here is a classical example of a kind of situation that has contributed to the development of mathematics and science—a representation or a solution that worked well at some point becomes less useful at another point, and so a new approach or method is needed.

Table 7.22 represents another way of solving the problem: by looking at the number of outcomes in different events. I have filled in the fourth row of the table, which shows the frequency of five events in the case of 4 children: 4G0B, 3G1B, 2G2B, 1G3B, 0G4B. Take a few minutes to make sense of the table. If you can make sense of the table, fill in the missing cells. If you don't understand the table, then read on.

TABLE 7.22

NUMBER OF GIRLS							Number of children
6	5	4	3	2	1	0	
							1
							2
							3
		1	4	6	4	1	4
							5

What this table shows is that if you have 4 children, there is 1 way to have 4 girls, there are 4 ways to have 3 girls (and 1 boy), there are 6 ways to have 2 girls (and 2 boys), there are 4 ways to have 1 girl (and 3 boys), and there is 1 way to have no girls (that is, all boys). Do the same for 1 child, 2 children, 3 children, and then 5 children. If you didn't understand the table before, fill out the table now before reading on. . . . 

What patterns do you see? Describe them as though you were talking to someone on the phone. . . . 

There are small (local) patterns; for example,

- The top-right-to-bottom-left diagonal contains only 1s.
- The column representing 0 girls contains only 1s.
- The column representing 1 girl is the counting numbers: 1, 2, 3, 4, 5.

There are also large (global) patterns. For example, there is a connection between each row and the next row. Look at the third row and the fourth row. What do you see? Think and then read on. . . . 

By now you may be smiling because you realize that you have seen this pattern before. Yes, this is simply Pascal's triangle, which you first saw in Investigation 1.6! Connecting this problem to Pascal's triangle makes our work much less tedious. If a couple has five children, what is the probability of 3 girls and 2 boys? Think and then read on. . . . 

The cell representing 3 girls and 2 boys contains 10 outcomes, and there are 32 outcomes in all; therefore, the probability of 3 girls and 2 boys is $10/32$, or about 31%. Similarly, the probability of having 2 girls and 3 boys is also $10/32$. So now what is the probability that the couple will have 3 girls and 2 boys or 2 girls and 3 boys?

Mutually exclusive events We can simply add the two probabilities together. The justification for this goes back to set theory. The two events (3B, 2G) and (2B, 3G) represent disjoint sets. In probability language, we say that they are **mutually exclusive events**

If events A and B are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$.

There is another global pattern that comes out of the table above. It arises from adding a new column headed "Total number of outcomes." If you have not yet seen this pattern, add that new column and fill in the numbers. What do you see? Think and then read on. . . . 

This column simply can be described most succinctly (in English) as "powers of 2."

INVESTIGATION 7.23 Probability of Having at Least 1 Girl

Let us examine another situation. Let's say a couple is planning to have 4 children. What is the probability that they will have *at least* 1 girl? Work on this and then read on. . . . 

DISCUSSION

STRATEGY 1: Refer to the table or tree diagram and count

From the tables you created in Investigation 7.21, count those outcomes in which there is at least 1 girl and divide by 16.

STRATEGY 2: Connect this problem to sets

Let us explore this strategy because it has implications for other probability situations. What if you were asked, "What is the probability of having no girls?" Do you see how the answer to this question enables us to answer the question "What is the probability of having at least 1 girl?"

Do you see why the following formula is true: $P(\text{no girls}) + P(\text{at least 1 girl}) = 1$?

Consider the sample space for having 4 children, and put the outcomes into two subsets:

A = the set of outcomes containing no girls

B = the set of outcomes containing at least one girl

Complementary events In this case, all outcomes go into one subset or the other. Not only are the two subsets disjoint, their union is equal to the whole sample space. Since every outcome is in either set A or set B , that means that $A = \bar{B}$. In other words, these are complementary sets. In probability language, we say that these are **complementary events**. Not surprisingly, if we have two events such that event A and event B are complementary, then $P(A) + P(B) = 1$.

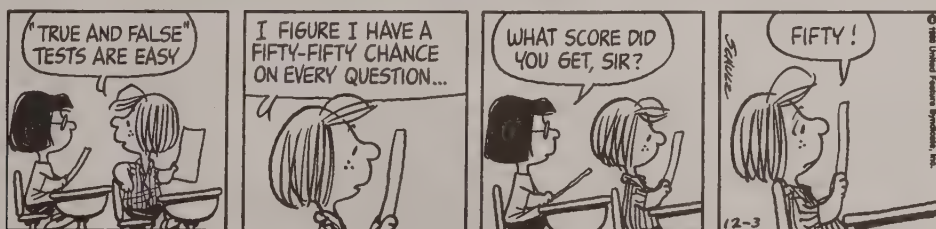
If we recall Investigation 7.22, we see that complementary events represent a special kind of mutually exclusive event.

In this case, the probability of A (the probability of having no girls) is much easier to determine than the probability of B (the probability of having at least 1 girl), and we can use basic algebra to transform the equation $P(A) + P(B) = 1$ into the equation $P(B) = 1 - P(A)$. We can now determine the probability of having at least 1 girl:

$$P(B) = 1 - P(A) = 1 - \frac{1}{16} = \frac{15}{16}$$

Let us now make use of our understanding of probability to investigate two related problems.

INVESTIGATION 7.24 50-50 Chance of Passing



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FIGURE 7.38

Poor Peppermint Patty. Today there is a true-false quiz with 10 questions. She forgot to read the story on which the questions are based, and so she has to guess at every question (see Figure 7.38). Assuming that she has a 50% chance at each question, what is the probability that she will get 70% or better on the quiz? Think about this and then read on. . . . 

DISCUSSION

STRATEGY 1: Make a tree diagram

In order to use a tree diagram for this problem, you would need a very big sheet of paper and a lot of time. However, the tree diagram points to another strategy. What is it?

STRATEGY 2: Be systematic and look for patterns

There is only one way she can get all 10 right. Why is this?

Now, how many ways can she get 9 out of 10 right? That is, how many different outcomes are there for getting 9 questions right? There are several ways to determine the number. Work on this question yourself before reading on. . . . 

Some students reason like this: “She could get all but the last one (number 10) right, she could get all but number 9 right, she could get all but number 8 right, and so on.” What pattern do you see here?

Some students see the systematic nature of this approach better by looking at, and filling out, the diagram below:

RRRRRRRRRW

RRRRRRRRWR

RRRRRRRWRR

etc.

Regardless of how we saw this problem, we find that there are 10 different ways in which she could get 9 out of 10 correct. More formally, we would say that there are 10 outcomes in the event “9 correct answers.”

Now does the following row of Pascal’s triangle make sense?

1 10 45 120 210 252 210 120 45 10 1

If not, analyze a 4-item test in which she guesses each time: There is 1 way to get all 4 correct, there are 4 ways to get 3 right, there are 6 ways to get 2 right, and so on. Can you use Pascal’s triangle now to determine the probability of Patty’s getting a score of 70% or greater? Think and then read on. . . . 

If you have understood the application of Pascal’s triangle to this problem, you have done the computation

$$\frac{(120 + 45 + 10 + 1)}{1024} = 0.17$$

Can you express the answer to the problem in a full sentence? Do so and then read on. . . . 

There are many valid ways to express the answer. Here are two.

- “Peppermint Patty has a 17% chance of getting at least 70% right.”
- “Peppermint Patty has about 1 chance in 6 of getting at least 70% right.”

Does the answer surprise you? Is it higher or lower than you expected? Many teachers do not believe in true-false tests because they believe that a lucky student can appear to know more than that student really does. Does this investigation make you feel more or less disposed toward true-false tests? What if there were 20 questions and the student guessed at each one? Will the probability of getting at least 70% be greater or less than on a 10-item quiz?

INVESTIGATION 7.25 What Is the Probability of Rolling a 7?

 If we roll one die, the probability of each outcome (that is, 1, 2, 3, 4, 5, or 6) is $1/6$ (assuming a fair die). Suppose we roll 2 dice and find the sum of the numbers on the dice. What is the probability of rolling a 7? Work on this problem and then read on. . . . 

DISCUSSION

There are several ways to be systematic, and there are several ways to represent this problem. Look at each of the following partially completed representations. How would you finish them? . . .  Now, complete the tables.

HISTORY

The development of probability theory was spurred by gamblers' questions. The great Italian scientist Galileo (1564–1642), who is better known for getting in trouble with the Inquisition by insisting that the earth revolved around the sun, was asked why one is more likely to get a sum of 10 than a sum of 9 when rolling three dice and adding the numbers. The foundations for probability as a mathematical discipline were laid by the work of Blaise Pascal (1623–1662) and Pierre Fermat (1601–1665) as they responded to several questions asked by a professional gambler, Chevalier de Mere. One of those questions was: Why would one lose money by betting (even money) that you would get at least one double 6 when throwing 2 dice 24 times?

Many people object to students studying gambling theory in school, so we will de-emphasize the gambling connections between dice and probability. This is not hard because dice are also used in many board games.

REPRESENTATION 1

Table 7.23 was created by thinking systematically.

TABLE 7.23

1, 1	2, 1	?, ?
1, 2	2, 2	
1, 3	?, ?	
1, 4		
1, 5		
1, 6		

REPRESENTATION 2

Table 7.24 was created by thinking systematically in a slightly different way.

TABLE 7.24

1, 1	1, 2,	?, ?
2, 1	2, 2	
3, 1	3, ?	
4, 1		
5, 1		
6, 1		

REPRESENTATION 3

Figure 7.39 is yet another tree diagram!

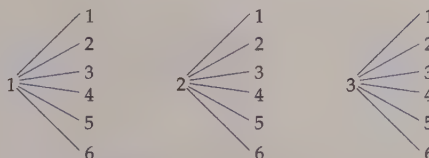


FIGURE 7.39

In each of these cases, we find that there are 36 distinct possible outcomes. This is a useful time to present formally the **multiplication principle**, which can be stated in outcomes or in probabilities:

Stated in number of outcomes If event A consists of a outcomes and event B consists of b outcomes, then the event “ A then B ” consists of $a \cdot b$ outcomes. In this case, event A is rolling a die and event B is rolling a die; both consist of 6 outcomes. Thus, the event “roll a die and then roll it again” consists of $6 \cdot 6 = 36$ outcomes.

Stated in probabilities If the probability of event A is x and the probability of event B is y , then the probability of the event “ A then B ” is $x \cdot y$.

We can now count the number of outcomes that produce a 7 and answer the original question. The probability of rolling a 7 is $6/36$, or $1/6$.

What if we wanted to know the probability of *each* event—that is, of rolling a 2, a 3, a 4, and so on? How can you use your work on the previous question to answer this question? What patterns do you see in your previous work that would help you to make the table? Describe the patterns you see in your table and then read on. . . . 

One way of describing what we find is to say that each time the sum increases, the number of outcomes increases by 1 until we get to (the event) 7. Then the number of outcomes decreases by 1 (for each event) until we get to 12 (see Table 7.25).

TABLE 7.25

Number (event)	Frequency
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

INVESTIGATION 7.26 What Is the Probability of Rolling a 13 with 3 Dice?

Let us extend the investigation now to three dice. Suppose we rolled 3 dice and added the 3 numbers; what is the probability of rolling a 13? Work on this question before reading on. . . 

DISCUSSION

First, we can use the multiplication principle to determine the total number of outcomes from rolling 3 dice—that is, the size of the sample space. There are $6 \cdot 6 \cdot 6 = 216$ possible outcomes!

How many of those outcomes produce a 13? Work on this and then read on. . . 

As you work, make note of what problem-solving strategies help you do the problem, help to keep you on track, and help you check your solution. Virtually all solutions to this problem require the solver to be systematic. However, there are different ways to be systematic, and you can use many different representations that will help answer this question. Below are the beginnings of two different strategies.

REPRESENTATION 1

Table 7.26 is a table created by thinking systematically.

TABLE 7.26

661	652	643	553	?
616	625	?	?	
166	562			
	526			
	265			
	256			

REPRESENTATION 2

Table 7.27 is another table created by thinking systematically.

TABLE 7.27

661	562	463	?
652	553	?	
643	544		
634	535		
625	?		
?			

Regardless of the representation used to solve the problem, we find that there are 21 outcomes that produce a sum of 13. Therefore, the probability of rolling a 13 is $21/216$, or slightly under 1 in 10.

An alternative strategy is to construct the entire sample space. The completed tables are shown at the end of this chapter on pages 439–440. If you find all 216 outcomes in the sample space, then you can make use of patterns to count the number of outcomes in the event called “sum of the 3 dice equals 13.” Making the table is an exercise in thinking about and looking for patterns that some people find fun and others find tedious. If you are in the latter group, you may look at the two different tables, each containing all 216 outcomes, at the end of the chapter. Look for patterns—my students and my children have found hundreds!

INVESTIGATION 7.27 “The Lady or the Tiger”

There is a famous story, written by Frank Stockton, called “The Lady or the Tiger,” which I have modified slightly to make an interesting problem. It seems that the king and queen had arranged for their daughter to be married to a prince, but she fell in love with a peasant. When the king discovered this affair, he ordered that the peasant be thrown into a room full of tigers. However, in response to his daughter’s pleas, he agreed to have the peasant walk through a maze to one of two rooms (see Figure 7.40). The princess would be waiting in one of the rooms, and the tigers would be in the other room. The princess asked if she could choose the room in which she would wait; the king agreed, for he believed that the chances were equal. If you were the princess, which room would you choose? Think about this and then read on. . . .

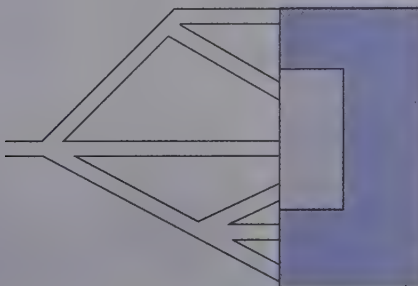


FIGURE 7.40

DISCUSSION

STRATEGY 1: Solve this as a tree diagram

Can you take your understanding of the tree diagrams used in other problems and apply it to this problem? Try to do so now; determine the probabilities of each event (the princess or the tigers), and then read on. . . 

Figure 7.41 shows 6 different doors (1 through 6) that open into two different rooms (A and B). The probability that the peasant will end up in room A = $1/6 + 1/3 + 1/9 = 11/18$. Do you see why?

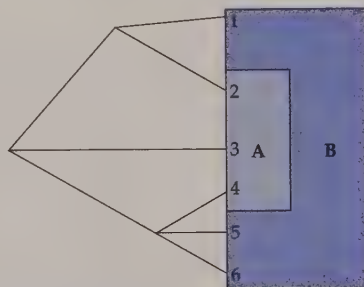


FIGURE 7.41

We can see why by breaking this problem into parts. We are assuming, of course, that at each point there is an equally likely chance that the peasant will take any path. The probability that, at the first junction, he will take the top path is $1/3$; and the probability that, at the second junction, he will take the top path is $1/2$. Therefore, using the multiplication principle, the probability that he will end up at door number 1 is $(1/3)(1/2) = 1/6$. We can determine the probabilities of arriving at the other five doors in a similar manner.

Because A and B are complementary events, we can conclude that the probability that he will end in room B is $1 - 11/18$, or $7/18$. Thus the princess should choose to wait in room A.

STRATEGY 2: Use an area model

At the beginning of the maze, there are three paths. Assuming that each is equally likely to be taken, we can represent them as having the same area (see Figure 7.42).

Top path
Middle path
Lower path

FIGURE 7.42

A	B	
A		
A	B	B

FIGURE 7.43

Figure 7.43 represents the outcomes of all of the paths with respect to the probabilities. Do you understand the diagram? How would you explain it?

The fraction of the area of the diagram that is allotted to the three As represents the probability that the peasant will end up at room A—that is, $1/6 + 1/3 + 1/9 = 11/18$. Do you see why?

This next situation is somewhat different from the previous ones. Let us first explore the problem and then examine what is different about it.

INVESTIGATION 7.28 Gumballs

Josie asks her dad for money to buy 2 gumballs, 1 for her and 1 for her brother. There are only 4 white gumballs and 2 red gumballs left in the machine (see Figure 7.44), and Josie really likes the red ones because they

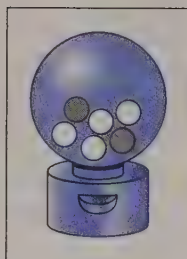


FIGURE 7.44

are spicy. What is the probability that she will get at least 1 red gumball? Work on this question and then read on. . . 

DISCUSSION

STRATEGY 1: Do a simulation

We need to be careful, for there are ways to go wrong. However, if you write R on two slips of paper and W on four slips of paper and then simulate taking out two gumballs, after a number of trials your experimental probability will be more and more likely to be close to the theoretical probability. Some students find that they don't really understand the problem at first, but after several trials with the simulation, they suddenly understand the problem more clearly and can then go back and determine the theoretical probability.

STRATEGY 2: Use reasoning

One student claims that there are 4 possible outcomes—RR, RW, WR, WW—and so the probability of getting at least 1 red is $\frac{3}{4}$. What do you think?

Plausible as this sounds, it is wrong. The mistake here is similar to the mistake that students often make in the boys-and-girls problem when they reason that there are five possible ways to have 4 children: 4G0B, 3G1B, 2G2B, 1G3B, and 0G4B. This line of reasoning is valid only if each outcome is just as likely to occur as any other.

If we think back to that problem (or to the dice problems) and look at ways to apply what we learned there to this problem, a tree diagram comes to mind. Explore a tree diagram before reading on. . . 

STRATEGY 3: Make a tree diagram

Figure 7.45 shows one way to represent this problem with a tree diagram. There are 6 trees because there are originally 6 gumballs. Each tree shows the outcomes if that gumball comes out first. That is, if a red gumball comes out first, then there are 5 gumballs left, 1 red and 4 white. However, if a white gumball comes out first, then there are 2 reds and 3 whites left.

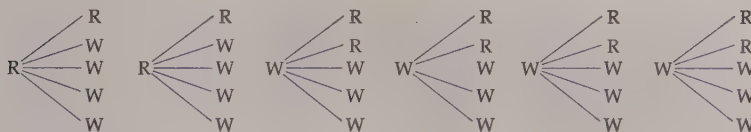


FIGURE 7.45

From Figure 7.45, we find that there are 30 outcomes and that the event “at least 1 red” occurs 18 times. Thus, the probability of getting at least 1 red gumball is $\frac{18}{30}$, or 0.6.

STRATEGY 4: Make a more sophisticated tree diagram

We have seen that many developments in mathematics and the invention of many notations were spurred by a desire to find faster ways to determine answers and more efficient ways to communicate them. In this case, some people look at the tree diagram in Figure 7.45 and say, “There’s a simpler way.” This simpler tree diagram is shown in Figure 7.46. Study it and try to make sense of it. Can you see the solution contained within the diagram?

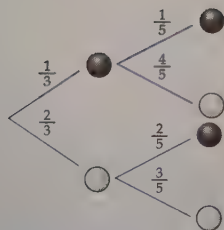


FIGURE 7.46

When we put the money in the machine, the probability of getting a red is $1/3$, and the probability of getting a white is $2/3$. Let's say we get a red the first time. In this case, the probability of getting a red is now $1/5$, and the probability of getting a white is $4/5$. Why is this? Similarly, let's say we get a white the first time. In this case, the probability of getting a red is now $2/5$, and the probability of getting a white is $3/5$.

We can use the multiplication principle to determine the probabilities of each branch of the tree, and we can check our work, because the sum should be 1. Do you see why?

$$P(RR) = \frac{1}{3} \left(\frac{1}{5} \right) = \frac{1}{15}$$

$$P(RW) = \frac{1}{3} \left(\frac{4}{5} \right) = \frac{4}{15}$$

$$P(WR) = \frac{2}{3} \left(\frac{2}{5} \right) = \frac{4}{15}$$

$$P(WW) = \frac{2}{3} \left(\frac{3}{5} \right) = \frac{6}{15}$$

These events are mutually exclusive (why is this?), so we can conclude that the probability of getting at least 1 red is

$$\frac{1}{15} + \frac{4}{15} + \frac{4}{15} + \frac{9}{15} = 0.6$$

Dependent and independent events Now let us return to the question posed at the beginning of the investigation: How is this scenario different from the ones we have explored before? Think and then read on. . . . 

With the gumballs, the probability of obtaining a red or white gumball the second time depended on what happened the first time. In the case of having boys and girls or throwing dice, the probability of the second stage of the activity does not depend on what happened first; that is, its probability is independent of the first result.

Two events are **dependent** if the probability of the second event is affected by the outcome of the first event.

Two events are **independent** if the probability of the second event is not affected at all by the outcome of the first event.

Fair Games

Another application of probability has to do with the idea of fair games and expected value, which has important real-life applications. Let us begin with a simple situation.

INVESTIGATION 7.29 Is This a Fair Game?

C Consider the following game for two players. Each person spins the spinner (see Figure 7.47). If the spinner lands on the same animal twice, the first player wins. If it lands on a different animal each time, the second



FIGURE 7.47

player wins. If you could choose, would you want to be the first or the second player, or doesn't it matter? Think about this and then read on. . . .

DISCUSSION

Can you see similarities between this problem and others we have investigated in this section? If not, look back and try to find connections. Then answer the question and read on. . . .

This problem is similar to the questions about the probability of having boys and girls. Do you see why?

Applying these similarities, we find that there are four equally likely outcomes in this game: (1) cat, cat, (2) dog, dog, (3) cat, dog, and (4), cat, dog. Thus each player has the same theoretical probability of winning.

Using probability language, we say that this is a **fair game**. How would you define *fair game* at this point? Write down your thoughts and then read on. . . .

INVESTIGATION 7.30 What About This Game?

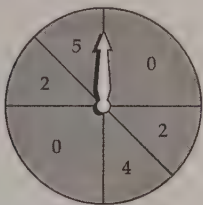


FIGURE 7.48

A school is having a carnival to raise funds. In the following game, the player spins the spinner (Figure 7.48) and receives the dollar amount of the number on which the arrow lands. If it costs \$2 to play the game, is this a fair game? What do you think?

DISCUSSION

STRATEGY 1: Make a simulation

You could make a spinner like the one in Figure 7.48 and simulate 100 games. If you made approximately \$200, the game is probably a fair game. If you didn't have a spinner, how else could you simulate this game?

First, we need to determine the probability of each outcome. That is, $P(0) = 1/2$, $P(2) = 1/4$, $P(4) = 1/8$, $P(5) = 1/8$. So you could make 8 pieces of paper and write 0 on four of them, 2 on two of them, 4 on one of them, and 5 on one of them, and then draw, making sure to put the piece of paper back each time.

STRATEGY 2: Determine the theoretical probabilities

We have spoken of theoretical and experimental probabilities earlier. Can we apply the concept of theoretical probability to theoretical winnings?

We can compute the theoretical winnings per turn by using the concept of weighted average (from Section 7.2):

$$0.125(5) + 0.125(4) + 0.25(2) + 0.5(0) = 0.625 + 0.5 + 0.5 = 1.625$$

That is, theoretically, $1/8$ of the time the player will win \$5; $1/8$ of the time \$4; $1/4$ of the time \$2; and $1/2$ of the time nothing. Some students own this solution, whereas others rent it. If this solution falls into the category of "it sort of makes sense," what might you do to make it "really make sense"? Think and then read on. . . .

BEYOND THE CLASSROOM

Most states now have lotteries. In one lottery scenario, players pick four numbers between 1 and 50. The state then randomly selects four numbers from the sample space—for example, 2, 5, 23, and 47. In another lottery scenario, the players pick four one-digit numbers between 0 and 9. The state then randomly selects one digit at a time, replacing the selected number before drawing the second number. In this case, 1, 1, 2, 2 is a possible winning combination. The first situation is an example of dependent events, and the second is an example of independent events.

One strategy would be to act it out. For example, let's say the person played the game 80 times. Theoretically, how much money will the player win in 80 turns? (Do you see why I chose 80 turns instead of 50 or 100 turns?)

Theoretically, $1/8$ of the time, or 10 different times (because 10 is $1/8$ of 80), the player will win \$5, and thus the player will win a total of \$50 on those 10 turns. When we apply this strategy to the \$4 and \$2 outcomes, we find that theoretically the player's total winnings will be \$130. Do you see that this is equivalent to the weighted average we obtained above?

If the player wins \$130 after 80 turns, the winnings are an average of \$1.625 per turn. Referring now to the original question, we find that theoretically, the player will lose money playing this game because it costs \$2 to play the game. It is important to note that we are not saying that the player will lose. The player might be very lucky and play 10 games and win \$50. However, playing a large number of games makes losing money more likely than winning it.

Can you revise your original definition of *fair game* so that it works for Investigations 7.29 and 7.30?

Expected Value

There is a term for the theoretical winnings per turn: **expected value**. The concept of expected value is an aspect of probability that has many real-life applications. For example, car insurance premiums are based on the insurance company's estimate of the probabilities of different groups of people having accidents. Our knowledge of expected value also arose from gambling; gamblers wanted to make sure that, over time, they would be more likely to make money than lose money. If an experiment (activity) consists of n events, and each event has a specific probability and a specific payoff, we can determine the expected value of that experiment (activity) in the following manner:

Let us denote each event as $E_1, E_2, E_3, \dots, E_n$.

Let us denote the probability of each event as $P_1, P_2, P_3, \dots, P_n$.

Let us denote the payoff of each event as $X_1, X_2, X_3, \dots, X_n$.

The **expected value** = $P_1 \cdot X_1 + P_2 \cdot X_2 + P_3 \cdot X_3 + \dots + P_n \cdot X_n$

The concept of expected value is used to define a fair game formally. Before reading on, try to describe whether a game is fair or not using the concept of expected value. . . 

A game is **fair** if and only if the expected value of the game equals the cost of playing the game.

Let us now extend the concept of expected value to a real-life application.

INVESTIGATION 7.31 Insurance Rates

Let's say that an insurance company is determining the rates for car insurance for the next year. The company has determined that for every 10,000 policyholders:

- 1 is likely to have an accident for which the company will have to pay a \$200,000 claim.
- 10 are likely to have an accident for which the company will have to pay a \$100,000 claim.

- 20 are likely to have an accident for which the company will have to pay a \$50,000 claim.
- 100 are likely to have an accident for which the company will have to pay a \$10,000 claim.
- 1000 are likely to have an accident for which the company will have to pay a \$1000 claim.
- The remainder of the policyholders will have no claims for the year.

Do you understand how the insurance company will use the concept of expected value in determining its rates? If so, determine the expected value of the claims and then read on. . . . 

If you don't see the connection between this situation and the others, please go back and reread the previous investigations and the definition of expected value. Do you see connections now?

DISCUSSION

This investigation has many connections to Investigation 7.30, in which each player asks: "Will the money I win (take in) be greater than the money that I put in?" Because the events did not all have the same probability, it was necessary to use the concept of weighted average, which was developed in Section 7.2.

In order to compute the expected value in the insurance situation, the insurance company wants to get an idea of the "average" payout per customer. This problem is a model of what insurance companies do, not what they actually do. Given the existence of powerful computers, they can use more sophisticated techniques than we are using here. However, this simplified model is useful in understanding the concept of expected value and how it applies to the business world. When we say that 20 people are likely to have an accident for which the insurance company will have to pay \$50,000, we are really saying that the company expects about 20 out of 10,000 policyholders to have accidents that will cost it in the neighborhood of \$50,000. Thus, even before computing the expected value, there has already been some averaging and approximating.

In order to determine the expected value, the incidence of each event must be converted to probability language. For example, "1 in 10,000 claims will be in the neighborhood of \$200,000" translates to "0.0001 probability of a \$200,000 claim."

Thus,

$$\begin{aligned}\text{Expected payout} &= 0.0001(200,000) + 0.001(100,000) + 0.002(50,000) \\ &\quad + 0.01(10,000) + 0.1(1000) = 420\end{aligned}$$

What does this 420 mean? Can you explain what it means in one or two sentences, as though you were talking to a fellow student who missed that class and doesn't quite own this concept? Write your rough draft and then read on. . . . 

It means that if the accidents occur at the rate and amount that the insurance company is predicting, it will pay out an average of \$420 per policy (per year). The actual rate the company charges will be determined by this expected value, the company's overhead (salaries, cost of operations, etc.), and the profit margin.

As before, some students may better understand the problem by determining the expected payoff for the 10,000 policies.

$$\$200,000(1) + \$100,000(10) + \$50,000(20) + \dots$$

Do you see how this strategy and the previous strategy are connected?



BEYOND THE CLASSROOM

One of my sisters is an insurance agent. As I was consulting with her when writing this material, she told me something that surprised me. I had always thought that insurance companies would ideally like to have as large a percentage of policyholders as possible. That is, if a company had 35% of the market in a city or state, it would strive to keep increasing this percentage to get it higher and higher. However, this is not the case. Can you imagine why an insurance company would not want to have 100% of the policyholders in an area? Think and then read on. . . . 

The reason companies don't want to have all the clients has to do with

natural disasters. Several major natural disasters have recently brought even some of the big companies to their knees: Hurricane Andrew in 1992 in Florida, the Midwest floods in the summer of 1993, the California earthquake in 1994, and the severe winter of 1995–1996 in the Northeast. In southern California, for example, the total damage from the earthquake came to over \$30 billion. Had even the largest insurance company had most of the homeowners' policies in this region, it would have gone bankrupt. Therefore, insurance companies determine the maximum percentage of total policyholders in any particular area that they want to insure.

Having performed several explorations and investigations on probability in this section, you are likely to have found that when we try to make predictions based on a small sample, we may or may not be close to the actual probability. For example, if you perform only 10 trials of having 4 children, you might get 2 boys and 2 girls 50% of the time, but that is not close to the theoretical probability of $3/8$. In other words, you have found that if you want to be more confident of your answer, you need to perform more trials. However, this will not necessarily get you closer. For example, you could flip a coin 10 times and find 5 heads but flip the coin 100 times and find 60 heads. In this case, the experimental probability from the larger sample is not closer to the actual probability. Thus, we *cannot* say that the more trials we perform, the closer we will get to the actual probability. What we *can* say is that as we perform an experiment over and over, the experimental probability will converge on a fixed number. This generalization is called the **Law of Large Numbers**.

Summary

In this section, we have developed some basic language for exploring and describing probability situations. Probabilities can be expressed as rational numbers between 0 and 1 and can be expressed using percents, ratios, and odds. We can discuss probabilities using set language, in which case the set of all possibilities is called the sample space. We distinguish between outcomes, which are elements of the sample space, and events, which are subsets of the sample space.

We have learned other distinctions between different kinds of probability situations.

- In some cases, all outcomes are equally likely; in other cases, this is not true.
- Some events are independent and some are dependent.
- In all situations, we can determine the experimental probability of an event or an outcome by collecting data; in some cases, we can determine theoretical probabilities using different problem-solving tools.

We have examined several rules and formulas that have come out of making sense of situations and understanding connections between probability and other areas, such as sets, weighted average, and our understanding of operations.

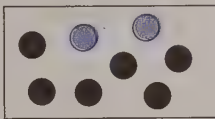
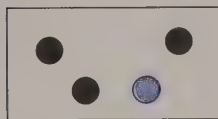
- Complementary events: $P(A) + P(B) = 1$
- Mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$
- the multiplication principle: If $P(A) = x$ and $P(B) = y$, then $P(A \text{ then } B) = x \cdot y$
- Expected value: $P_1 \cdot X_1 + P_2 \cdot X_2 + P_3 \cdot X_3 + \cdots + P_n \cdot X_n$

We have found that there are many different techniques we can use to solve probability problems: tree diagrams, formulas, area models, and simulations. These techniques work better when they are used with other problem-solving tools: making tables, looking for patterns, being systematic, and using reasoning.

Some real-life situations are similar to ones we have studied here, for example, the probabilities associated with genetics, as in Exploration 7.11. Some real-life situations are more complex than the ones we have discussed here, and additional tools are required to solve those problems.

EXERCISES 7.3

1. a. Which of the following would you choose if you were picking 1 ball and wanted black? Justify your choice.



- b. What if you were picking 2 balls (without replacement) and wanted 2 black ones?
2. A drawer has 4 red socks and 4 blue socks.
- If 2 are drawn, what is the probability of a match?
 - If 3 are drawn, what is the probability of a match?
 - What is the probability of having all one color after 4 draws?
3. a. Drawing from a regular deck of playing cards, what is the probability of drawing a face card (jack, queen, or king) 3 times in a row? (Assume that jokers are taken out, and assume no replacement.)
- b. How would the probability change if you assumed replacement?
- c. Marla believes that the answer to the second question is $3/13 + 3/13 + 3/13$. Describe how you might convince her that this answer is incorrect.
4. If it snows, there is a 33% chance that schools will close. The weather person says that there is a 25% chance of snow tomorrow. What is the probability that there will be no school tomorrow?
5. The American Lung Association states that approximately 1 out of every 17 deaths in the United States last year was from lung cancer. If there are 34 students in a

class, does that mean that 2 people in this class will die from lung cancer?

6. Let's say there is a 15% chance of getting strep if you are exposed to a person with strep. Let's say there is a 50% chance that you were exposed.
- What is the probability that you will get strep?
 - Let's say each person in your (4-person) family had a 50% chance of being exposed. What is the probability that you will all get strep?
7. Consider the following game: Each person rolls two standard dice and makes a proper fraction with the two numbers.
- If the fraction is is in simplest form, the first player wins.
 - If the fraction is not in lowest terms, the second player wins.
 - If the two numbers are identical, the second player wins.
- Is this a fair game?
8. Consider a pair of pyramid-shaped dice (that is, dice containing the numbers 1 through 4, rather than 1 through 6 as on regular dice. If you roll two dice at a time, what is the most likely sum and what is the probability of that sum?
9. What are the possible results from rolling two dice and subtracting the number showing on the face of one from that showing on the face of the other in such a way that a nonnegative number is obtained? What result is most likely, and what is its probability?
10. The following experiment consists of rolling a 12-sided die. The outcome set is {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

What probability would you assign to each of the following outcomes?

- a. An even number turns up.
 - b. A prime number turns up.
 - c. A divisor of 12 turns up.
 - d. What is the most probable event if you roll two such dice and add the numbers? What is its probability?
11. Determine the probability of getting 8 heads in a row when flipping a coin 8 times. Give the answer both in decimal form and in ratio form. In decimal form, round the answer to two digits past the string of zeros, and write the decimal you obtained in words. By ratio form, I mean “the chance is 1 in ____.”
 12. If you flip a coin 10 times, what is the probability of getting at least 4 heads?
 13. In the board game Monopoly, if you roll three doubles in a row, you go to jail. What is the probability of rolling three doubles in a row? Describe three ways in which you could represent this probability. Then select the one that you think would be most understandable to the general population. For example, if you were writing an article for the daily newspaper about math in society, which representation would you choose and why?
 14. When my children were young, they played Junior Monopoly, which uses only one die. One day, Josh rolled a 2, then Emily rolled a 2, then I rolled a 2. Josh and Emily thought this was great fun. What is the probability of rolling three 2s in a row on three tosses of a die?
 15. If we look at Table 7.32 (p. 440) and add the number of outcomes in each column, the numbers for the first six columns are 1, 3, 6, 10, 15, 21 which is like the Handshakes Problem in Exploration 1.1. However, the sum of the outcomes in the seventh column is 25 (not the 28 we got with handshakes). Can you explain why we get 25 instead of 28?
 16. Make 2 six-sided dice so that only even sums are possible and each sum is equally likely. Your solution must be other than the trivial solution of “the same odd number on all 12 faces.”
 17. a. The following dates all have a similar pattern. Describe that pattern.
6/12/72 3/25/75 5/19/95
b. What is the probability of such a date in the year 1996?
c. In which year of the century will such dates be most probable?
 18. Recall the gumball investigation. What if Josie spent a third quarter? Do you think $P(WWR) = P(WRW) = P(RWW)$? Why or why not?
 19. Let's say a four-year-old child sits down at the word processor and strikes four letters at random. What is the probability that the child will spell the word *math*? What assumptions did you make in answering this question? For example, is the probability of striking every key the same? That is, do you assume that they are all equally likely outcomes?
 20. A popular Hanukkah game is called Dreidel. Each player begins with a number of pieces of gold (actually chocolate in a round, gold-colored wrapper in the shape of a coin). The game consists of spinning a top that has four sides.
If the top lands on the side labeled gimel, the player gets all the gold in the pot.
If the top lands on the side labeled shin, the player has to put one coin in the pot.
If the top lands on the side labeled hay, the player gets half of the gold in the pot.
If the top lands on the side labeled nun, the player gets nothing.
a. What is the probability of getting no gimels after 4 turns?
b. What is the probability of the top landing on each of the four sides on your first 4 turns?
 21. Consider a multiple-choice quiz consisting of 5 questions with 3 possibilities per question. What is the total number of possible outcomes? What does “outcome” mean here?
 22. Write a program to simulate flipping a coin 1000 times. Analyze the data, using what you learned about statistics in Section 7.2. For example, are the data normally distributed? Do 68% of the data lie within 1 standard deviation of the mean? Are the mean, median, and mode identical?
 23. Let's say 1000 people play the carnival game in Investigation 7.30 over the course of the carnival. Theoretically, how much money will the school make from this game?
 24. Let's say you wanted to make the carnival game in Investigation 7.30 a fair game. Describe one situation that would be a fair game. Explain why that game would be fair.
 25. Write a computer program to simulate the carnival game in Investigation 7.30 and analyze the results.
 26. There is a carnival game called chuck-a-luck. The game works this way.
The player picks a number from 1 to 6. The operator then rolls 3 dice.
If the player's number comes up all 3 times, the player wins \$3.
If the player's number comes up 2 times, the player wins \$2.
If the player's number comes up 1 time, the player wins \$1.
If the player's number doesn't come up, the player pays the operator \$1.
If the game costs \$1, is this a fair game?
 27. Let's say a state is considering introducing a new lottery game. The state will charge \$1 for a ticket. Each week, one person's name will be selected at random. The

amount of the person's prize will be determined in the following manner: Five Ping-Pong balls—four white and one black—will be placed in a hopper. The balls will be selected one at a time until all five balls are in a line.

- If the black ball is first, the person wins \$1,000,000.
 - If the black ball is second, the person wins \$100,000.
 - If the black ball is third, the person wins \$10,000.
 - If the black ball is fourth, the person wins \$1,000.
 - If the black ball is fifth, the person wins \$500.
- a. Is this a fair game?
 - b. The state expects to sell 250,000 tickets each week. Do you think this game will be a good money raiser? Why or why not?
28. Just as insurance companies use the concept of expected value in determining insurance rates, oil companies use expected value in determining whether or not to drill for oil. Let's say that an oil company is considering drilling for oil in a certain area. The company estimates that it will cost an average of \$200,000 to drill each oil well in this area. It has determined that there are three kinds of outcomes: a dry well, a "medium" strike that would produce about \$750,000, and a big strike that would produce about \$2,500,000. The company has estimated that the probability of a medium strike is $\frac{1}{5}$ and the probability of a big strike is $\frac{1}{25}$. Should the company drill for oil? Determine the expected value, and then describe your conclusion as though you were talking to a group of speculators who don't understand expected value.
29. Describe a real-life scenario whose occurrence is a probability and whose probability you would be interested in determining.
- a. Define the scenario.
- b. Describe how you might go about determining the probability.
 - c. Describe some aspects of the problem that would influence the actual probability.
- I have sketched one scenario below; this would be the beginning of a good response.
- a. The probability that I will get a job after I graduate.
 - b. Interview professors who work with teacher certification and average their guesses; interview principals in this area and average their guesses; contact those students who got certified at KSC last summer and determine how many got jobs.
 - c. Probabilities will differ depending on whether I will look only in this area, what my certification is (early childhood, elementary, middle, etc.), whether I am 22 or 32, and so on.
30. The inspiration for this problem came from one of my Middle School Mathematics Methods students. Let's say you have a bag that contains 10 colored beads. You can pull out one bead at a time and then put it back and then shake up the bag. Let's say you have done this five times and have pulled out a red bead three times, a blue bead, and a green bead. What are the facts and what inferences do you have about the contents of the bag? List and justify your facts and inferences.
31. The inspiration for this problem came from a question asked of me by a parent. Let's say the local middle school has five sixth-grade mathematics teachers, two of whom are female, four seventh-grade mathematics teachers, one of whom is female, and four eighth-grade mathematics teachers, one of whom is female. Assuming that the students' placement is random, what is the probability that a student will have at least one female mathematics teacher during the student's three years at the school?

SECTION

7.4

COUNTING AND CHANCE

WHAT DO YOU THINK?

- What does "counting" mean in the context of probability?
- What is the difference between a combination and a permutation?

In many probability problems, simply determining the size of the sample space can be quite tedious. In this section, we will examine some algorithms that can make solving probability problems much easier. As has been the case throughout the book, the goal is not simply to "get" the algorithms, but to understand why they work. There is a reasonable probability that some of you will never use the algorithms developed in this section with your students. However, there is a reasonable probability that all of you who go on to teach elementary mathematics will have your students engage in simpler counting investigations. What then, can you expect to get out of this section?

These investigations will help you to apply problem-solving tools to probability concepts. They focus on representing a problem in useful ways and solving a problem by making a similar, simpler problem and then generalizing from the simpler problem to the actual problem. They call on your skills in recognizing patterns and then using those patterns to solve the problem. They build on your ability to communicate and reason mathematically. They encourage you to see connections among probability problems and connections to other problems we have already investigated.

INVESTIGATION 7.32 How Many Ways to Take the Picture?

Let's say your college has a chapter of Kappa Delta Pi, the national education society. Let's say this was a new society in your college, and in the first year there were 9 members. Let's also say that the members decided to have a group picture taken and wanted only one row—that is, they wanted everyone in the first row. In how many different ways can they line up for the picture? Work on this question yourself before reading on. . . . 

DISCUSSION

This is one of the many problems in mathematics where many people find that if they see a solution, they see it right away, and if they don't, they're stuck. Even if you were systematic, you could become stuck trying to do this one:

1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 9 8
etc.

Stuck, that is, unless you have problem-solving tools. If you are stuck, look at the steps for problem-solving on the inside front cover of the *Explorations* volume. What strategies might help? Then read on. . . . 

STRATEGY 1: Make a simpler problem

There are several different strategies that can be successfully applied to this problem. The first one we will discuss is “make a similar, simpler problem and then work up.” If you didn't make much progress on your own and you like this method, use it to finish the problem.

Let's see how it works. Let's start with a club with only 3 members:

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1

This gives us a total of 6 different arrangements. Where did the 6 come from?

Most people see 3 groups of 2, and hence 3×2 . Some people can jump to the solution of the original problem from here; others need more data from which to generalize.

If you don't yet feel confident that you can connect this to the original problem, try a club with 4 members. If you haven't done this, try it on your own before reading on. . . . 

1 2 3 4
1 2 4 3
1 3 2 4
1 3 4 2
1 4 2 3
1 4 3 2

This gives us 6 possibilities in which person number 1 is on the left. Either by reasoning or by working out the other possibilities yourself, you can see that there will be 6 outcomes in which each of the other 3 members is on the left. This gives us a total of 24 outcomes. Where did the 24 come from?

As before, we have 4 groups, each of which consists of 2 groups of 3. If we put the numbers in order, we have either $4 \times 3 \times 2$ or $2 \times 3 \times 4$.

If you feel that you can jump from here to the solution to the problem, do so. If not, then you can either determine the number of possibilities for a group with 5 members or try another strategy.

STRATEGY 2: Make a tree diagram

In this case, a tree diagram (Figure 7.49) is not terribly helpful for most students. In Section 7.3, we found that a tree diagram was helpful when we were looking at having 2 or 3 or 4 children. However, as the number of children increased, the tree diagram became less helpful and more tedious. I mention this here because many students who struggle in math often think that the better students and the teacher *always* know what to do and *always* pick the “right” strategy, and this is not true. I know many students who have done well in the course who admit that they frequently found that their first attempt at a challenging problem was not productive and that they needed to try another strategy.

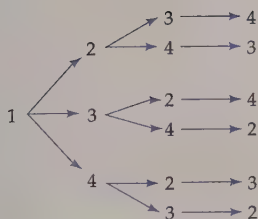


FIGURE 7.49

STRATEGY 3: Connect this problem to something familiar

What about the multiplication principle? Many people understand this strategy better if we simultaneously use the “act it out” strategy.

Let’s begin with the first person. We have nine possibilities. Once we have that person set, how many choices do we have for the second position? There are eight possibilities. Do you see why? Do you see how to finish the problem using this line of reasoning? Work on it before reading on. . . 

Using this line of reasoning, the number of possibilities will be 9 times 8 times 7 times 6 and so on, so that the total number of possibilities is:

$$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$$

Do you see how the first strategy hinted at this solution?

New terminology will make our discussion of this problem and other probability problems easier. How would you describe the computation above to someone—let’s say a friend in class who didn’t have this book? Do so and then read on. . . 

One way: “Start with 9, then 8, then 7, and keep reducing the number by one till you can’t anymore. Now compute the product.”

This kind of computation is common in probability and has a name: **factorial**. That is, $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$ can be written as 9!

Mathematicians insert a 1 at the end of this product, and define any number n **factorial**, $n!$, as

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1$$



LANGUAGE

Yes, a factorial contains an exclamation point; maybe the person who discovered this pattern was so surprised or excited that he or she decided to denote this algorithm with an exclamation point!

INVESTIGATION 7.33 How Many Different Election Outcomes?

Let’s say Kappa Delta Pi has its first election: for president and treasurer. How many possible ways can a president and treasurer be elected from a pool of 9 people? Work on this and then read on. . . 

DISCUSSION

STRATEGY 1: Be systematic, make a table, and look for patterns**TABLE 7.28**

1, 2	2, 1	
1, 3	2, 3	
1, 4	2, 4	
1, 5	2, 5	etc.
1, 6	2, 6	
1, 7	2, 7	
1, 8	2, 8	
1, 9	2, 9	

Note that the numbers in Table 7.28 represent possible outcomes. For example, 1, 2 represents the outcome “person 1 as president and person 2 as treasurer.” Why is {2, 1} not simply a duplicate of {1, 2}? Why did we skip {2, 2}? Can you finish from here?

STRATEGY 2: Make a tree diagram

Some students find a tree diagram (Figure 7.50) very helpful. Can you finish from here?

**FIGURE 7.50****STRATEGY 3:** Connect to previous problems

We can use the multiplication principle again. Do you see how?

There are 9 different possibilities for president (each member), but only 8 possibilities for treasurer (assuming that no one will be both president and treasurer).

Without too much difficulty, most students can see this as 8 groups of 9, or 9×8 .

This is “sort of like” the beginning of factorial.

Electing 3 officers Now let’s see how confident you are about this idea. What if the group decided to have a president, a treasurer, and a secretary? How many different outcomes are there for this scenario? Work on this and then read on. . . .

Each of the three strategies from before still applies (see Figure 7.51).

			8 groups					
7 groups	{	123	132	192	This process repeats 9 times; that is, there is another group of 8×7 outcomes that begins with 2, then another group of 8×7 outcomes that begins with 3, and so on.	213	231	291
		124	134	193		214	234	293
		125	135	194		215	235	294
		126	136	195		216	236	295
		127	137	196		217	237	296
		128	138	197		218	238	297
		129	139	198		219	239	298

FIGURE 7.51

The total number is $9 \cdot 8 \cdot 7$.

This question (about president, treasurer, secretary) is more complex than the question in Investigation 7.32 about the picture. Instead of looking at all

the possible permutations of a set of specified size, which we did in the picture investigation, we are now asking how many different *permutations* of specified size we can make from a set of a given size. In this situation, the outcome "1 2 3" is considered to be different from the outcome "1 3 2." Do you see why?

Using symbols, we say that our election question seeks the number of different subsets of size r that we can make from a set of size n . We can also refer to this amount as the number of **permutations** of n things taken r at a time. The shorthand for this situation is ${}_nP_r$. (Note: Some books use the notation P_r^n .)

In the case of the different possibilities for president and treasurer, we were looking for ${}_9P_2$.

In the case of the different possibilities for president, treasurer, and secretary, we were looking for ${}_9P_3$.

Can you now generalize a formula for the number of permutations of a subset of size r from a set of size n ? Try to do so and then read on. . . 

The generalization is

$${}_nP_r = n(n-1) \cdots (n-r+1)$$

If this makes sense, great. If not, substitute 9 for n and 2 (and then 3) for r so that it makes sense.

Some authors prefer a different formula, one that helps the learner to see how many terms to include and/or what the last term is. This formula rests on the following observation:

$$9 \cdot 8 \cdot 7 = \frac{9!}{6!}$$

(Active readers will make sure that they believe this equivalence!) In this situation, 6 is the number of people who will not be elected. This observation leads to an alternative formula:

$${}_nP_r = \frac{n!}{(n-r)!}$$

INVESTIGATION 7.34 How Many Outcomes This Time?

Suppose Kappa Delta Pi decided to elect two members to send to the national convention. How many possible outcomes are there now? Think and then read on. . . 

DISCUSSION

Do you understand why this is not the same as the problem with the president and treasurer? If so, explain why. If not, think and then read on. . . 

Combinations Using the notation of the investigation with the president and treasurer, the outcomes {1, 2} and {2, 1} represent different outcomes. In the first outcome, student 1 is the president and student 2 is the treasurer, whereas in the second outcome, student 2 is the president and student 1 the treasurer. However, delegates {1, 2} and {2, 1} represent the same outcome. Mathematically, we distinguish between permutations and **combinations**. In probability situations in which order matters, we speak of permutations; when order doesn't matter, we speak of combinations. For example, if outcomes "abc" and

"bac" are considered to be different, we are in a permutation situation; if they are considered to be the same, we are in a combination situation.

Using symbols, we can write the number of combinations of n things taken r at a time as ${}_nC_r$.

Now let us discuss the solution to how many outcomes there are for this question.

First, we will be systematic and look for patterns (see Table 7.29).

TABLE 7.29

1, 2	2, 3	3, 4	
1, 3	2, 4	3, 5	
1, 4	2, 5	3, 6	
1, 5	2, 6	3, 7	etc.
1, 6	2, 7	3, 8	
1, 7	2, 8	3, 9	
1, 8	2, 9		
1, 9			

Can you finish the problem? Does this problem remind you of a problem or a pattern that we have encountered before? Think before reading on. . . 

If you thought about the handshakes problem (Exploration 1.1) or the sum of the first 100 numbers (Investigation 1.5), you are right.

We find that there are 36 different outcomes—that is, $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$. We would like to be able to generalize this procedure so that we can use it in a variety of situations; for example, what if we had a set of 20 elements (instead of 9) and we wanted all outcomes consisting of 3 elements at a time (instead of 2)? One way to determine the general procedure is to analyze the results of the simpler problems. Thus, we will explore where the 36 comes from. Work on this before reading on. . . 

From one perspective, we can see that 36 comes from $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$. Thus, we could generalize that if there were 20 members and we wanted all outcomes of two at a time, there would be $19 + 18 + 17 + \cdots + 3 + 2 + 1$. What if we wanted all possible outcomes of three at a time? In order to develop a procedure that will work in more situations, we have to look more deeply. What other ways do you see to determine the 36? Think before reading on. . . 

Using the algorithm developed in Exploration 1.1, we can say that

$${}_9C_2 = \frac{9(8)}{2} = 36$$

Now this should look similar to the permutation algorithm:

$${}_9P_2 = 9(8) = 72$$

Why do we divide by 2? Think about this before reading on. . . 

Because each combination is connected to two permutations, we can see that in this case, the number of combinations will be half the number of permutations.

Do you feel able to make a hypothesis for the general combination algorithm, ${}_nC_r$? If so, make it and then read on. . . 

If not, work through the next problem and then try to make a hypothesis.

If 3 people go to the convention What if 3 people could go to the convention? How many different combinations are there? Complete Table 7.30 before reading on. What patterns do you see that could help you to complete the table or help you to determine the total?

TABLE 7.30

1 2 3	1 3 4	1 4 5	1 5 6	1 6 7	1 7 8	1 8 9
1 2 4	1 3 5	1 4 6	1 5 7	1 6 8	1 7 9	
1 2 5	1 3 6	1 4 7	1 5 8	1 6 9		
1 2 6	1 3 7	1 4 8	1 5 9			
1 2 7	1 3 8	1 4 9				
1 2 8	1 3 9					
1 2 9						
2 3 4	2 4 5	2 5 6	2 6 7	2 7 8	2 8 9	
2 3 5	2 4 6	2 5 7	2 6 8	2 7 9		
2 3 6	2 4 7	2 5 8	2 6 9			
2 3 7	2 4 8	2 5 9				
2 3 8	2 4 9					
2 3 9						
3 4 5	3 5 6	3 6 7	3 7 8	3 8 9		
3 4 6	3 5 7	3 6 8	3 7 9			
3 4 7	3 5 8	3 6 9				
3 4 8	3 5 9					
3 4 9						

In the first set of outcomes, where person number 1 is first, we have 28 total outcomes, and this number is the sum of the sequence $7 + 6 + 5 + 4 + 3 + 2 + 1$. In a similar fashion, the number of outcomes in the second set is equal to $6 + 5 + 4 + 3 + 2 + 1 = 21$. From these patterns, it is reasonable to conclude that we can continue in this manner. Doing so, we find that there are 84 different outcomes for this situation—that is, $28 + 21 + 15 + 10 + 6 + 3 + 1$. In everyday language, we say that there are 84 possible ways to send 3 people to the convention.

Determining a way of connecting this 84 to a formula that we can use is not obvious, but there are connections. What do you think? Can you generalize from the handshakes problem in Exploration 1.1? Work on this before reading on. . . . 

There are many possible paths that will enable us to see the general procedure. Let us take one path.

First, let us summarize what we know:

We know that ${}_9P_2 = 9 \cdot 8 = 72$, that ${}_9C_2 = 9 \cdot 8/2$, and that there is a similarity between these two.

We know that ${}_9P_3 = 9 \cdot 8 \cdot 7 = 504$.

Therefore, we might suspect that ${}_9C_3$ might be something like $(9 \cdot 8 \cdot 7)/x$.

But we know that ${}_9C_3 = 84$, so we can make an equation and solve for x . That is, ${}_9C_3 = 84 = (9 \cdot 8 \cdot 7)/x$.

Solving the equation, we find that $x = 6$. This still prompts the question "Why do we divide by 6?" What do you think? Work on this before reading on. . . . 

In this case, each combination outcome is connected to 6 permutation outcomes.

$$\left. \begin{array}{l} 1\ 2\ 3 \\ 1\ 3\ 2 \\ 2\ 3\ 1 \\ 2\ 1\ 3 \\ 3\ 1\ 2 \\ 3\ 2\ 1 \end{array} \right\} 1\ 2\ 3$$

Can you make the leap? Do you see a pattern?

$${}_9C_2 = \frac{9 \cdot 8}{2}$$

$${}_9C_3 = \frac{9 \cdot 8 \cdot 7}{6} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3}$$

$${}_nC_r = ? \quad \text{What do you think?}$$

$${}_nC_r = \frac{n \cdot (n-1) \cdots (n-r+1)}{r!} = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)! r!}$$

INVESTIGATION 7.35 Pick a Card, Any Card!

Consider a standard deck of playing cards. Pick two cards. What is the probability of getting two queens? Can we use any of the algorithms? Can you explain why one of the algorithms applies or why none of them applies? Work on this problem and then read on. . . . 

DISCUSSION

This is a combination (not a permutation) problem. Do you see why?

$${}_{52}C_2 = \frac{52 \cdot 51}{2} = 1326$$

Does this mean that the probability is $1/1326$? What do you think?

The answer to this question is not $1/1326$. The number 1326 means that there are 1326 different possible combinations. Because we want to know the probability of getting two queens, we also have to determine how many of those 1326 combinations involve two queens. What do you think? Do this now before reading on. . . . 

One strategy is to list them systematically—for example, SD, SH, SC, HD, HC, DC. Another strategy is to use the appropriate combination algorithm appropriately. Do you see how ${}_4C_2$ applies to this part of the problem?

In either case, we find that there are 6 different combinations that involve 2 queens.

Thus, we find that the probability of a queen is

$$\frac{6}{26 \cdot 51} = \frac{1}{13 \cdot 17} = \frac{1}{221}$$

INVESTIGATION 7.36 So You Think You're Going to Win the Lottery?

**BEYOND THE CLASSROOM**

There are some facts about state lotteries that are troublesome.

In California, the poor spend 15 times more of their income, as a percentage, on lottery tickets than the rich do. . . . A Detroit study showed that in tough times, middle- and upper-class folk stopped playing the lotteries—but poor urban dwellers kept right on playing. . . . Other studies show that players are disproportionately minority people with less than average income and education.¹³

Since 1980, the number of states with lotteries has increased considerably. What factor(s) do you think might have led to the increase in states having lotteries?

During the 1980s, federal aid to states and cities shrank considerably. Many states “solved” this problem by establishing or increasing lotteries and designating the profits from the lottery for expenses such as education. The state lotteries have been called voluntary taxes. Can you see why?

Let's begin with a rather straightforward lottery in which players select four digits. The winning number is commonly selected in the following fashion: 10 Ping-Pong balls (each with a digit 0 through 9 written on it) are placed in a hopper. One Ping-Pong ball is selected. That digit now goes in the first place. That Ping-Pong ball is put back in the hopper, and the process is repeated three times.

What is the probability of picking a winning number? Work on this and then read on. . . . 

DISCUSSION**STRATEGY 1: Connect to the multiplication principle**

Each time, there are 10 possibilities, so the sample space is $10 \times 10 \times 10 \times 10$. Because all outcomes are equally likely, the probability of any outcome is $1/10,000$.

STRATEGY 2: Act it out

What are some possible winning numbers?

Possibilities include 1234, 4612—in fact, any number from 0000 to 9999. How many combinations is that?

Can you extend this problem or think of similar problems?

Summary

In this section, we have examined a number of situations known as counting situations. That is, we have looked for patterns and order in those situations in order to find easier ways to count the number of outcomes in the sample space. Many children find such counting problems to be interesting. We learned the difference between combinations and permutations, and we examined how the formulas work.

Once again, it is important to emphasize that the algorithms themselves are useless if you do not understand what they mean. I strongly believe that there is too much blind application of algorithms in our schools. I see this regularly in schools that I visit, and I see many of my students doing this also, either out of math phobia or out of apathy. It is my hope that over time, more and more students and teachers will strive to make sense of situations in which mathematics applies.

¹³ *Utne Reader*, November/December 1993, p. 19.

EXERCISES 7.4

- In how many ways can the following students' names be displayed on a poster: Amy, Betty, Carl, Ed, Frank, Gisela?
- In how many ways can the names of 4 candidates be listed on a ballot for an election?
- Find the number of different ways in which 4 flags can be displayed on a flagpole, one above the other, if 10 different flags are available.
- Mrs. Olson has 7 brands of cat food and 3 cats, each of which receives 1 can per day. In how many different ways can she serve the cats on any day, assuming that no 2 cats get the same brand?
- Those students who celebrate Christmas and come from large families may find this problem to be familiar. Let's say that a family of 5 decides that instead of each person buying a present for each other person in the family, all the family members will put their names in a hat. Each person then takes one name out of the hat and buys a present for that person. What is the probability of every family member getting his or her own name?
- Several years ago I received as a gift a book of different animals. However, the picture of each animal's body had been cut into three pieces, and the pages of the book enabled me to make different combinations of animals. The name of the book was *Por-gua-can* because in the creature shown on the cover, the head was a porcupine's, the torso was an iguana's, and the feet were a pelican's. There were 9 animals. How many different animal combinations are possible?
- The German club has 12 members. In how many different ways can a subset of 3 members be selected to go on a field trip?
- A basketball team has 9 players, 5 of which are in the starting lineup.
 - How many different starting lineups are possible?
 - How many different starting lineups are possible if the star must be in the lineup?
- Let's say you went out for ice cream. The store had 9 different flavors, and you wanted a triple-decker ice cream cone. How many different triple-decker ice cream cones are possible at this store?
- Janine's boss has allowed her to have a flexible schedule. Her boss says she can pick whatever 5 days a week she will work.
 - How many different work combinations can be made?
 - How many choices give her consecutive days off?
 - How many choices give her Wednesday off?
- This unfortunate event happened to me when my two children were little. Each of them had a friend over for lunch. After lunch, we had popsicles for dessert. There were 9 popsicles in the freezer and 3 flavors: 3 grape, 3 cherry, and 3 orange. What is the probability that each child will get the popsicle that child wants?
- Let's say a company has 10 members on its management team: 7 men and 3 women. There is a conference in Hawaii, and the company has decided to send 3 people. If the selection of the 3 people to go to Hawaii is determined by lottery, what is the probability that at least 1 woman will be selected?
- You and a friend are at Paul and Elizabeth's Restaurant for a night of fine dining. You can choose among 2 appetizers, 4 main dishes, and 3 desserts.

Appetizer: soup or salad
Main dish: fish, chicken, beef, vegetarian
Dessert: chocolate cake, apple pie, ice cream

 - How many possible different dinners are there?
 - If there are 3 choices of salad dressing and 4 choices of ice cream, how many possible dinner combinations are there?
- If you pick a card from a regular deck of cards, what is the probability of getting a face card 3 times in a row? Assume that the face card is put back in after each draw, so that you are selecting from a full deck each time.
- If you select 2 cards from a deck, what is the probability of getting 2 matching cards (2 kings, 2 fives, and so on)?
- If you select 5 cards from a deck, what is the probability that you will get 4 of a kind? 3 of a kind? 2 of a kind?
- Recall Investigation 7.36. Suppose the Ping-Pong ball was not replaced. Now what is the probability?
- Explain why these two formulas are equivalent.

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{and} \quad {}nP_r = n(n-1) \cdots (n-r+1)$$
- What do you think will be the value of ${}_nP_n$?
- Little Caesar's pizza had a commercial on television that said that you could buy one pizza and get another one free, with up to 5 toppings per pizza. The dialogue on the commercial went something like this. The older person said, "5 toppings per pizza; that's 10 different pizzas to choose from." The little guy then said, "No, it's 1,048,576 combinations." What do you think?
- Interview a nearby automobile dealership. Ask the people responsible for ordering how they determine which cars to order to have on hand. Compare the selection process for a small dealership that can stock only 5 Plymouth Voyager minivans with that for a large dealership that can stock 50 Plymouth Voyager minivans.
- How many possible words can be spelled using the letters from the word *mathematics*?
 - What word can you find with the highest ratio of actual (possible) words to the theoretical number of words?

CHAPTER SUMMARY

1. There are many ways to represent data. In some cases, what graph to use is a matter of preference; in other cases, some graphs are more appropriate than others.
2. Different graphs can give different impressions of a set of data. Graphs can be constructed to give the reader a distorted impression of the data!
3. When examining data that others have gathered, we need to examine the data carefully, thinking about the reliability and validity of the data, and about whether it is fair, appropriate, or distorted.
4. The terms *mean*, *median*, and *mode* all represent an "average." Each of the three terms has a conceptual base:
 - The mean is the center of gravity of a set of data.
 - The median is the numerical middle of the set of data.
 - The mode is the datum that occurs most often.
5. In some distributions, the mean, median, and mode are equal. In others, they may be very different, and some may not even be appropriate. In those cases, we need to examine which measure is the best representative for "average."
6. Averages are not always highly representative of a population. In most cases, when we are examining a population, we need to have more information about the population, depending on the questions we are asking. We may want to know the center of the data, or we may want to know about the spread of the data, in which case knowing the extreme values, the range, and the standard deviation is important.
7. When a set of data is normally distributed, we can make powerful generalizations and predictions about that set of data.
8. Random and haphazard are not the same thing. Many events that are random can be quantified.
9. When working with probability situations, sometimes we can determine both the theoretical and experimental probability of an event happening. Sometimes we can only determine the experimental probability.
10. Probability formulas and rules are easier to understand when we take the time to be clear about the terms being used, e.g. events vs. outcomes, dependent vs. independent events, complementary events, and permutations and combinations.

11. Care needs to be taken when conducting surveys so that the results are reliable.

Basic Concepts

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When a set of data is normally distributed, we can make the following conclusions:
 68% of the data lie within 1 standard deviation from the mean.
 95% of the data lie within 2 standard deviations from the mean.
 99.8% of the data lie within 3 standard deviations from the mean.

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Completed probability tables for Investigation 7.26, p. 417

TABLE 7.26

6 6 1	6 5 2	6 4 3	5 5 3	4 4 5
6 1 6	6 2 5	6 3 4	5 3 5	4 5 4
1 6 6	5 6 2	3 6 4	3 5 5	5 4 4
	5 2 6	3 4 6		
	2 5 6	4 6 3		
	2 6 5	4 3 6		

TABLE 7.27

6 6 1	5 6 2	4 6 3	3 6 4	2 6 5	1 6 6
6 5 2	5 5 3	4 5 4	3 5 5	2 5 6	
6 4 3	5 4 4	4 4 5	3 4 6		
6 3 4	5 3 5	4 3 6			
6 2 5	5 2 6				
6 1 6					

Two different representations of the sample space of rolling 3 dice

Note: The first column is simply to make communication easier; for example, we can talk about patterns in rows 19 to 24.

TABLE 7.31

1	111	211	311	411	511	611
2	112	212	312	412	512	612
3	113	213	313	413	513	613
4	114	214	314	414	514	614
5	115	215	315	415	515	615
6	116	216	316	416	516	616
7	121	221	321	421	521	621
8	122	222	322	422	522	622
9	123	223	323	423	523	623
10	124	224	324	424	524	624
11	125	225	325	425	525	625
12	126	226	326	426	526	626
13	131	231	331	431	531	631
14	132	232	332	432	532	632
15	133	233	333	433	533	633
16	134	234	334	434	534	634
17	135	235	335	435	535	635
18	136	236	336	436	536	636
19	141	241	341	441	541	641
20	142	242	342	442	542	642
21	143	243	343	443	543	643
22	144	244	344	444	544	644
23	145	245	345	445	545	645
24	146	246	346	446	546	646
25	151	251	351	451	551	651
26	152	252	352	452	552	652
27	153	253	353	453	553	653
28	154	254	354	454	554	654
29	155	255	355	455	555	655
30	156	256	356	456	556	656
31	161	261	361	461	561	661
32	162	262	362	462	562	662
33	163	263	363	463	563	663
34	164	264	364	464	564	664
35	165	265	365	465	565	665
36	166	266	366	466	566	666

TABLE 7.32

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	111	112	113	114	115	116	126	136	146	156	166	266	366	466	566	666
2		121	122	123	124	125	135	145	155	165	256	356	456	556	656	
3		211	131	132	133	134	144	154	164	246	265	365	465	565	665	
4			212	141	142	143	153	163	236	255	346	446	546	646		
5			221	213	151	152	162	226	245	264	355	455	555	655		
6			311	222	214	161	216	235	254	336	364	464	564	664		
7				231	223	215	225	244	263	345	436	536	636			
8				312	232	224	234	253	326	354	445	545	645			
9				321	241	233	243	262	335	363	454	554	654			
10				411	313	242	252	316	344	426	463	563	663			
11					322	251	261	325	353	435	526	626				
12					331	314	315	334	362	444	535	635				
13					412	323	324	343	416	453	544	644				
14					421	332	333	352	425	462	553	653				
15					511	341	342	361	434	516	562	662				
16						413	351	415	443	525	616					
17						422	414	424	452	534	625					
18						431	423	433	461	543	634					
19						512	432	442	515	552	643					
20						521	441	451	524	561	652					
21						611	513	514	533	615	661					
22							522	525	542	624						
23							531	532	551	633						
24							612	541	614	642						
25							621	613	623	651						
26								622	632							
27								631	641							

Geometry as Shape

- ◆ 8.1 Basic Concepts of Geometry
- ◆ 8.2 Two-Dimensional Figures
- ◆ 8.3 Three-Dimensional Figures

In the next three chapters, we will focus on three aspects of geometry: shapes, transformations, and measurement. Look at the pictures in Figure 8.1. What do you see?



FIGURE 8.1 Three natural spirals: (a) leaves of the sago palm, (b) horns of a mountain sheep, (c) the chambered nautilus.

We see a shape that is called a spiral occurring in different parts of the world: in a plant and on two different animals. How is this shape formed? Why is this shape formed in these situations — that is, what purpose or function does this shape serve? These are questions that mathematicians, biologists, physicists, and others have asked and answered. Essential to answering these questions is a deep understanding of these shapes, which also involves understanding their measures.

It is very possible that the study of geometry is as old as the study of numbers. Geometry is an important part of human life, both from an aesthetic and from a practical perspective. Geometry helps explain why some paintings, sculptures, and shapes, and even some music, appeal to more people than others. Geometry helps us to determine which shapes are more useful than others, and which are stronger than others. In Chapter 1, we saw that patterns is a theme that runs through all of mathematical thinking. In the next three chapters, we will examine the patterns that arise from shapes. In one sense, shapes are patterns. “Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is possible and desirable at all levels.”¹

Actually, there are many geometries in mathematics: Euclidean, non-Euclidean (where parallel lines meet and where the sum of the angles of a triangle is not 180 degrees), projective geometry (used in art and architecture), coordinate geometry, topology (which includes understanding networks), and fractal geometry (one of the more recent geometries). We will be studying a subset of what is generally called Euclidean geometry (shapes of two- and three-dimensional objects), a subset of transformational geometry, which is the study of changes we can make in congruent shapes (slide, flip, turn) and when shapes are similar, and a subset of the mathematics of measurement, focusing largely on measures of geometric figures.

There are two dispositions, which you have experienced during the course of this book, that become crucial in the next three chapters:

1. Deconstructing shapes

Just as numbers can be taken apart and put together (decomposed and composed), so too can shapes. If you see that any polygon can be decomposed into triangles, the formula about the sum of interior angles of polygons, $180(n - 2)$, actually makes sense. If you see that any parallelogram can be decomposed into two congruent triangles, the relationship between the area of a parallelogram, $(A = bh)$, and the area of a triangle, $(A = \frac{1}{2}bh)$, makes sense.

2. Looking for generalizations

This comes largely from looking for patterns and connections. From generalizations come rules and a deeper understanding of mathematical structure.

▶ This is a good point to jump on the Web and read the NCTM standards on geometry. Read the overview for Pre-K–12, and then read material on the Pre-K–2 and grades 3–5 standard on geometry. You will find that these descriptions are both readable and interesting. The NCTM encourages hands-on exploration of ideas. You will find explorations in the NCTM standards and in the “Geometry and Spatial Sense” book in the NCTM Addenda Series for Grades K–6 that are very similar to ones in the next two

¹ Branko Grünbaum, cited in Marjorie Senechal, *On the Shoulders of Giants: New Approaches to Numeracy*, ed. Lynn A. Steen (Washington, D.C.: National Academy Press, 1990), p. 139. Reprinted with permission from *On the Shoulders of Giants: New Approaches to Numeracy*. Copyright © 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington D.C.

chapters of your *Explorations* volume. Many of my students tell me geometry is their least favorite part of mathematics. After the next two chapters, many of these same students report that it is now their favorite part of mathematics!

Geometry and Spatial Thinking

It is important to note that “school geometry” is only one of many subsets of a much larger set called spatial thinking. Let us examine some of the other subsets so that you can have a better sense of the larger goal and a vision of geometry in school mathematics.

One aspect of spatial thinking is **hand-eye coordination**. Most young children are still very much in the process of developing this ability. There are many activities in the field of geometry that can help children to develop this ability, such as copying shapes that the teacher or other students have made with pattern blocks, geoboards (such as those at the end of the *Explorations* volume), or other manipulatives.

Another important spatial thinking ability is called **figure-ground perception**, the ability to identify a figure against a complex background. One example that most of my students remember from elementary school is a picture for which the directions ask, “Can you find the 12 monkeys hidden in this picture?” The activity is far more than just a fun exercise. This ability to discriminate what you are looking for from extraneous lines or information is important in many professions.

Another important ability is called **perceptual constancy**, the ability to recognize figures and objects when they are not in their familiar orientation. For example, many students see the triangle on the left in Figure 8.2 as an isosceles triangle but will say that the triangle on the right is not an isosceles triangle. Similarly, other students will see that the triangle on the right is an obtuse triangle but will fail to see that the triangle on the left is also an obtuse triangle. In fact, the two triangles are congruent.

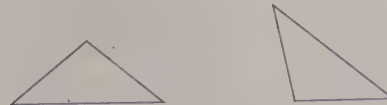


FIGURE 8.2

Visual discrimination is the ability to find similarities and differences between or among objects. You have probably seen tasks developing this ability in *Sesame Street*, commercial activity books, and schoolbooks. For example, five similar objects are shown, and the student is asked to pick the two that are identical.

Visual memory has to do with a person’s ability to describe or draw accurately an object that is no longer in view. In Section 8.2, there will be an investigation that asks you to look at an object for only a few seconds and then try to draw that object from memory.

These are but a few of the many kinds of abilities that come under the larger heading of spatial thinking and that are important in everyday life and crucial in many occupations. Douglas Clements and Michael Battista state that “much of the thinking required in higher mathematics is spatial in nature,”² and they note that Einstein said that he thought not so much in words as in

² Douglas H. Clements and Michael T. Battista, “Geometry and Spatial Reasoning,” in *Handbook of Research on Mathematics Teaching and Learning*, ed. Douglas A. Grouws (New York: Macmillan, 1992), p. 442.

images. Ultimately, we want students to realize that virtually all fields of mathematics have spatial aspects and applications.

The van Hiele Levels of Geometric Thinking

Two Dutch educators, Pierre and Dina van Hiele-Geldorf, were concerned with how geometry was generally taught. In the late 1950s, they developed what is now known as the van Hiele model. Ignored for many years in the United States, it is now gaining more prominence. Let us look and see what the van Hiele model says about learning geometry. The van Hieles found that there are levels, or stages, in the development of a person's understanding of geometric ideas and concepts. Each level describes the sophistication of the learner's thinking tools for learning geometric ideas. The following descriptions are meant as an introduction and overview; the complete model is rather sophisticated and worth more attention.

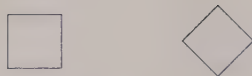


FIGURE 8.3

Level 1: Visual (recognition) At this level, the person's primary thinking tool is direct visual observation. A student at this level can recognize and name figures but does so more by focusing globally—for example, "This is a square because it looks like one." Although students at this level may be made aware of the various properties of geometric objects (for example, that a square has four equal sides), such awareness can be overridden by other factors. For example, if we turn a square on its side, as in Figure 8.3, the student may insist that it is no longer a square but is now a diamond.

Level 2: Analysis At this level, the person can go beyond mere appearance and recognize and describe shapes by their properties, such as a square's four right angles, equal sides, parallel opposite sides, and so on. However, a student at this level does not spontaneously and regularly look at *relationships* between figures. A student who argues that a figure is not a rectangle because it is a square is showing level 2 thinking.

Level 3: Relationships At this level, the student can start to use more abstract reasoning processes *consistently*. The student can now classify figures more powerfully; for example, the student can now see that squares are a special kind (subset) of rectangles. Students at this level can begin to harness the power of "if-then" reasoning and are able to follow and make informal proofs. However, they are not yet able to appreciate fully the need for formal, axiomatic geometry.

Level 4: Formal proof This is the level of high school geometry—axioms, definitions, postulates, theorems, and so on. Students thinking at this level can understand and appreciate the need for a more rigorous system of logic and are able to work with abstract statements and make conclusions based on logic rather than just on intuition.

Level 5: Mathematical systems This is the level of college geometry. At this level, students are able to focus successfully on the systems themselves, not just on the deductions in the given system.

It is important to reemphasize that these are not descriptions of *what* students know, but rather of the ascending levels of *how* they are able to think—that is, the kinds of thinking tools they can use to learn geometry. It is expected that by the end of middle school, students are able to think at level 3 and are "ready" for level 4 tasks and investigations.

▶ The NCTM supports this notion of building formal knowledge from investigations that are at a level appropriate for the learner:

"Geometry should focus on intuitive, 'common sense' investigations of geometric concepts in such a way that general properties emerge and are used as the basis for conjectures and deductions. Later, observations and deductions can be studied more formally as part of a mathematical system" (*Professional Standards*, p. 137).

As you will soon discover, our understanding of geometry has been most strongly influenced by people living in Greece over 2000 years ago who sought to go beyond observing that certain things seemed to be true to proving *why* they were true. In doing so, they looked to define many of the words that were used to describe shapes. However, *shape* itself is an undefined term in the sense that we do not even today have a precise definition for this idea. "We know that shapes may be alike in some ways and different in others. A football is not a basketball, but both are smooth closed surfaces; a triangle is not a square, but both are polygons. . . . We know that shapes can change and yet be in some way the same: our shadows are always our shadows, even though they change in size and contour throughout the day."³

Before examining important geometric ideas and shapes, let us take a few moments to examine some of the "big ideas"⁴ in our exploration of the geometry of shape.

1. Shapes are found everywhere, and virtually all shapes have functional value ("form follows function") and/or aesthetic value.
2. Each geometric shape represents the common characteristics of a set of objects. This is similar to the idea of lowest common denominator: What do all of the objects have in common? For example, there are many different shapes that all have the name *triangle*. The definition of *triangle* specifies what all of these shapes have in common. In this chapter, we will focus on common two- and three-dimensional shapes and terms and on the relationships between two-dimensional representations of three-dimensional objects.
3. Every shape has multiple attributes—size, angles, lengths of sides, ratios between and among sides, etc. This means that we can classify any set of shapes in multiple ways. This tendency to classify is crucial and innate; young children do it naturally and enjoy it. A common classification investigation in pre-school is to sort buttons, and there are multiple possibilities, such as number of holes, shape, size, color, and plain or textured.
4. This act of classifying leads us to a deeper understanding of mathematical structure, which leads to greater mathematical power, including the ability to solve problems.
5. Seeing relationships within and between shapes also leads to a deeper understanding of mathematical structure. For example, relationships among "different" triangles enables us to understand better how triangles are named; and relationships among quadrilaterals help us to see how they are related—when we prove something about one, it is true for others in that "family."

³Marjorie Senechal, in *On the Shoulders of Giants*, ed. Lynn A. Steen, p. 140. Reprinted with permission from *On the Shoulders of Giants: New Approaches to Numeracy*. Copyright © 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, D.C.

⁴I distinguish between *big ideas* and *competencies*. Both are important. Your instructor will determine the specific competencies to be addressed in the course at your college. There is also a national dialogue and debate concerning the competencies of teachers in general and the mathematical competencies of elementary teachers in particular.

SECTION

8.1

BASIC CONCEPTS OF GEOMETRY

WHAT DO YOU THINK?

- Why do we not define terms like *point*, *line*, and *plane*?
- What does the name we give a line (or angle) tell us about it?
- Why are there 360 degrees in a circle?

Early geometry Archaeologists tell us that over 5000 years ago, the Egyptians and Babylonians had given names to many geometric figures and had discovered many geometric relationships. By 3000 B.C., the Egyptians had developed rather sophisticated understandings about geometry. For example, the relationship between the lengths of the three sides of a right triangle ($a^2 + b^2 = c^2$) had been known to the Egyptians at least 1500 years before and to the Chinese for at least 1000 years.⁵ They used this knowledge in many ways—for example, to make the pyramids and to redraw boundaries each year after the Nile river flooded the fields. The ancient Greek historian Herodotus wrote:

Seostris . . . made a division of the soil of Egypt among the inhabitants. . . . If the river carried away any portion of a man's lot . . . the king sent persons to examine, and determine by measurements the exact extent of the loss. . . . From this practice, I think, geometry first came to be known in Egypt, whence it passed into Greece.⁶

However, our records indicate that the Egyptians' knowledge was intuitive and inductive. Before deductive reasoning was developed, they could not be sure of their conclusions, and in fact many of their formulas were wrong, such as those for the area of a circle and the volume of some solids.

Geometry as a mathematical system Modern mathematics was born in ancient Greece, for it was the Greeks who pushed mathematics from the mainly empirical level to the deductive level. It was Thales (who lived in the sixth century B.C.) first, and then his student Pythagoras and his students, who not only asked *how* and *what* but also asked *why*. These early Greeks were not content to know that $a^2 + b^2 = c^2$ seemed to be true for all right triangles; they were driven to be able to prove that it *must* be true. This change in thinking (analogous to moving from van Hiele level 3 to level 4) "symbolizes the circumstances under which the foundations not only of modern mathematics but also of modern science and philosophy were established."⁷ It is not surprising that the word *geometry* comes from two Greek words: *geo*, meaning "earth," and *metria*, meaning "measurement."

Over many hundreds of years of exploration, the Greeks came to invent what we call mathematical systems. About 300 B.C. a Greek mathematician, Euclid, published *The Elements*, a set of 13 books that essentially organized what the Greeks had learned about mathematics up to that time. It contained over 600 theorems resting upon 10 fundamental postulates. *The Elements* were viewed by most mathematicians as a model of deductive reasoning for nearly 2000 years.

Although a formal treatment of geometry is not the focus of this chapter, I believe that an awareness of mathematical systems and their value in modern society is important. It is essential that your future students develop a good mathematical foundation in elementary school (van Hiele levels 1, 2, and 3) so that they can build confidently on this knowledge in high school and beyond. Geometry is only one of many modern mathematical systems. Some others are matrix algebra, abstract algebra, analysis, differential equations, probability, statistics, and chaos theory.

This notion of mathematical systems has taken thousands of years to develop and is one of the primary reasons for the explosion of science and tech-

⁵ H.A. Freebury, *A History of Mathematics* (New York: Macmillan, 1961), p. 35.

⁶ Quoted in Julian L. Coolidge, *A History of Geometrical Methods* (New York: Dover, 1963), pp. 8–9.

⁷ Dirk J. Struik, *A Concise History of Mathematics* (New York: Dover, 1948), p. 41.

nology in the past 400 years. The truth of any mathematical system rests upon certain terms that are left undefined and certain ideas whose truth is assumed (postulates). This is because we cannot define every term that we use. The futility of such an approach is humorously treated in the B.C. cartoon in Figure 8.4.



Source: By permission of Johnny Hart and Creators Syndicate; © 1985 News America Syndicate.

FIGURE 8.4



LANGUAGE

A convention among mathematicians is to use capital letters to refer to points and lowercase letters to refer to lines and planes.

Before we examine basic geometric definitions and concepts, let us briefly describe three fundamental undefined terms. In everyday life, we refer to the starting point of a race or a point on a map. Geometrically, a **point** has no dimension (length or width), but it does have a location. In everyday life, we refer to lines on a paper and lines of longitude. Geometrically, a **line** is straight and extends infinitely in two directions. We can also think of a line as an infinite collection of points that indicate a straight path. In everyday life, we refer to planes when we talk about floors and countertops. Geometrically, a **plane** is considered to be a flat surface that extends infinitely in all directions.

INVESTIGATION 8.1 Point, Line, and Plane

The purpose of this investigation is to have you come to appreciate the kind of thinking the Greeks started. You may find it helpful to use the point of your pencil or pen to represent a **point**, a ruler (or other object with a straight edge) to represent a **line**, and a piece of posterboard to represent a **plane**. By cutting slits in the posterboard, as shown in Figure 8.5, you can more concretely investigate the questions asked below.

Points and lines

- Draw a point on a piece of paper. How many different lines can you draw that go through that point?
- Draw two points on a piece of paper. How many different lines can you draw that go through both points?

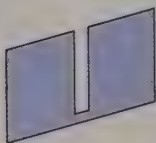
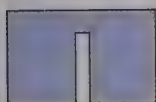


FIGURE 8.5



FIGURE 8.6



FIGURE 8.7



FIGURE 8.8

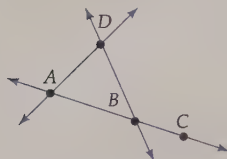


FIGURE 8.9

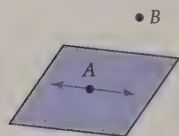


FIGURE 8.10

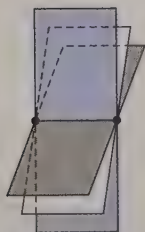


FIGURE 8.11

- C. Draw three points on a piece of paper. How many different lines can you draw that go through all three points?

If you answer any of the questions, “it depends,” can you be more specific? Write down your responses before reading on. . . .

DISCUSSION

A. An infinite number of lines will go through any point (see Figure 8.6).

B. Only one line will go through any two points. That is, if we draw *any* two points, there will always be one and *only* one line that contains those two points (see Figure 8.7).

This powerful, but generally partially understood, conclusion is often stated as follows: **Two points determine a line.** Do you understand the equivalence of the two statements?

C. How many lines we can draw through three points depends on the points. If the three points are on a line, then we can draw only one line (see Figure 8.8). However, if the three points are not on a line, then three points will determine three lines (see Figure 8.9).

This leads us to our first true definition: **Collinear** points are points that lie on the same line. As we have just seen, any two points are collinear.

On the other hand, if three (or more) points do not lie on a single line, then those points are said to be **noncollinear**. In Figure 8.9, points A, B, and C are collinear points, whereas points A, B, and D are noncollinear points.

There is a common misconception about collinear points that needs to be addressed. Draw a point on a line and place your pencil somewhere above the desk. If the point on the paper is point A and the tip of the pencil is point B (see Figure 8.10), are points A and B collinear? Think before reading on. . . .

Yes; *any* two points are said to be collinear. Even when a line is not already drawn, if it is *possible* to draw such a line, then the points are collinear.

Part 2 of Exploration 8.4 addresses similar issues.

Planes

A. Now draw a point on a piece of paper. How many different planes can go through that point?

B. Draw two points on a piece of paper. How many different planes can go through both points?

C. Draw three points on a piece of paper. How many different planes can go through all three points?

If you answer any of the questions, “it depends,” can you be more specific? Write down your responses before reading on. . . .

DISCUSSION

A. An infinite number of planes will go through any one point.

B. An infinite number of planes will go through any two points (see Figure 8.11).

 The answer for three points depends on where you put the points. If we draw three noncollinear points, there is only one plane that will contain all three points. Do you see why? Think and then read on. . . 

One of the best ways to demonstrate this requires three people. First, draw one point on a sheet of paper. Next, the first person puts a pencil in the air (tip up). Then the second person puts another pencil in the air (tip up), making sure only that the three points are noncollinear. Now the third person takes a model of a plane (cardboard or a hard-cover book). How many different planes contain all three points?

When two or more points lie on the same plane, they are said to be **coplanar** points. As you have just seen, any two or three points are coplanar, because we can always find one plane that will contain them.

There is another way to see this concept concretely. Make a model of a three-legged stool like the one shown in Figure 8.12. You can be creative; for example, use three pencils, some glue, and a piece of cardboard. Put the model on a flat surface. Then make a model of a four-legged stool like the one shown and put it on a flat surface. What do you notice? Think before reading on. . . 



FIGURE 8.12

The three-legged stool will never rock. Many four-legged stools will. Why? How would you explain why to a friend who is not in this course?

Because the three points at the bottom of the stool are not collinear, there is only one plane that will contain them. However, the four points at the bottom of the four-legged stool may or may not all lie on the same plane. Thus, we have a famous property: **Three noncollinear points determine a plane.**

Line Segments and Rays

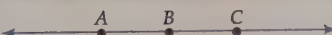
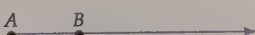
In most everyday situations, we don't work with lines; instead, we work with line segments and rays.

We define a **line segment** to be a subset of a line that contains two points of the line (which we call the **endpoints**) and all points between those two points.

We define a **ray** as the subset of a line that contains a specific point (the **endpoint**) and all the points on the line that are on the same side of that point.

Table 8.1 shows the differences among a line, a line segment, and a ray.

TABLE 8.1

Term	Diagram	Notation
Line		$\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$
Line segment		$\overline{AB}$ or $\overline{BA}$
Ray		$\overrightarrow{AB}$ but not $\overrightarrow{BA}$

The notation to represent lines, line segments, and rays is a convention that has evolved. The conventional way to denote a line is by using two points on the line and a double arrow above the two letters. In Table 8.1, it would not be

incorrect to refer to the line as $\overleftrightarrow{ABC}$, but it would be unconventional, like offering to shake someone's hand with your left hand.

The conventional way to denote a line segment is to use two letters, representing the endpoints, and a bar above the two letters.

Naming rays requires careful thought, for there is unconventional and then there is incorrect! For example, $\overrightarrow{AB} \neq \overrightarrow{BA}$. Why is this? The convention for naming a ray is to use two letters, the first of which is the endpoint of the ray. The second letter can be any other point on the ray and indicates the direction of the ray. For example, in Figure 8.13, we can speak of $\overrightarrow{DA}$, which is different from $\overrightarrow{AD}$. On the other hand, $\overrightarrow{DA}$ and $\overrightarrow{DM}$ are considered to be the same ray. Why is this?

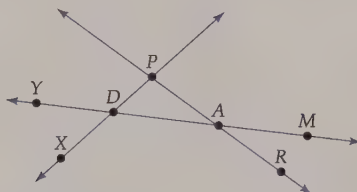


FIGURE 8.13



FIGURE 8.14

Kinds of Lines

There are many possible ways in which two or more lines may be related to each other. For example, they may **intersect**, as in Figure 8.14. Two lines are said to intersect if they have exactly one point in common.

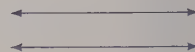
If two intersecting lines form right angles, they are said to be **perpendicular**. (See Figure 8.15.)



Perpendicular lines



Concurrent lines



Parallel lines

FIGURE 8.15

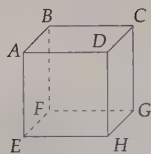


FIGURE 8.16

If three or more lines intersect at a single point (that is, have a point in common), they are said to be **concurrent** lines. (See Figure 8.15.)

As you know, not all lines intersect. If two lines lie in the same plane and never intersect, they are said to be **parallel** lines. (See Figure 8.15.)

There is a fourth possibility: two lines that do not intersect because they do not lie in the same plane. Such lines are called **skew** lines. For example, in Figure 8.16, $\overrightarrow{AD}$ and $\overrightarrow{EF}$ are skew lines.

Angles

When two or more lines intersect, many different kinds of **angles** are formed. How would you define an angle in your own words? Try to do so before moving on. . . 

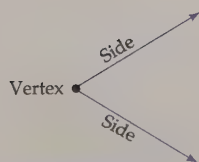


FIGURE 8.17

A common mathematical definition is that an **angle** consists of the union of two rays that have a common endpoint, which we call the **vertex** of the angle. Each of the rays is called a **side** of the angle (see Figure 8.17).

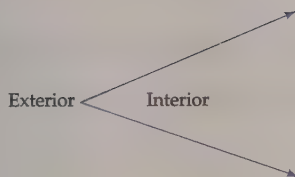


FIGURE 8.18

Does this make sense, or is it just more mathematical jargon? Is your definition consistent with this one? Think before reading on. . . 

An angle partitions a plane into three disjoint sets: the angle itself, the **interior** of the angle, and the **exterior** of the angle (see Figure 8.18).

Now that we have defined angles, let us examine how they are named. Before reading on, though, make sure that you are actively thinking while reading. What do you know about naming angles? Write down what you know before reading on. . . 

Naming Angles

We often refer to an angle by using its vertex. For example, we can call the angle at the left in Figure 8.19 angle A . We can also refer to angles by using a numbering system. In the diagram at the right in Figure 8.19, we can talk about angle 1, angle 2, angle 3, and angle 4.



FIGURE 8.19

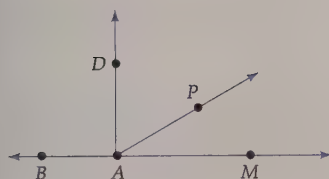
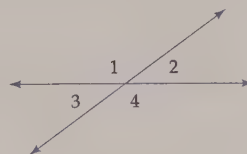


FIGURE 8.20

Both ways of referring to angles have limitations. Look at Figure 8.20. If someone asks you to look at angle A , what are the possibilities? How might we resolve this dilemma? Think before reading on. . . 

One way out of this dilemma is to use the symbol $\angle$ and three letters to name an angle. The order of the three letters is important. We start with a point on one side of the angle, then give the vertex, and then give a point on the other side of the angle (see Figure 8.21). These three points determine an angle in much the same way that three noncollinear points determine a plane.

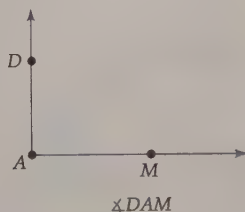
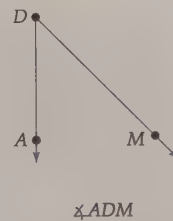


FIGURE 8.21



Thus, in Figure 8.20, we can distinguish among $\angle DAM$, $\angle PAD$, and $\angle PAM$.

$\angle PAM = \angle MAP$. Do you see why?

$\angle DAM \neq \angle ADM$. Do you see why?

Measuring Angles

We measure angles by how “open” they are, and the tool we use is a **protractor**. The most common unit of measurement for angles is a **degree**.



HISTORY

You may recall that there are 360 degrees in a circle. Do you know where the 360 comes from?

It comes from the Babylonians who, for reasons unknown to us, decided (over 3000 years ago) to divide a circle into 360 parts. We know that 360 was an important number for them, perhaps because there are approximately 360 days in a year.

When the French converted measures of length, mass, and other quantities to the metric system, why

didn't they convert measures of a circle? We know that the French attempt to metrify the week (10 days in a week and 10 hours in a day) failed. Perhaps the 24-hour day and 7-day week were just too deeply ingrained, and perhaps this was also true with angles. Actually, 360 has advantages over 100. Consider some common angles—for example, 60° and 45° . If there were 100 degrees in a circle, what would these angles be?

Although protractors come in a variety of styles, the correct use of all protractors requires the following:

1. The vertex of the angle must lie at the center of the protractor, which is not always the center of the bottom edge of the protractor (as shown in Figure 8.22).
2. One ray of the angle must lie directly under the line that goes through a 0 point of the protractor (which is not always labeled).
3. The measure of the angle is read by looking at the number corresponding to where the other ray of the angle crosses the number line that goes around the protractor.

In the right-hand figure, the protractor has been moved in order to measure an angle neither of whose vertices is parallel to the bottom of the page. Measure the two angles and then check your measurements below or with a friend.

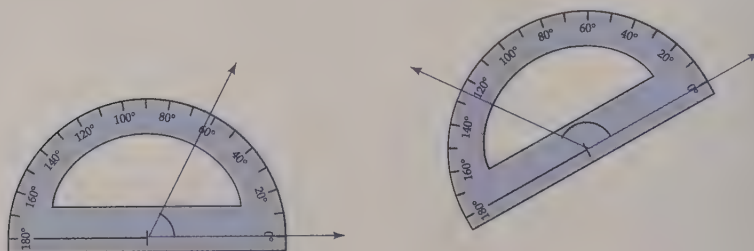


FIGURE 8.22

The measure of the angle at the left is 63 degrees, and the measure of the angle at the right is 126 degrees.

For the most part, the level of precision of measuring angles in this course will consist of rounding the measure to the nearest whole number. However, an angle can be divided into 60 minutes (denoted by $'$), and a minute can be divided into 60 seconds (denoted by $"$). Can you think of a situation in which this kind of precision would be necessary? For example, an angle of 0 degrees and 1 minute is $1/21,600$ of a circle, too small to show on a piece of paper.

Classifying Angles

We have names for different kinds of angles. We can classify angles with respect to an angle whose measure is one-fourth of a circle—that is, 90 degrees. This is a **right angle**. Just as $1/2$ is a reference fraction, a right angle is a reference angle.

An angle whose measure is less than 90 degrees is called an **acute angle**.

An angle whose measure is greater than 90 degrees but less than 180 degrees is called an **obtuse angle**.

If angles are seen from a static perspective, these terms are often sufficient, but angles can also be seen from the dynamic perspective of **turns**. For example, someone might be told to open a valve “one-quarter of a turn,” or 90 degrees. A half turn would be 180 degrees, a full turn would be 360 degrees, and one and a half turns would be 540 degrees. It sometimes makes sense to speak of angles with measures of 180 degrees or greater.

An angle whose measure is 180 degrees is called a **straight angle**.

An angle whose measure is greater than 180 degrees is called a **reflex angle**. Determine the measure of the reflex angle in Figure 8.23. You might want to compare notes with another student because there is more than one way to do it.

The measure of the angle is 225° . One way to determine the measure is to extend the horizontal ray to the right and measure the resulting acute angle and then add 180 to the measure of the acute angle. Another way is to measure the obtuse angle and then subtract it from 360. Do you understand both ways? Did you determine the measure in a third way?

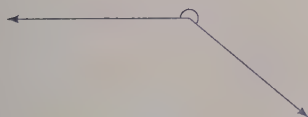


FIGURE 8.23



CHILDREN

When we say that one angle is bigger than the other, we mean that its measure is greater.

However, some children will say that angle 2 in Figure 8.24 is bigger than angle 1.

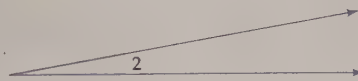


FIGURE 8.24

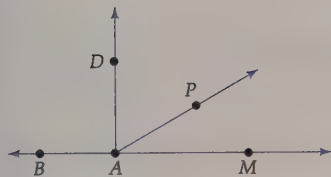


FIGURE 8.25

Sometimes the relative location of two angles is important. For example, we speak of adjacent angles. What do you think adjacent angles are? Can you draw two adjacent angles? Can you write a definition? If you do, give it to someone who doesn't know the term and see whether that person understands the term in the way that you meant it. Then read on. . .

Adjacent means “next to,” but more specifically, two angles are **adjacent** if and only if they have the same vertex and share a side.

Thus, in Figure 8.25, $\angle DAP$ and $\angle PAM$ are adjacent angles. There are other adjacent angles in the diagram. How many pairs of adjacent angles can you find in the diagram?

We sometimes speak of complementary and supplementary angles.

If the sum of the measures of two angles is 90 degrees, we call them **complementary** angles.

If the sum of the measures of two angles is 180 degrees, we call them **supplementary** angles.



LANGUAGE

Many of my students struggle to understand *adjacent*. The following has been helpful to many of them. In addition to the angles' having a common vertex and a common side, the interiors of the two angles must be disjoint; that is, no point can be in the interior of both angles.

In Figure 8.25, $\angle DAP$ and $\angle PAM$ are complementary, and $\angle BAP$ and $\angle PAM$ are supplementary. It is important to note that angles do not have to be adjacent angles in order to be complementary or supplementary.

Look at Figure 8.26. Which pairs of angles appear to be equal? Could you justify their being equal or prove why they are equal?

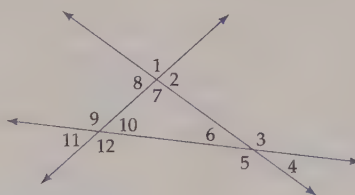


FIGURE 8.26

LANGUAGE

Many students argue convincingly that their lives would be much easier if such angles were called opposite angles.

What other relationships between pairs of angles do you see?

Whenever two lines intersect, four angles are formed. The angles that are opposite each other—for example, $\angle 1$ and $\angle 7$ in Figure 8.26—are called **vertical angles**. How would you define *vertical angles* formally? This will be left as an exercise.

At a van Hiele level 1 perspective, it certainly seems reasonable that vertical angles are always equal. An example of a level 4 activity would be a formal proof. Though this text is not emphasizing proofs, I believe a few proofs are helpful so that the elementary teacher can see what the various levels look like. How might we prove that the vertical angles *must* be equal? Think about this before reading on. . . .

A key to the proof is to realize that each pair of adjacent angles in Figure 8.26 is supplementary. For example, $\angle 1$ and $\angle 2$ are supplementary. Do you see why? If you didn't see this, can you now see how we might show that $\angle 2$ and $\angle 8$ must be equal?

If you see the relationships, the proof is fairly straightforward:

Statement	Justification
$m\angle 1 + m\angle 2 = 180$	Together they form a straight angle.
$m\angle 7 + m\angle 2 = 180$	Together they form a straight angle.
$m\angle 1 + m\angle 2 = m\angle 7 + m\angle 2$	Transitive property; both sums are equal to 180.
$m\angle 1 = m\angle 7$	Algebra—we subtracted the same amount from both sides.

Again, it is this type of thinking (level 4) that made what the Greeks did over 2000 years ago so different from what had been done before.

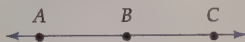
Summary

In this section, we have focused on the building-block language and concepts of Euclidean geometry: point, line, and plane. Line segments can intersect and join together to make plane figures, also called two-dimensional figures, which we shall explore in Section 8.2. We have examined different subsets of lines (line segments and rays) and different kinds of lines (parallel, perpendicular, concurrent, and skew). Plane figures can intersect to make space figures, also called three-dimensional figures, which we will explore in Section 8.3.

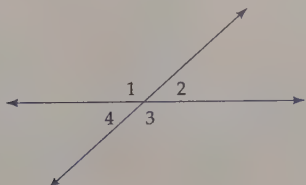
We have examined the different kinds of angles that can be formed by intersecting lines. We can classify angles by their measure: acute, right, obtuse, straight, and reflex. We can also classify angles by their relationship to other angles: complementary, supplementary, and vertical.

EXERCISES 8.1

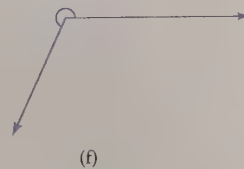
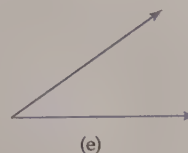
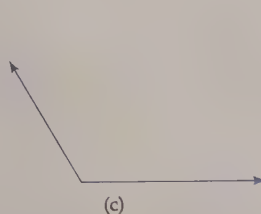
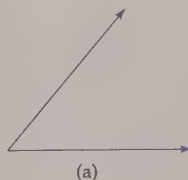
1. Name all the possible different rays that can be formed from the three points below.



2. How many different rays can be formed from four collinear points?
3. Sketch four lines such that three are concurrent and two are parallel.
4. Sketch a pair of angles whose intersection is
- exactly two points or explain why this is not possible.
 - exactly three points or explain why this is not possible.
 - exactly four points or explain why this is not possible.
5. True or false? If true, briefly explain why. If false, provide a counterexample.
- If two distinct lines do not intersect, then they are parallel.
 - If two lines are parallel, then they lie in the same plane.
 - If two lines intersect, then they lie in the same plane.
 - If two planes have a point in common, then they have infinitely many points in common.
 - If a line is perpendicular to a plane, then it is perpendicular to all lines in that plane that go through the point at which the line intersects the plane.
 - If a plane contains a line segment, then it contains the line of which the segment is a subset.
 - If three lines are concurrent, then they are also coplanar.
 - If two planes intersect, then the intersection is either a point or a line.
 - Three different planes can have at most one point in common.
 - Two distinct planes either intersect or are parallel.
 - Two intersecting lines determine one and only one plane.
 - Vertical angles can be supplementary.
6. When two lines in a plane intersect, four distinct regions are formed.

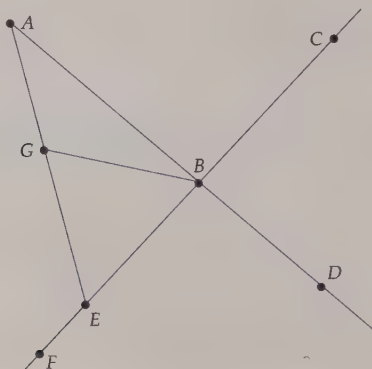


- If three lines intersect in a plane, what is the maximum number of regions that can be formed?
 - If four lines intersect in a plane, what is the maximum number of regions that can be formed?
 - If five lines intersect in a plane, what is the maximum number of regions that can be formed?
 - If n lines intersect in a plane, what is the maximum number of regions that can be formed?
7. Estimate the measure of the following angles. Describe your reasoning process. Then measure the angles with a protractor and determine your percent error.

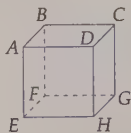


8. Make at least five more angles. Next estimate their measure, determine their measure, and determine your percent error. Then graph your percent error for the angles in Exercises 7 and 8. If you note that the percent error seems to be decreasing, can you articulate any thinking processes that you are developing that are making your estimates more accurate?
9. Using only a straightedge and reasoning, try to make angles with the following measures. Then describe your reasoning process. After making each angle, check your work with a protractor and determine the percent error.
- 30°
 - 45°
 - 150°
 - 300°
 - 67°

10. In the figure below, $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are perpendicular lines.



- Name two complementary angles.
 - Name two supplementary angles.
 - Name two vertical angles.
 - Name two adjacent angles.
11. Refer to the cube pictured below, and use symbols such as AB to name the following:



- Two parallel line segments
 - Two line segments that do not lie in the same plane
 - Two intersecting line segments
 - Three concurrent line segments that do not lie in a single plane
 - Two skew line segments
 - A pair of supplementary angles
 - A pair of perpendicular line segments
 - Are points A , B , and H coplanar points? Why or why not?
12. This exercise focuses on communication. For each of the following terms, suppose a friend in the class has come to you and told you that the term just doesn't make sense. Describe how you would help that person to make sense of the term. Your description must include your own definition of the term.
- Collinear points
 - Concurrent lines
 - Adjacent angle
 - Skew lines
13. a. Find the sum of the following two angles: $45^\circ 32'$ and $73^\circ 57'$.
b. Write $18^\circ 45'$ in decimal form, to two decimal places.
c. Express $128^\circ 24'$ in decimal form, to two decimal places.

14. Clock problems.

- How many times a day will the minute hand be directly on top of the hour hand?
- What times could it be when the two hands make a 90-degree angle?
- What angle do the hands make at 7 o'clock?
- What angle do the hands make at 3:30?
- What angle do the hands make at 2:06?
- Make up and answer a problem like these. Then swap problems with a friend, and have the friend check your work and you check the friend's work.

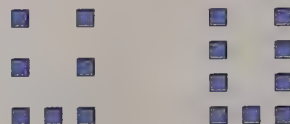
15. How many different pairs of vertical angles are formed by three concurrent lines?

16. Dot-matrix printers, television screens, and college marching bands all create pictures from dots. For example, the diagram below shows three ways in which the letter U might be formed on a dot-matrix printer.

- Using a 5 by 5 grid, make designs for each of the (capital) letters of the alphabet.



- The diagram below shows two different arrangements of band members that form the letter U. Determine the minimum number of band members necessary to spell MSU on a field.



- Determine the minimum number of band members necessary to spell your college's initials on a field.
17. Draw a Venn diagram to show the relationships among intersecting, perpendicular, parallel, concurrent, and skew lines. Let U be the set of all lines. Explain your diagram.
18. Construct two rays AB and CD for which $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is a set containing one point and $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \emptyset$.
19. Define the term *vertical angles* formally.
20. In this exercise, our universal set will be the 26 capital letters of the alphabet. Separate the letters into two or more groups so that each group is alike in some way. Describe each subset and give it a name. If you know of a mathematical term that fits the description, use it; if you don't, then be creative. In how many different ways can you arrange the letters into subsets? For each classification, remember to describe the subset, give it a name, and list the members of that subset.

SECTION

8.2

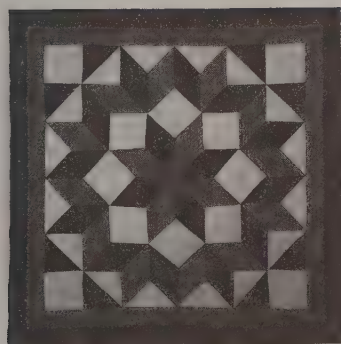
TWO-DIMENSIONAL FIGURES

WHAT DO YOU THINK?

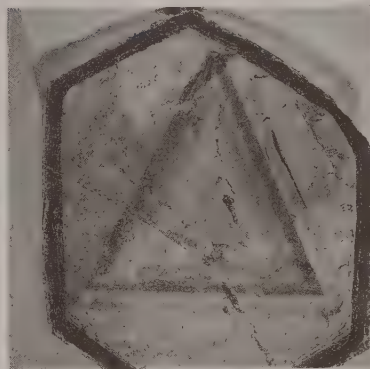
- In what ways are triangles and quadrilaterals different? In what ways are they similar?
- How are circles and polygons related?
- Can every polygon be broken down into triangles? Why or why not?
- Why do we use two words to name different triangles, but only one word to name different quadrilaterals?

Think of geometric figures that people generally find pleasing, such as those in Figure 8.27. What words would you use to explain why these objects are interesting or appealing? When you look at the various objects and pictures, what similarities do you see between certain objects and shapes—for example, triangles and hexagons?

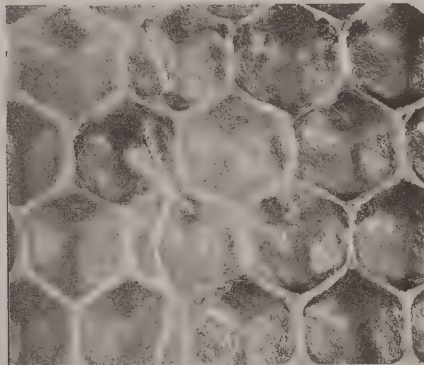
When we discuss similarities and differences in my class, many geometric terms emerge in the discussion. Some students talk about similar shapes—for example, hexagons in honeycombs and snowflakes, squares in some pictures, and triangles in others. Some students observe that many of the shapes are symmetrical. In explaining similarities, some students talk about the angles, the length of sides, or the fact that some figures look similar. Many students observe that even the more complex shapes can be seen as being constructed from simpler shapes, such as triangles and quadrilaterals.



(a)
Carpenter's Wheel quilt



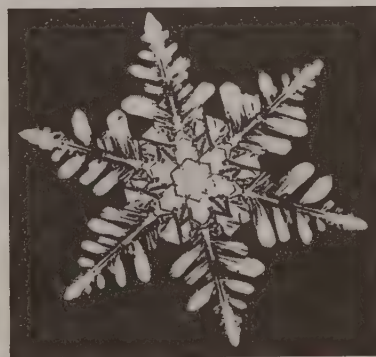
(b)



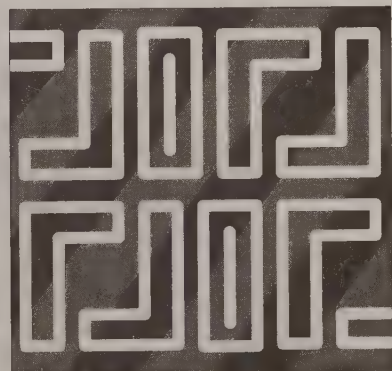
(c)



(d)
lamp post



(e)
snowflake



Source: Le Roy H. Appleton, *American Indian Design & Decoration* (New York: Dover, 1950).

(f)
Native American basket weaves

FIGURE 8.27

In this section, we will learn more about geometric figures and shapes, and you will learn and discover some of the secrets of the patterns in the objects you have seen. During this time, you will learn quite a bit of vocabulary and many of the properties that make these patterns possible. In Chapter 9, we will examine how shapes can be put together to make interesting designs you have seen, such as those in quilts, plants and animals, and buildings.

INVESTIGATION 8.2 Classifying Figures

Before we examine and classify important two-dimensional shapes, we first need to investigate the kinds of possible two-dimensional shapes. Although most of elementary students' exploration of two-dimensional figures will involve polygons and circles, it is important to know that these figures represent but a small subset of the kinds of figures that mathematicians study. Both circles and polygons are curves. A mathematical curve can be thought of as a set of points that you can trace without lifting your pen or pencil. If you watch young children making drawings, you discover that they make all sorts of curves!

As you might expect, if we look at any set of curves, there are many ways to classify them. As I have done throughout this book, rather than giving you the major classifications, I will engage you in some thinking before presenting them. Look at the 13 shapes in Figure 8.28 and classify them into two or more groups so that each group has a common characteristic. Do this in as many different ways as you can, and then read on. . . .

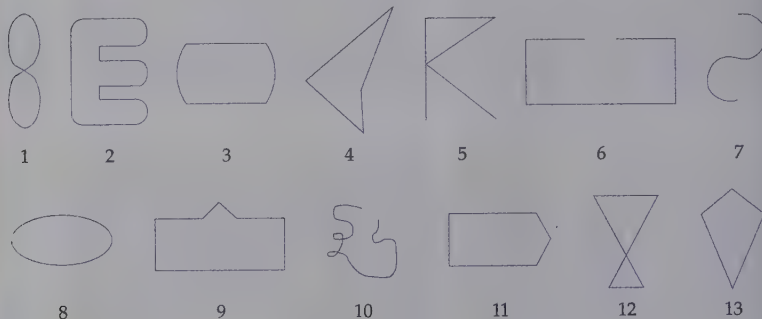
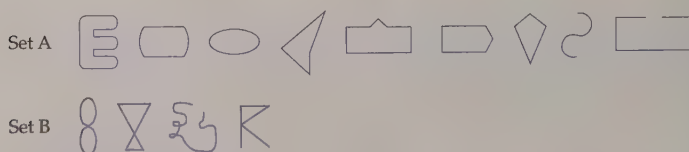


FIGURE 8.28

DISCUSSION

One way to sort the figures is shown below. How would you describe the figures in set A and the figures in set B? Do this before reading on. . . .



The figures in set A are said to be simple curves. We can describe simple curves in the following way: A figure is a **simple curve** if we can trace the fig-

LANGUAGE

What other words might you use to describe the intersecting and not intersecting subsets?

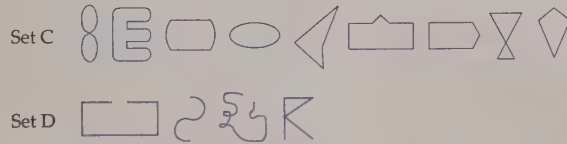
Some students use the phrase “trace over,” and others talk about figures that “run over themselves” or “cross themselves.” Other students talk about the set of figures that contain two smaller regions within each figure.

CHILDREN

Using the terms developed in this investigation, how would you classify dot-to-dot pictures?

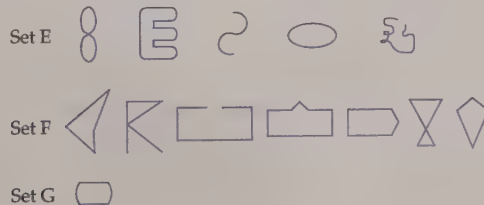
ure in such a way that we never touch a point more than once. If you look at the figures in set A, you can see that they all have this characteristic; and all the figures in set B have at least one point where the pencil touches twice, no matter how you trace the curve.

Now look at the curves in sets C and D. How would you describe the figures in set C and the figures in set D? Do this before reading on. . . .



The figures in set C are said to be closed curves. We can describe closed curves in the following way: A figure is a **closed curve** if we can trace the figure in such a way that our starting point and our ending point are the same. If you look at the figures in set C, you can see that they all have this characteristic; and no matter how you try, you cannot trace the figures in set D with the same starting and ending point.

Now look at the curves in sets E, F, and G. How would you describe the figures in set E, the figures in set F, and the figures in set G? Do this before reading on. . . .



These three sets are interesting for two reasons. First, these sets are likely to be generated in the classroom—both your classroom and the elementary classroom. Second, the language used to describe the three sets poses a challenge, for most people describe the figures in set E as consisting only of curvy lines, the figures in set F as consisting only of straight line segments, and the figure in set G as having both curvy and straight line segments. The challenge here is that when mathematicians use the word *curve*, this word encompasses both curvy and straight line segments—a curve is a set of points that you can trace without lifting your pen or pencil. There is nothing wrong with students’ use of the terms *curvy* and *straight*. What is important is the realization that we are using the words *curve* and *curvy* in different ways. We do this all the time in everyday English. Recall the various uses of the word *hot* in Chapter 1: “It sure is a hot day.” “I love Thai food because it is hot.” “This movie is really hot!”

Most of our investigations of curves will focus on simple, closed curves. Looking at the descriptions above, try to define the term *simple closed curve* before reading on. . . .

We will define a **simple closed curve** as a curve that we can trace without going over any point more than once while beginning and ending at the same point. The set of polygons is one small subset of the set of simple closed curves.

At this point, you might want to do the following activity with another student.

- Draw a simple closed curve.
- Draw a simple open curve.
- Draw a nonsimple closed curve.
- Draw a nonsimple open curve.

Exchange figures with another student. Do you both agree that each of the other's drawings matches the description? If so, move on. If not, take some time to discuss your differences.

There is an important mathematical theorem, known as the *Jordan curve theorem*, after Camille Jordan: Any simple closed curve partitions the plane into three disjoint regions: the curve itself, the interior of the curve, and the exterior of the curve. See the example in Figure 8.29.

HISTORY

Generally, when a theorem is named after a person, it is named after the person who first proved the theorem. In this case, Jordan's proof was found to be incorrect, but the theorem is still named after him!

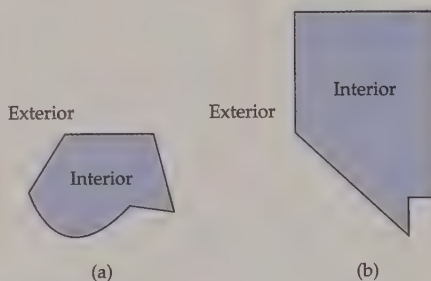


FIGURE 8.29

I know that many students' reaction to this theorem is "Why do we need to prove something that is so obvious?" As mentioned before, being critical ("to examine closely") is an attitude that I invite. In the examples in Figure 8.29, deciding whether a point is inside or outside is easy. However, look at Figure 8.30. Although this figure is a simple closed curve, it is a rather complicated figure, and such complicated shapes are encountered in some fields of science. Is point A inside or outside? How would you determine this? Think before reading on. . . .

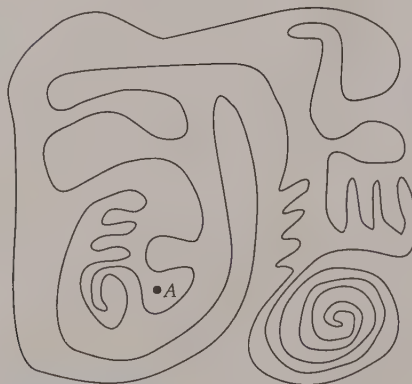


FIGURE 8.30

Some wag once remarked that mathematicians are among the laziest people on earth because they are always looking for shortcuts and simpler ways to solve problems. Thus, you may be wondering whether someone has

MATHEMATICS AND CHILDREN

The active and curious reader here might realize that this procedure might have applications for solving mazes, and there is a branch of mathematics that does analyze mazes. Another active and curious reader might realize and/or remember that most children enjoy the challenge of solving mazes.

found an easier way to solve these problems. Indeed, someone has: I can use the shape in Figure 8.31 to illustrate the method. Start at a point that is clearly outside the shape and draw a line segment connecting that point to the point you are looking at; it helps if you pick an outside point so that the line segment will cross the curve in as few points as possible. Each time you cross a point, it's like a gate—if you were outside, you are now inside; if you were inside, you are now outside. Thus, it is a relatively simple matter to determine that point B is inside the curve.

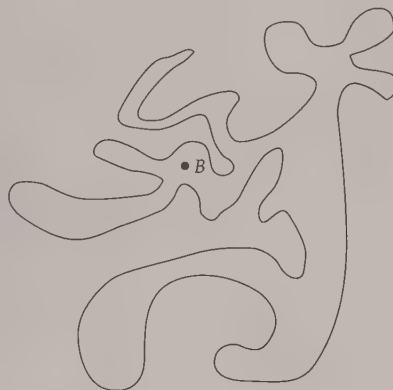


FIGURE 8.31

An active reader might go back to the first shape to see whether point A is inside or outside. Even more active would be the reader who then worked on the figure to verify to his or her satisfaction that the method just described really does work.

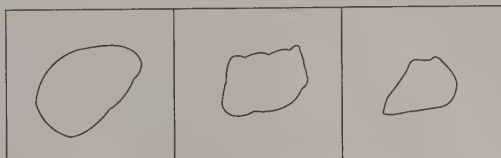


CHILDREN

One way in which children can learn mathematical ideas is by copying figures. The drawings in Figure 8.32 represent common attempts by four-year-olds to copy a circle, a square, and a triangle. What do you notice?

The drawings show that what the children notice is that all are simple closed curves. What they do not seem to notice are other attributes that seem so obvious to us: the number of sides,

the size of the angles, and so on. (This discussion connects to a field of mathematics called *topology*, one of the many fields of mathematics that we will not study in this text.) These same children as six-year-olds (with no "training") will generally copy the figures fairly accurately. Their minds are developing at a rapid rate at this period of their lives.



Source: From *How Children Learn Mathematics*, 3/e by Copeland, Richard W., © 1979. Reprinted by permission of Prentice-Hall, Inc., Upper Saddle River, NJ.

FIGURE 8.32

Polygons

A **polygon** is defined as a simple closed curve composed only of line segments. Thus, the simple closed curve in Figure 8.29(a) is not a polygon, whereas the simple closed curve in Figure 8.29(b) (which looks like the state of Nevada) is a polygon. On any polygon, the point at which two sides meet is called a **vertex**, the plural of which is **vertices**. The line segments that make up the polygon are called **sides**.

The word *polygon* has Greek origins: *poly-*, meaning “many,” and *-gon*, meaning “sides.” You are already familiar with many kinds of polygons. Just as we found in Chapter 2 that the names we give numbers have an interesting history, so do the names we give to polygons. The most basic naming classification involves the number of sides (see Table 8.2).

TABLE 8.2

Number of sides	Name
3 sides	Triangle
4 sides	Quadrilateral
5 sides	Pentagon
6 sides	Hexagon
7 sides	Heptagon
8 sides	Octagon
n sides	n -gon

Congruence

As you saw in Part 5 of Exploration 8.1, questions sometimes arise about whether two figures are “the same” or not. Such observations and questions deal with the idea of congruence.

At an informal level, we can say that two figures are congruent iff they have the same shape and size. An informal test of congruence is to see whether you can exactly superimpose one figure on top of the other. This is closely connected to how children initially encounter the concept and is related to the dictionary definition: “coinciding exactly when superimposed.”⁸ That is, if one figure can be superimposed over another so that it fits perfectly, then the two figures are congruent. (Henry Ford changed our world by having the inspiration to make cars out of sets of congruent parts rather than one at a time. For example, the left front fender of a 1997 Dodge Caravan in Minnesota is congruent to the left front fender of any other 1997 Dodge Caravan.)

Formally, we say that two polygons are **congruent** iff all pairs of corresponding parts are congruent. In other words, in order for us to conclude that two polygons are congruent, two conditions have to be met: (1) Each corresponding pair of angles must have the same measure, and (2) each corresponding pair of sides must have the same length. We use the symbol $\cong$ to denote congruence.

For example, in Figure 8.33, triangle CAT and triangle DOG are congruent iff $\angle C \cong \angle D$, $\angle A \cong \angle O$, $\angle T \cong \angle G$, $\overline{CA} \cong \overline{DO}$, $\overline{AT} \cong \overline{OG}$, and $\overline{TC} \cong \overline{GD}$.

⁸ *The American Heritage Dictionary of the English Language* (Boston: Houghton Mifflin, 1996). Copyright © 1996 by Houghton Mifflin Company. Reproduced by permission from *The American Heritage Dictionary of the English Language*, Third Edition.

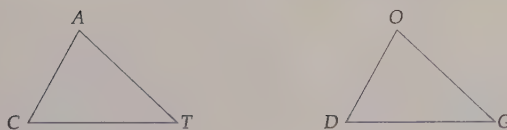


FIGURE 8.33

The notions of congruent and equal are related concepts. We use the term *congruence* when referring to having the same shape, and we use the term *equal* when referring to having the same numerical value. Thus, we do not say that two triangles are equal; we say that they are congruent. Similarly, when we look at line segments and angles of polygons, we speak of congruent line segments and congruent angles. However, when we look at the numerical value of the line segments and angles, we say that the lengths of two line segments are equal and that the measures of two angles are equal.

**BEYOND THE CLASSROOM**

When do we need congruence in everyday life or in work situations? Take a few minutes to think about this before reading on. . . . 

Congruence is important in manufacturing; for example, the success of assembly-line production depends on being able to produce parts that are congruent. One of the

differences between a decent quilt and an excellent one is being able to ensure that all the squares are congruent. This is quite difficult when using complex designs. Most of the manipulatives teachers use with schoolchildren (pattern blocks, unifix cubes, Cuisenaire rods, and fraction bars) have congruent sets of pieces.

INVESTIGATION 8.3 Re-creating Shapes from Memory

For this investigation, you will want to have a pencil and an eraser.

A. Turn to page 489. Keep the book open for about 1 second and then draw Figure 8.76 from memory.

Check the picture again for 1 second. If your drawing was incomplete or inaccurate, change your drawing so that it is accurate. Check the picture again for 1 second. Keep doing this until your drawing is complete and accurate.

Now go back and try to describe your thinking processes as you tried to re-create the figure. From an information processing perspective, your eyes did not simply receive the image from the paper; your knowledge of geometry helped determine *how* you saw the picture. What did you hear yourself saying to help you remember the picture? Compare your thinking processes with another student. Then read on. . . . 

DISCUSSION

Some students see a diamond and 4 right triangles. Other students see a large square in which the midpoints of the sides have been connected to make a new square inside the first square. Yet other students see four right triangles that have been connected by “flipping” or rotating them.

B. Now look at Figure 8.77 (on page 489) for several seconds. Then try to draw it from memory. As before, check the picture again for a couple of seconds. If your drawing was incomplete or inaccurate, fix it. Keep doing this until your drawing is complete and accurate. Then go back and try to describe your thinking processes as you tried to re-create the figure. 

DISCUSSION

This figure was more complex. Some students see a whole design and try to remember it.

Some decompose the design into four black triangles and four rectangles as shown in Figure 8.34.

The figure can also be seen as being composed of 9 squares, which can also be seen in Figure 8.34. The four corner squares have been cut to make congruent right triangles. Each of the other four squares on the border has been cut into two congruent rectangles.

Some students re-create this figure by seeing a whole square and then looking at what was cut out (see Figure 8.35). That is, they saw that they needed to cut out each corner, and they saw that they needed to cut out a rectangle on the middle of each side. Finally, they remembered to cut out a square in the center.

When I did this exercise myself, I saw a “square race track” with a similar pattern on each corner (Figure 8.36).

This is a famous quilting pattern called the Churn Dash. When people made their own butter, the cream was poured into a pail called a churn. Then the churner rolled a special pole back and forth with his or her hands. At the end of the pole was a wooden piece called a dash, which was shaped like the figure you saw. If you make many copies of this pattern and put them together, you can see what a Churn Dash quilt looks like.

There are several implications for teaching from this investigation. How a person re-creates the figure is related to the person’s spatial thinking preferences and abilities. Different people “see” different objects. That is, not everyone sees the figure in the same way. Although there are differences in how people re-create the figure, very few people can re-create the figure without doing some kind of decomposing—that is, without breaking the pattern into smaller parts. Being able to do this depends partly on spatial skills and partly on being able to use various geometric ideas (congruent, triangle, square, rectangle) at least at an intuitive level.

Although some people manage to live happy, productive lives at the lowest van Hiele level, an understanding of basic geometric figures and the relationships among them is often helpful in everyday life (for example, in home repair projects and quilting) and in many occupations. Now that your interest in geometric figures has been piqued by this investigation and the pictures at the beginning of the section, let us examine the characteristics and properties of basic geometric shapes.

Triangles

There are many kinds of triangles. As before, you will probably better understand the names and be able to apply your knowledge of triangles if you actively engage in the following investigation.

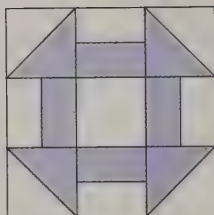


FIGURE 8.34

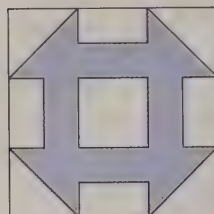


FIGURE 8.35

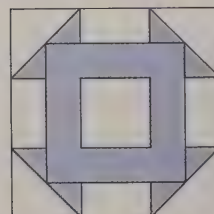


FIGURE 8.36

INVESTIGATION 8.4 Classifying Triangles

You will find nine triangles in Figure 8.37. Copy them and cut them out, and then separate them into two or more subsets so that each subset has a common characteristic. Can you come up with a name for each subset? What names might children give to different subsets? Do this in as many different ways as you can before reading on. . . .

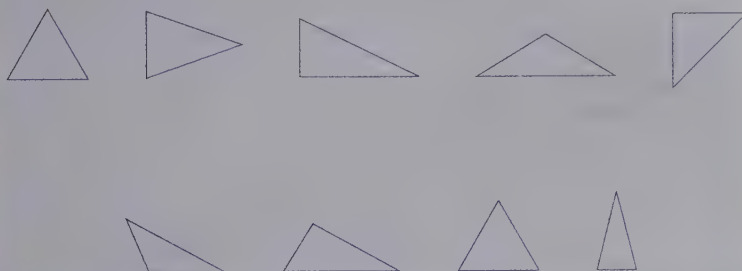


FIGURE 8.37

DISCUSSION

STRATEGY 1: Consider sides

One way to classify triangles is by the length of their sides: all three sides having equal length, two sides having equal length, no sides having equal length. There are special names for these three kinds of triangles.

- If all three sides have the same length, then we say that the triangle is **equilateral**.
- If at least two sides have the same length, then we say that the triangle is **isosceles**.
- If all three sides have different lengths—that is, no two sides have the same length—then we say that the triangle is **scalene**.

Which of the triangles in Figure 8.37 are scalene? Which are isosceles? Which are equilateral?

STRATEGY 2: Consider angles

We can also classify triangles by the relative size of the angles—that is, whether they are right, acute, or obtuse angles. This leads to three kinds of triangles: right triangles, obtuse triangles, and acute triangles.

- We define a **right** triangle as a triangle that has one right angle.
- We define an **obtuse** triangle as a triangle that has one obtuse angle.
- We define an **acute** triangle as a triangle that has three acute angles.

Many students see a pattern: A right triangle has 1 right angle, an obtuse triangle has 1 obtuse angle, yet an acute triangle has 3 acute angles. What was the pattern? Why doesn't it hold? Think before reading on. . . .

The key to this comes from looking at the triangles from a different perspective: Every right triangle has 2 acute angles, and every obtuse triangle has 2 acute angles; thus a triangle having more than 2 acute angles will be a different kind of triangle. This perspective is represented in Table 8.3. Does it help you to understand better the three definitions given above?

TABLE 8.3			
First angle	Second angle	Third angle	Name of triangle
Acute	Acute	Right	Right triangle
Acute	Acute	Obtuse	Obtuse triangle
Acute	Acute	Acute	Acute triangle



FIGURE 8.38

STRATEGY 3: Consider angles and sides

This naming of triangles goes even further. What name would you give to the triangle in Figure 8.38?

This triangle is both a right triangle and an isosceles triangle, and thus it is called a right isosceles triangle or an isosceles right triangle. How many possible combinations are there, using both classification systems? Work on this before reading on. . .

There are many strategies for answering this question. First of all, we find that there are nine possible combinations (see Figure 8.39). We can use the idea of Cartesian product to determine all nine. That is, if set S represents triangles classified by side, $S = \{\text{Equilateral, Isosceles, Scalene}\}$, and set A represents triangles classified by angle, $A = \{\text{Acute, Right, Obtuse}\}$, then $S \times A$ represents the nine possible combinations.

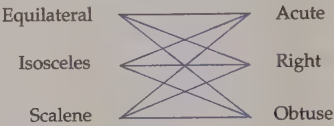


FIGURE 8.39

However, not all nine combinations are possible. For example, any equilateral triangle must also be an acute triangle. (Why is this?) Therefore, “equilateral acute” is a redundant combination. However, it is possible to have scalene triangles that are acute, right, and obtuse. Similarly, we can have isosceles triangles that are acute, right, and obtuse.

Name the two triangles in Figure 8.40. Then read on. . .



FIGURE 8.40

**MATHEMATICS**

Did you notice the geometric balance in Figure 8.39, which represents the Cartesian product of the two sets of triangles?

Both are obtuse, isosceles triangles. The orientation on the left is the standard orientation for isosceles triangles. Any time we position a triangle so that one side is horizontal, we call that side the **base**. As stated at the beginning of the chapter, students often see only one aspect of a triangle; for example, they see the triangle at the left as isosceles but not also obtuse, and they see the triangle at the right as obtuse but not also isosceles.

Special Line Segments in Triangles

There are four special line segments that have enjoyed tremendous influence in Euclidean geometry: angle bisector, median, altitude, and perpendicular bisector.

An **angle bisector** is a line segment that bisects an angle of a triangle.

In triangle ABC (Figure 8.41), $\overline{AD}$ is an angle bisector. Hence, $m\angle BAD = m\angle DAC$.

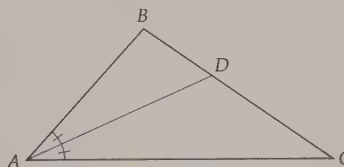


FIGURE 8.41

A **median** is a line segment that connects a vertex to the midpoint of the opposite side.

In triangle SAT (Figure 8.42), $\overline{TR}$ is a median. Hence, $SR = RA$.

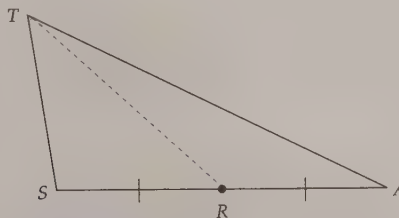


FIGURE 8.42

An **altitude** is a perpendicular line segment that connects a vertex to the side opposite that vertex. In some cases, as in $\triangle SAT$ below, we need to extend the opposite side to construct the altitude.

In triangle ABC (Figure 8.43), $\overline{BF}$ is an altitude. Hence, $m\angle BFA = m\angle BFC = 90^\circ$.

In triangle STA , $\overline{TP}$ is an altitude. Hence, $m\angle TPA = 90^\circ$.

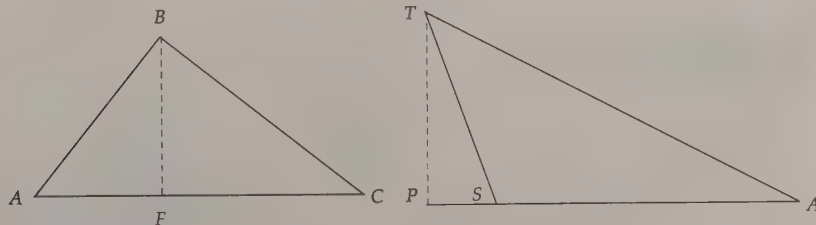


FIGURE 8.43

Many students have trouble with the idea of an altitude being outside the triangle. This happens when we have an obtuse triangle oriented with one side of the obtuse angle as the base of the triangle. If you are having trouble connecting the definition of *altitude* in these situations, I recommend the following: Trace the triangle and cut it out. Stand it up so that $\overline{SA}$ is on the plane of your desk and T is above that plane. Now draw a line from T that goes “straight down.” What do you notice?

If triangle STA were large enough so that you could stand with your head at point T , the length of line segment $\overline{TP}$ would tell you how tall you were!

A **perpendicular bisector** is a line that goes through the midpoint of a side and is perpendicular to that side. In triangle PEN (Figure 8.44), $\overline{MX}$ is a perpendicular bisector of side $\overline{PN}$, because M is the midpoint of $\overline{PN}$ and $\overline{MX}$ is perpendicular to $\overline{PN}$.

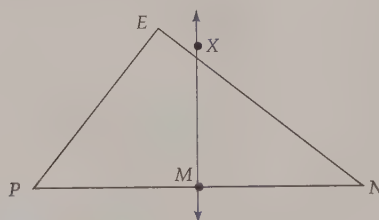


FIGURE 8.44

Triangle Properties

You probably remember that the sum of the measures of the angles in any triangle is 180 degrees. How could you convince someone who does not know that?

This is a case in which the van Hiele levels are instructive. A level 2 activity would be to have the students measure and add the angles in several triangles. If their measuring was relatively accurate, one or more students would see the pattern and offer a hypothesis that the sum is always 180.

An example of a level 3 activity, which I recommend if you have never done it, is to cut off the three corners of a triangle as in Figure 8.45 and then put the three corners together. What do you see?



FIGURE 8.45

An example of a level 4 activity would be a formal proof.

We now turn our attention to polygons having four sides—that is, quadrilaterals.

Quadrilaterals

Do you recall Investigation 2.1, where we classified quadrilaterals in order to develop the idea of subsets? You may wish to refresh your memory by reviewing that investigation before reading on.

In this book, we will define the following kinds of quadrilaterals:

- A **trapezoid** (Figure 8.46) is defined as a quadrilateral with at least one pair of parallel sides.

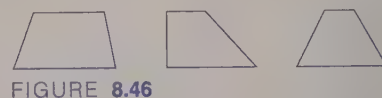


FIGURE 8.46

- A **parallelogram** (Figure 8.47) is defined as a quadrilateral in which both pairs of opposite sides are parallel.

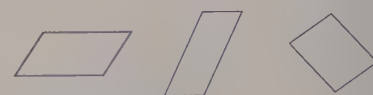


FIGURE 8.47

BEYOND THE CLASSROOM

How many of each kind of quadrilateral can you find in everyday life?



LANGUAGE

Most of my students do not realize that there is not a universal dictionary of mathematical terms. With respect to defining quadrilaterals, there are some mathematicians who prefer to define trapezoids as quadrilaterals with exactly one pair of parallel sides. I prefer to define trapezoids as quadrilaterals with at least one pair of parallel sides for reasons that you will see later in this chapter.

- A **kite** (Figure 8.48) is defined as a quadrilateral in which two pairs of adjacent sides are congruent.
- A **rhombus** (Figure 8.49) is defined as a quadrilateral in which all sides are congruent.
- A **rectangle** (Figure 8.50) is defined as a quadrilateral in which all angles are congruent.
- A **square** (Figure 8.51) is defined as a quadrilateral in which all four sides are congruent and in which all four angles are congruent.



FIGURE 8.48

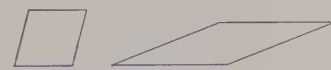


FIGURE 8.49

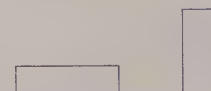


FIGURE 8.50



FIGURE 8.51

Diagonals One characteristic of all polygons with more than three sides is that they have diagonals. The more sides in the polygon, the more diagonals. This term is probably familiar to most readers. However, before reading the definition of *diagonal* below, stop and try to define the term yourself so that it works for all polygons, not just squares and other quadrilaterals. Then read on. . . .



A **diagonal** is a line segment that joins two nonadjacent vertices in a polygon.

Figure 8.52 shows two different diagonals. One of the exercises will ask you to find patterns to determine the number of diagonals in any polygon.

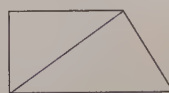


FIGURE 8.52

Angles in quadrilaterals We know that the sum of the measures of the angles of any triangle is 180 degrees. Will the sum of the measures of the four angles of *any* quadrilateral also be equal to one number, or will there be different numbers for different quadrilaterals? What do you think? What could you do to check your hypothesis? Once you believe that your hypothesis is true, how could you prove it? Think before reading on. . . .



It turns out that for any quadrilateral, the sum of the measures of the four angles is 360 degrees. The following discussion is an informal presentation of one proof. If we draw a generic quadrilateral *QUAD* and one diagonal, as in Figure 8.53, what do you notice that might be related to this proof?

If you see two triangles and *connect* this to your knowledge that the sum of the measures of the angles of a triangle is 180 degrees, you have the key to the proof. That is, you can conclude that the sum of all six angles must be 360 degrees. However, these six angles are equivalent to the four angles of the quadrilateral!

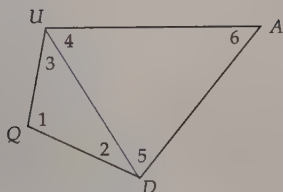


FIGURE 8.53

Relationships Among Quadrilaterals

If you did Part 6 of Exploration 8.1 (Quadrilaterals), you found that we can view the set of quadrilaterals in much the same way we view a family tree showing the various ways in which individuals are related to others. Figure 8.54 shows one of many ways to represent this family tree for quadrilaterals. Take a few moments to think about this diagram and to connect it to what you know about these different kinds of quadrilaterals. Write a brief description. Does it make sense? Does it prompt new discoveries in your mind? 

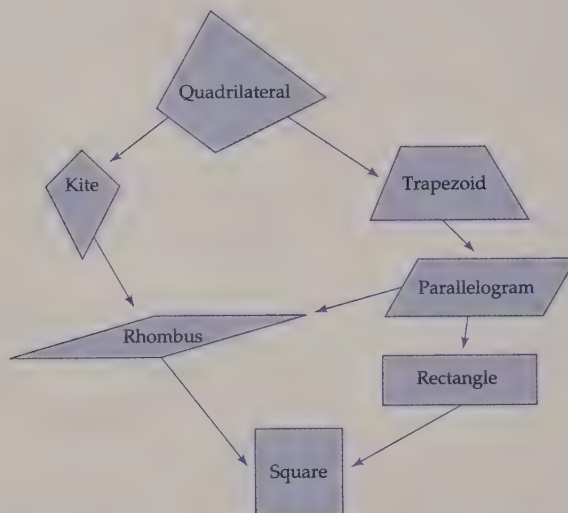


FIGURE 8.54

One way to interpret this diagram is to say that any figure contains all of the properties and characteristics of the ones above it. The quadrilateral at the top represents those quadrilaterals that have no equal sides, no equal angles, and no parallel sides; this is analogous to the scalene triangle. The kite and the trapezoid represent two constraints that we can make: two pairs of congruent, adjacent sides or one pair of sides parallel. If we take a kite and require all four sides to be congruent, we have a rhombus. If we take a trapezoid and require *both* pairs of opposite sides to be parallel, we have a parallelogram. If we require the angles in a parallelogram to be right angles, we have a rectangle. If we require all four sides of a parallelogram to be congruent, we have a rhombus. Both the rhombus and the rectangle can be transformed into squares with one modification—requiring the rhombus to have right angles or the rectangle to have congruent sides. Part 6 of Exploration 8.1 enables you to explore these relationships further. A key point is to begin to see connections and relationships among figures. Many students find, in this course, that their picture of geometry changes from looking like a list of definitions and properties to looking more like a network with connections among the various figures. This quadrilateral family tree can also help students to realize why mathematics teachers say that a square is a rectangle and it is a rhombus: It has all the properties of each! In everyday language, we say that a square is a special kind of rhombus and a special kind of rectangle. In mathematical language, we say that the set of squares is a subset of the set of rhombuses and a subset of the set of rectangles. The Venn diagram in Figure 8.55 illustrates this relationship.

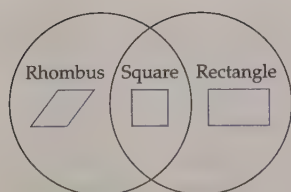


FIGURE 8.55

Convex Polygons

Another concept that emerges with polygons having four or more sides is the idea of *convex*. Before reading on, look at the two sets of polygons in Figure 8.56, convex and concave (not convex). Try to write a definition for *convex*. Then read on. . . .

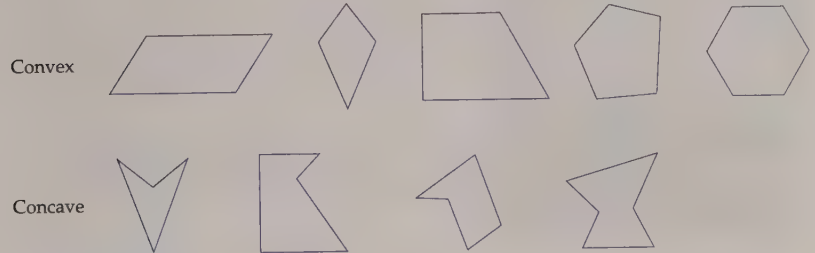


FIGURE 8.56

For many students, defining *convex* is much like defining balls and strikes in baseball. As one umpire once said, "I knows it when I sees it." Many students focus on the word *concave* and say that the figure has at least one part that is caved in. Such a description is acceptable at level 1 on the van Hiele model, but we need a definition that is not as vague as "caved in."

Examine the following definition to see whether it makes sense. Think before reading on. . . .

LANGUAGE

I have found rather different definitions of *convex* in different mathematics textbooks. Once again, mathematics is not as cut and dried as many people believe!

A polygon is **convex** iff the line segment connecting any two points in the polygonal region lies entirely within the region.

If a polygon is not convex, it is called **concave**.

Looking at diagonals is an easy way to test for concave and convex. If *any* diagonal lies outside the region, then the polygon is concave. In polygon $ABCDE$ in Figure 8.57, the diagonal AD lies outside the region.

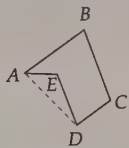


FIGURE 8.57

Other Polygons

Although most of the polygons we encounter in everyday life are triangles and quadrilaterals, there are many kinds of polygons with more than four sides. Stop for a moment and think of examples, both natural and human-made objects. Then read on. . . .

All of the figures in Figure 8.58 are polygons.

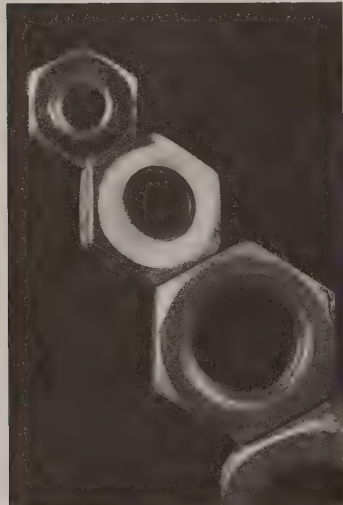
- The stop sign is an octagon—an eight-sided polygon.
- The common nut has a hexagonal shape—a six-sided polygon.
- The Pentagon in Washington has five sides.

Let us examine a few important aspects of polygons with more than four sides.

First, we distinguish between regular and nonregular polygons. What do you think a regular pentagon or a regular hexagon is? How might we define it? Think about this and write down your thoughts before reading on. . . .

A **regular polygon** is one in which all sides have the same length and all interior angles have the same measure.

What do we call a regular quadrilateral? What about a regular triangle? Is it possible for a regular polygon to be concave? Think about these questions before reading on. . .



A regular quadrilateral is called a square. A regular triangle is called an equilateral triangle. A regular polygon cannot be concave.

We found that the sum of the degrees of the interior angles of any triangle is 180, and the sum for any quadrilateral is 360. What do you think is the sum of the interior angles of a pentagon? Can you explain your reasoning? Can you find a pattern in this progression that will enable you to predict the sum of the interior angles of any polygon—for example, one with 10 sides or with 100 sides? Work on this before reading on. . .

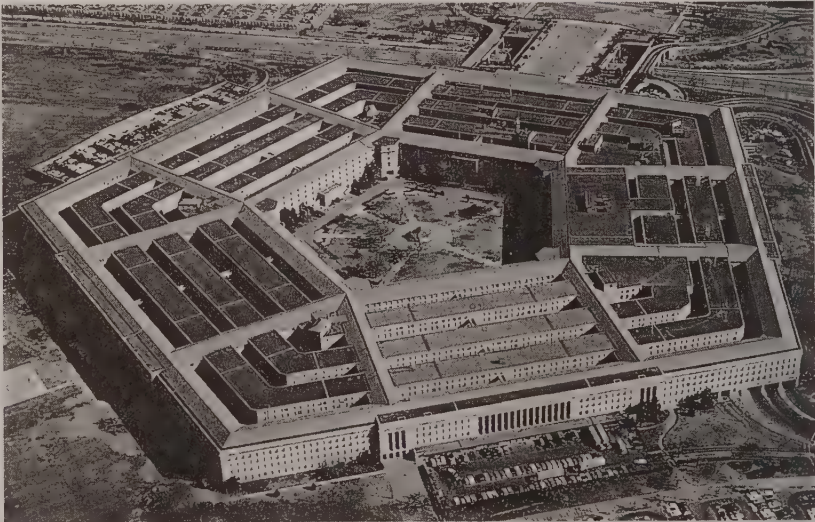


FIGURE 8.58

From Table 8.4, many students can see that the sum increases by 180 each time but cannot come up with the general case. The solution to this question comes from connecting the problem-solving tool of making a table to the seeing of patterns to seeing “increases by 180” as equivalent to “these are multiples of 180.” We can represent this equivalent representation in a fourth column that contains 180, $2 \cdot 180$, $3 \cdot 180$, and so on. What do you see now? Then read on. . .

TABLE 8.4

Sides	Sum	Reasoning
3	180	Increases by 180 each time
4	360	
5	540	
6	720	

The number we multiply 180 by is 2 less than the number of sides in the polygon. Therefore, the sum of the angles of a polygon having n sides will be equal to $(n - 2)180$

Curved Figures

There is one more class of two-dimensional geometric figures that we need to discuss: those figures that are composed of curves that are not line segments.

How many words do you know that describe such shapes? Think and then read on. . . .

There are many such geometric figures—for example, circle, semicircle, spiral, parabola, ellipse, hyperbola, and crescent (see Figure 8.59).



FIGURE 8.59

In this course, we will focus on the simplest of all curved geometric figures: the circle. Stop for a moment to think about circles. How would you define a circle? Try to do so before reading on. . . .

A **circle** is the set of points in a plane that are all the same distance from a given point, the center.

You are probably familiar with the following basic vocabulary for circles.

In Figure 8.60, C is the *center* of the circle.

The line segment $\overline{CA}$ is called a *radius*, the plural of which is *radii*.

The line segment $\overline{AB}$ is called a *diameter*.

The line segment $\overline{XY}$ is called a *chord*.

The line $\overleftrightarrow{PQ}$, which intersects the circle only at point D , is called a *tangent*.

An *arc* is any curved line that lies on the circle. We use two letters to denote an arc if the arc is less than half of the circle. However, for larger arcs, we use three letters. Do you see why?

We need this to distinguish between the arc at the left ($\widehat{AD}$) in Figure 8.61 and the arc at the right ($\widehat{ABD}$), because both of them have points A and D as endpoints.

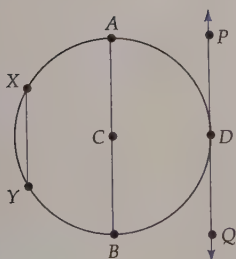


FIGURE 8.60

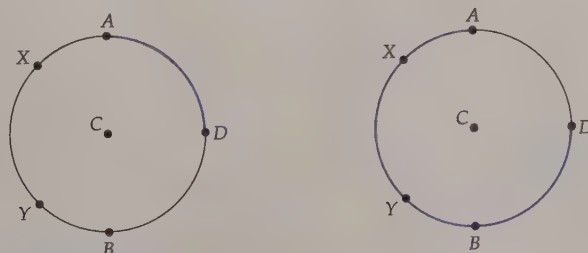


FIGURE 8.61

Based on these descriptions and your previous experience with circles, try to write a definition for each of these terms. Then compare your definitions with the ones below. . . .

A **radius** of a circle is any line segment that connects a point on the circle to the center.

A **diameter** of a circle is any line segment that connects two points of the circle and also goes through the center of the circle.

A **chord** is any line segment that connects two points of the circle. Thus, a diameter is also a chord.

A **tangent** line intersects a circle at exactly one point.

An **arc** is a subset of a circle—that is, a part of a circle.

Summary

Look back over this section and the related explorations that you did. What did you learn? In this section, we have examined basic building-block terms, concepts, and properties that are necessary if we are to explore and understand shapes in two dimensions. You now better understand why we have so many names for different triangles and quadrilaterals. More important, you can see the relationships among triangles and among quadrilaterals, so that terms like *right isosceles triangle* make sense and the statement that “a square is also a rectangle” makes sense. Knowing the terms and the properties and relationships gives us the background to understand the structures of three-dimensional shapes (in Section 8.3) and the many intricate two-dimensional designs we find in quilts and art (in Chapter 9).

EXERCISES 8.2

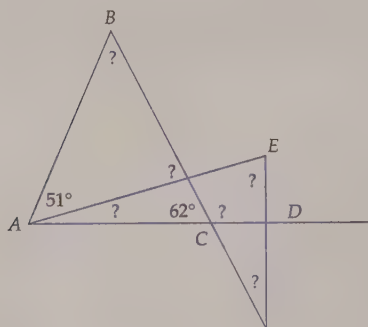
1. Consider the figure below. You are given the following information:

$\triangle ABC$ is an isosceles triangle with base $\overline{AC}$.

$\triangle ADE$ is a right triangle.

$\angle BAE = 51^\circ$

$\angle BCA = 62^\circ$



Determine the measures of the angles that have question marks.

2. Given that:

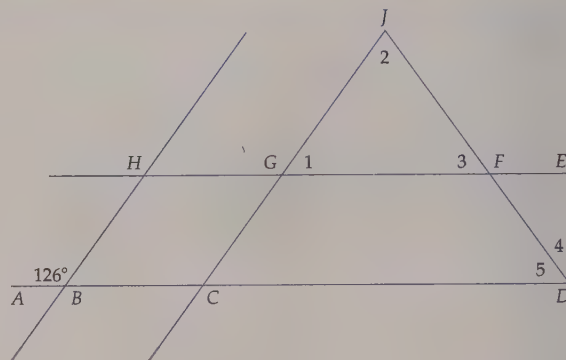
$\overline{AD} \parallel \overline{HE}$

$\overline{BH} \parallel \overline{CG}$

$\overline{HE} \perp \overline{DE}$

$\triangle CJD$ is an isosceles triangle with base $\overline{CD}$.

$\angle ABH = 126^\circ$



Find the measures of angles 1, 2, 3, 4, and 5.

3. The definition of a regular polygon states that all sides have the same length and all interior angles have the same measure. Why is the second part of the definition necessary? That is, why can't we just say that a polygon is a regular polygon if all the sides are the same length?
4. For each figure below, write “polygon” or “not a polygon.” If it is a polygon, also write “convex” or “concave.”



(a)



(b)



(c)

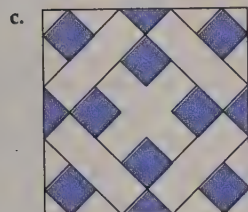
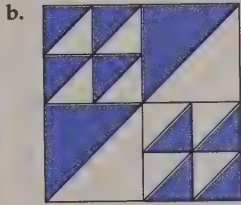
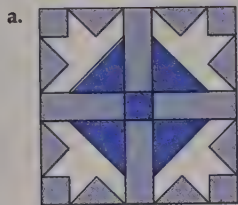


(d)



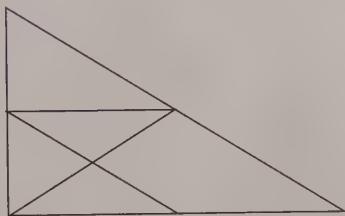
(e)

5. Describe all the geometric shapes you see in the quilt designs below:

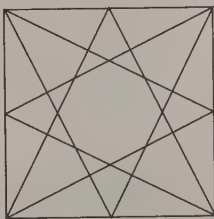


Source: Jinny Beyer, *The Quilter's Album of Blocks & Borders* (Delaplane, VA: EPM Publications, Inc., 1986), p. 185.

6. Name at least six different polygons that you can see in this shape. Trace and number each shape.



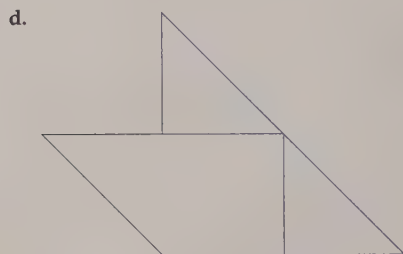
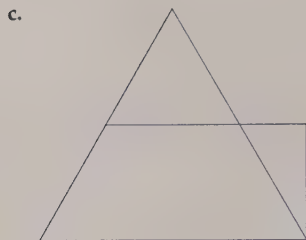
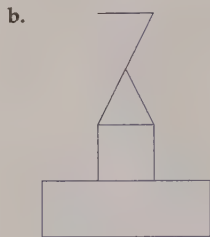
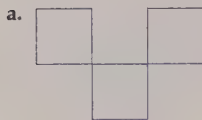
7. Look at the shape at the right.



- What do you see? (This is an open-ended question.)
 - Select several nonsimple shapes: name them and justify their name. For example, are the quadrilaterals in each of the four corners kites or do they just look like kites?
8. Draw each of the following or briefly explain why such a figure is impossible.
- An isosceles trapezoid
 - A concave quadrilateral
 - A curve that is simple and closed but not convex
 - A nonsimple closed curve
 - A concave equilateral hexagon
 - A concave pentagon having three collinear vertices
 - A hexagon with 3 acute angles and 3 obtuse angles
 - A pentagon that has 3 right angles and 1 acute angle
9. For the following questions, if the answer is yes, draw a picture; if it is no, briefly explain why not.
- Can a right triangle also be an isosceles triangle?
 - Can a quadrilateral have 3 obtuse angles?
 - Can a rhombus have 4 acute angles?
 - Can a square also be a rectangle?
10. Express the relationship between scalene, isosceles, and equilateral triangles with a Venn diagram.
11. Make a family tree to represent the relationships among different kinds of triangles.
12. Explain why it is impossible to have a triangle whose three sides are 6 cm, 8 cm, and 15 cm.
13. Is this figure a kite? Why or why not?



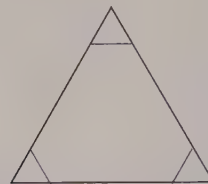
14. Describe all quadrilaterals that have these characteristics. If there is more than one, say so.
- A quadrilateral with opposite sides parallel
 - A quadrilateral with 4 right angles
 - A quadrilateral with all sides equal
 - A quadrilateral in which the diagonals bisect each other
 - A quadrilateral in which the diagonals are congruent
 - A quadrilateral in which adjacent angles are congruent
 - A quadrilateral in which opposite angles are equal
 - A quadrilateral in which no sides are parallel
 - A quadrilateral with 4 congruent sides and 2 pairs of congruent angles
 - A quadrilateral with 4 congruent angles and 2 pairs of congruent sides
15. Show that the family tree for quadrilaterals also holds for the characteristics of the diagonals.
16. Draw a Venn diagram to illustrate how squares, rectangles, and rhombuses are related.
17. In this section, we found that we could name some quadrilaterals in terms of their "ancestors." For example, we could say that a rhombus is a parallelogram with four equal sides. What other quadrilaterals could we describe in terms of their ancestors?
18. Write directions for each of the figures below. To assess the accuracy and clarity of your directions, your instructor will do exactly what you say. Before turning in your directions, you might want to have a friend who is not in this class try to draw these figures using only your directions.



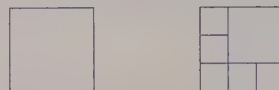
19. Just as we can speak about interior angles of polygons, we can speak about exterior angles. We simply need to define them. Take any regular polygon and select one interior angle. Extend one side of the angle outside the polygon. The exterior angle has the same vertex as the interior angle and shares a common side.

- Are these directions sufficient? That is, is there only one possible angle that can be made following these directions?
 - Draw and label the exterior angle of a triangle. What is the measure of the exterior angle?
 - Draw and label the exterior angle of a quadrilateral. What is the measure of the exterior angle?
 - Draw and label the exterior angle of a hexagon. What is the measure of the exterior angle?
 - What is the measure of each exterior angle of a regular polygon with n sides?
20. a. Sketch a pair of parallelograms whose intersection is exactly three points.
b. Sketch a pair of parallelograms whose intersection is exactly four points.
c. Sketch a pair of parallelograms whose intersection is exactly five points.
d. Describe the maximum number of points of intersection that two parallelograms can have and sketch your answer.

21. How can you draw three line segments within an equilateral triangle so that you form a regular hexagon? (See the triangle that follows.)



- Draw any square. Find the midpoint of each side. Connect the midpoints to make a new quadrilateral. What do you observe? Can you prove your generalization?
 - Draw any rectangle. Find the midpoint of each side. Connect the midpoints to make a new quadrilateral. What do you observe? Can you prove your generalization?
 - Draw *any* quadrilateral. Find the midpoint of each side. Connect the midpoints to make a new quadrilateral. What do you observe? Can you prove your generalization?
23. The diagram below shows a square that has been cut so that we now have 6 smaller squares. The more general statement of this problem is: Can you find a way to cut a square into x smaller squares?



- For which of the following x 's can you find a solution?
- For which of the following x 's can you find more than one solution?
- If you cannot find a solution, make an argument that this is impossible.
- Summarize strategies that you used and/or patterns that you saw that helped. For example, did you find strategies that you could modify to enable you to cut the square into a different number of smaller squares?

4	5	6	7
8	9	10	11
12	13	14	15

SECTION

8.3

THREE-DIMENSIONAL FIGURES

WHAT DO YOU THINK?

- What do pyramids and cones have in common?
- How are properties of two-dimensional objects and three-dimensional objects related?
- How can you represent three-dimensional objects in two-dimensional space?

Just as we discovered patterns and relationships among many two-dimensional objects, so too there are many patterns and relationships among three-dimensional figures, also called space figures. With respect to practical matters, understanding the geometry of human-made objects helps us to make them work better and, in the case of objects like bridges, overpasses, and airplanes, to make them work more safely. Geometry also helps us to understand natural phenomena better—for example, why certain animals have the shapes they have. An understanding of shapes has many applications in science. For example, many carcinogens are virtually identical in size and shape to other compounds, and thus they fool the body into thinking they are not harmful. The silicon chip has the same structure as the diamond, except that there are silicon atoms instead of carbon atoms at these positions.⁹ With respect to aesthetics, geometry helps us to understand why some shapes are so appealing to people and to understand patterns within those shapes (see Figure 8.62).



HISTORY

The massive pyramid at Gizeh is one of the seven wonders of the ancient world. Even today we are not certain how it was constructed. Many of the blocks weighed more than 10 tons! The pyramid was originally covered with

white marble and must have dazzled like a mirror in the desert. It was so well built that some of the edges fit together so well that a razor cannot be inserted into the space between two blocks.

INVESTIGATION 8.5 Connecting Polygons to Polyhedra

Just as we examined families (subsets) of triangles and quadrilaterals, we will now investigate families of three-dimensional geometric figures (see Figure 8.63 on page 479). If you did Exploration 8.8, you grappled with describing and classifying three-dimensional figures.

Let us explore the connection between polygons and *polyhedra*, which will be loosely defined (for now) as three-dimensional figures made up of polygons.

The second column of Table 8.5 (on page 479) describes several attributes of polygons, which we investigated in Section 8.2. Which of these attributes do you think hold for polyhedra or can be modified to describe different kinds of polyhedra? Fill in as much of the third column as you can. The questions below are given to help you focus on the connections between how we see and define two- and three-dimensional figures. After you have completed as much of the third column as possible, compare your hypotheses with another student. Then read on. . . 

⁹ Marjorie Senechal, *On the Shoulders of Giants*, p. 173. Reprinted with permission from *On the Shoulders of Giants: New Approaches to Numeracy*. Copyright © 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, D.C.



(a)
Pyramids at Gizeh



(b)
the Parthenon



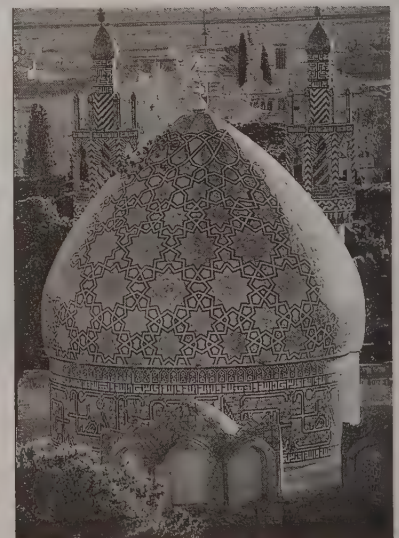
(c)
Nautilus shell



(d)
pyrite crystals



(e)
soccer ball



(f)
Shrine of Shah Nimatuollahi

FIGURE 8.62

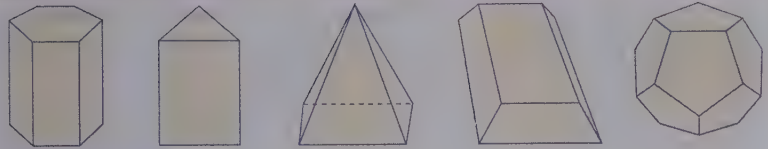


FIGURE 8.63

- If a polygon is defined as a simple closed curve, is there an analogous definition for a polyhedron?
- All polygons have vertices, line segments, and angles. Do these terms work for describing component parts of polyhedra? Do we also need new terms?
- Can we classify polyhedra by the number of sides?
- If there are regular polyhedra, how might they be defined?
- If there are convex polyhedra, how might they be defined?

TABLE 8.5

	Polygons (two-dimensional)	Polyhedra (three-dimensional)
What they are	Simple closed curves	
Definition	Union of line segments	
Component parts	Vertices	
	Line segments	
	Angles	
	Other?	
Classification	By number of sides	
	Regular vs. not regular	
	Convex vs. concave	

DISCUSSION

In Section 8.2, we began with simple closed curves that partitioned a plane (two dimensions) into three disjoint sets: the curve, inside, and outside.

Though we will not rigorously define simple closed surfaces, we can say that they partition space (three dimensions) into three disjoint sets: the surface itself, inside, and outside (see Figure 8.64).

We will use the term **space figure** to describe any three-dimensional object.

We will use the term **polyhedron** (the plural is **polyhedra**) to describe those simple closed surfaces that are composed of polygonal regions.

We will use the term **solid** to describe the union of any space figure and its interior.

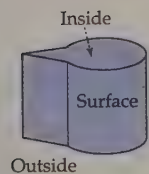


FIGURE 8.64

Component parts Just as the component parts of polygons have special names, so do those of polyhedra.

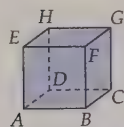


FIGURE 8.65

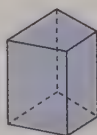
Each of the separate polygonal regions of a polyhedron is called a **face**; for example, square $ABFE$ is a face of the cube in Figure 8.65.

The sides of each of the faces are called **edges**; for example, $\overline{AB}$ is an edge of the cube in Figure 8.65.

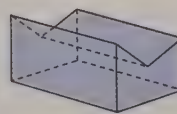
The **vertices** of the polyhedron are simply the vertices of the polygonal regions that form the polyhedron; for example, E and F are vertices of the cube in Figure 8.65.

Convex and concave Just as polygons can be convex or concave, so can polyhedra. Before reading the definition of a convex polyhedron, think back to the definition of a convex polygon and see whether you can modify that definition for three-dimensional objects. Then read on. . . 

A polyhedron is **convex** iff any line segment connecting two points of the polyhedron is either on the surface or in the interior of the polyhedron (see Figure 8.66).



Convex



Concave

FIGURE 8.66

Families of Polyhedra

Now let us investigate some of the families of polyhedra, which you may have explored in Exploration 8.8. Take a few minutes to examine the figures in Figure 8.67. How are these figures alike? How are they different? Write your thoughts in your notebook before reading on. . . 

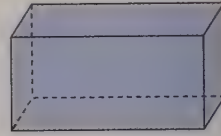
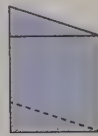
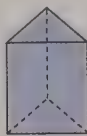
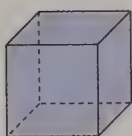


FIGURE 8.67

All of these figures have at least two sides that are parallel; some students would say that the top and bottom sides are parallel. And the faces are all polygons. All of these figures are called prisms. We use the word **prism** to describe all polyhedra that have two parallel **bases** that are congruent polygons. It is a convention to call the faces of prisms **lateral faces**.

What one shape can be used to describe the lateral faces of *all* prisms? In other words, all lateral faces of all prisms are _____. Think and read on. . . 

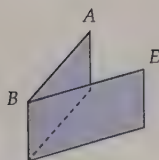


FIGURE 8.68

In all prisms, the lateral faces are parallelograms. In some cases, all of the lateral faces are rectangles. How would you describe the differences between those prisms whose lateral faces are non-rectangular parallelograms and those whose lateral faces are rectangles?

In the latter case, the plane of the base and the plane of the lateral faces are perpendicular (see Figure 8.68). We could also say that the *dihedral angle*

formed by either base and any face is a right angle. (A **dihedral angle** is simply a three-dimensional angle—that is, an angle whose vertex is a line and whose sides are planes.)

Thus, we can define a **right prism** as a prism in which the lateral faces are rectangles. Alternatively, we could define a right prism as a prism in which the angle formed by either base and any lateral face is a right dihedral angle.

A prism that is not a right prism is an **oblique prism** (see Figure 8.69).

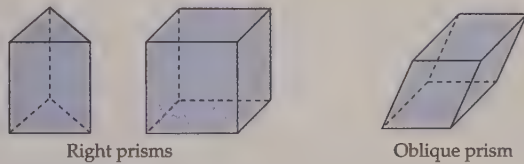


FIGURE 8.69

Long before they study formal geometry, many children know the names for two special kinds of prisms.

Though this is not a term mathematicians use, what we call a **box** is actually a prism in which all six faces are rectangles. If all six faces are squares, we call the figure a **cube** (see Figure 8.70).

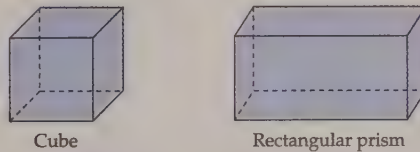


FIGURE 8.70

Pyramids

Let us consider now another family of polyhedra. You may recognize the polyhedra in Figure 8.71 as pyramids. How might we define that term? Make your own definition and then read on. . . . 

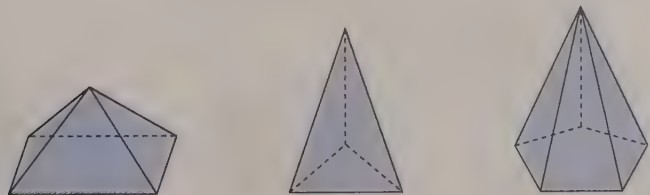


FIGURE 8.71

We use the word **pyramid** to describe those polyhedra whose base is a polygon and whose sides are triangles that have a common vertex. That common vertex is called the **apex** of the pyramid.

An alternative way to think of a pyramid is to start with any polygon and a point above the plane of the polygon. Now connect that point to each vertex of the polygon.

Most of the pyramids you have seen in pictures (or in person, if you are lucky) have square bases. However, the base can be any polygon. A pyramid is named according to its base: triangular pyramid, square pyramid, etc.

Regular Polyhedra

In Section 8.2, we discussed regular polygons. How might we define a regular polyhedron? Try to do so before reading on. . . 

One of the ways we classified polygons was by the number of sides: triangles, quadrilaterals, pentagons, hexagons, etc. We could speak of a regular hexagon and a nonregular hexagon. However, that doesn't work with polyhedra. Do you see why?

We define a **regular polyhedron** as a convex polyhedron in which the faces are congruent regular polygons and in which the numbers of edges that meet at each vertex are the same.

Stop! Does that definition make sense? I find that the concept of a regular polyhedron is one of the more abstract in the book. What does it mean to say that "the numbers of edges that meet at each vertex are the same"? If you are still not sure, read on, but then go back and check to see whether this definition jibes with that of regular polyhedra.

Which of the prisms and pyramids we have discussed so far do you think might be regular polyhedra? Think before reading on. . . 

A cube is a regular polyhedron. A triangular pyramid composed of equilateral triangles is a regular polyhedron and has a special name, **tetrahedron**. The origin of the name is Greek: *tetra* ("four") and *hedron* ("face").

A fact that surprises many people is that there are not a large number of regular polyhedra. In fact, there are only five regular polyhedra: the tetrahedron, the cube, the **octahedron** (with 8 triangular faces), the **dodecahedron** (with 12 pentagonal faces), and the **icosahedron** (with 20 triangular faces) (see Figure 8.72). The solids made from the regular polyhedra are called Platonic solids after the Greek philosopher Plato.



HISTORY

Over 2000 years ago (long before we knew about atoms), many Greek philosopher-scientists believed that there were four basic elements out of which all things arose: earth, air, fire, and water. Some of the Greeks believed that the smallest particle of earth had the form of a cube, the smallest particle of air had the form of an octahedron, the smallest particle of fire had the form of a tetrahedron, and the smallest particle of water had the form of an icosahedron. The dodecahedron was associated with the universe, probably because it was the last solid discovered, although it has been speculated that it is associated with the universe because it has twelve faces and there are twelve signs in the zodiac.¹⁰



Tetrahedron



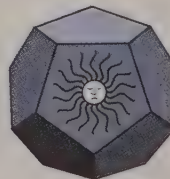
Octahedron



Cube



Icosahedron



Dodecahedron

Source: The five regular solids drawn by Johannes Kepler in *Harmonices Mundi*, Book II, 1619.

FIGURE 8.72

¹⁰ H.A. Freebury, *A History of Mathematics* (New York: Macmillan, 1961), p. 36; H.A. Eves, *An Introduction to the History of Mathematics* (New York: Holt, Rinehart and Winston, 1969), p. 68.

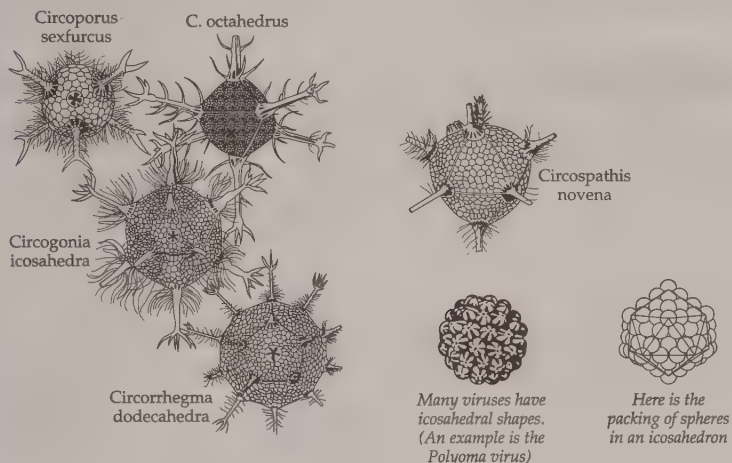


BEYOND THE CLASSROOM

These regular polyhedra occur in nature:

- Crystals of salt and of pyrite have been found in the shape of a cube.
- Crystals of chrome alum have been found in the shape of a tetrahedron.

- Crystals of pyrite have been found in the shape of an octahedron.
- Skeletons of microscopic sea animals have been found in the shape of a dodecahedron and in the shape of an icosahedron (see Figure 8.73). Honest, I didn't make this up!¹¹



Source: Radiolarians—*Kunstformen der Natur*, Vols. 1–10, Ernst Häckel des Bibliographischen Instituts, Leipzig, Germany, 1899–1904; Polyoma virus—Drawn after K. W. Adolph, D. L. D. Caspar, C. J. Hollingshed, E. E. Lattman, W. C. Phillips, and W. T. Murakami, *Science*, Vol. 203, p. 1117, 1979; Packing of spheres—Drawn after A. L. Mackey, *Acta Crystallographica*, Vol. 15, p. 916, 1962 as seen in *Symmetry: A Unifying Concept*, 1994.

FIGURE 8.73

Cylinders, Cones, Spheres

The polyhedra we have defined thus far have all been simple closed surfaces in which all the faces are polygons. There are three other kinds of three-dimensional objects that are commonly found and that elementary school children study. *Cylinders*, *cones*, and *spheres* are related to polyhedra we have studied. Before we examine these three, stop for a moment and consider which polyhedra are related to cylinders, which to cones, and which to spheres. Then read on. . . 

Think of a prism with more and more sides (see the prism at the left in Figure 8.74). At some point, a prism with a lot of sides begins to look more like a cylinder than a prism. From one perspective, we can think of a cylinder as a prism in which the bases are circles. Technically, this is not true because the bases of prisms are polygons and a circle is not a polygon.

Thus, we will describe a **cylinder** more formally as a simple closed surface that is bounded by two congruent circles that lie in parallel planes.

¹¹ *Historical Topics for the Mathematics Classroom: Thirty-first Yearbook* (Reston, VA: NCTM, 1969), p. 220.

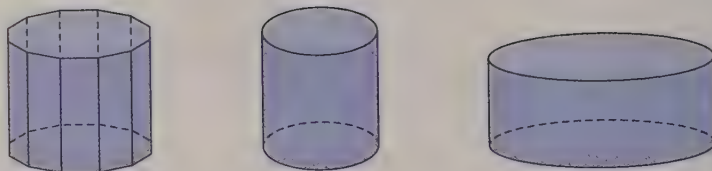


FIGURE 8.74

**MATHEMATICS**

We can also define a cylinder by using set language (remember Chapter 2?) and say that a cylinder is constructed by starting with a circle and a line segment that has one endpoint on the circle and the other endpoint not on the circle. A cylinder consists of the union of three surfaces: the circle, the union of all line segments congruent to and parallel to the given line segment, and the circle formed by connecting the other endpoints of each of the line segments.

Earlier, we talked about right prisms and right pyramids. A cylinder is a **right cylinder** iff the center of the upper circle is directly above the center of the lower circle. If a cylinder is not a right cylinder, it is called an **oblique cylinder**.

Now think of a pyramid with more and more sides (see the pyramid at the left in Figure 8.75 below). At some point, a pyramid with a lot of sides begins to look more like a cone than a pyramid. From one perspective, we can think of a cone as a pyramid in which the base is a circle. Technically, this is not true because the base of a pyramid is a polygon and a circle is not a polygon.

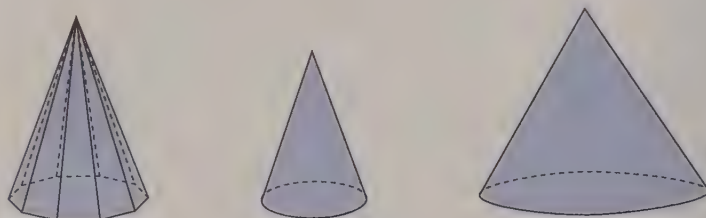


FIGURE 8.75

**MATHEMATICS**

As we did with cylinders, we can define the term *cone* by using set language and say that a cone is constructed by starting with a circle and a point not on the circle. A cone consists of the union of two surfaces: the circle and the union of all line segments connecting that point and the circle.

Thus, we will describe a **cone** more formally as a simple closed surface whose base is a circle and whose lateral surface slopes up to a vertex that we call the **apex**.

If the apex of the cone lies directly above the center of the base, then we call it a **right cone**. If a cone is not a right cone, it is called an **oblique cone**. In everyday life, we generally experience only right cones and right cylinders. Therefore, in this book, we will use the terms *cones* and *cylinders* unless referring to oblique cones or cylinders.

Finally, think of a polyhedron that has more and more and more faces. You might want to look at the five regular polyhedra in Figure 8.72 and imagine starting with a dodecahedron made of clay and then slicing the various faces at an angle to make more and more faces. Eventually, the figure would begin to resemble a sphere more than a polyhedron.

A sphere is also conceptually related to a circle. Can you apply the earlier definition of a circle to define the term *sphere*? Try to do so before reading on. . . 

A **sphere** is the set of points in space equidistant from a given point, which is called the **center**.

How would you define the radius and diameter of a sphere? Try to do so before reading on. . . 

Any line segment joining the center of the sphere to a point on the surface is called a **radius**.

Any line segment whose endpoints lie on the surface of the sphere and that contains the center is called a **diameter**.

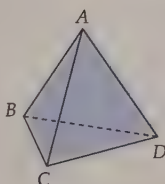
Summary

In this section, we have named and classified the basic kinds of three-dimensional geometric figures. From the investigations and explorations, you can see that a variety of seemingly different figures are actually prisms. You have seen that prisms and cylinders are closely related. You have seen that pyramids and cones are closely related. We will take advantage of these similarities when we examine formulas for surface areas and volumes of figures in Chapter 10. In Explorations 8.7, 8.8, and 8.9, you explored some relationships between two-dimensional and three-dimensional figures.

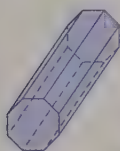
EXERCISES 8.3

1. Given the tetrahedron below, name the following:

- A face
- A vertex
- An edge



2. Name the figures below.



a.



b.

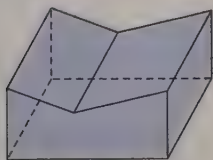


c.

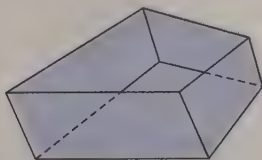


d.

3. Which of the polyhedra below are convex?



a.



b.

4. Draw a nonconvex rectangular or pentagonal prism.

5. We defined a regular polyhedron as a convex polyhedron in which the faces are congruent regular polygons and in which the numbers of edges that meet at each vertex are the same. Carlos says that instead of saying that the numbers of edges that meet at each vertex are the same, we could have said that the numbers of faces that meet at each vertex are the same. What do you think? Support your choice.
6. Can you make a pyramid in which the triangular faces are not all congruent?

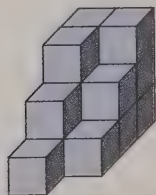
7. Write a definition of *diagonal* for polyhedra.

8. There is a relationship between the number of diagonals and the base of a prism. That is, triangular prisms have a certain number of diagonals, square prisms have a certain number of diagonals, and so on.

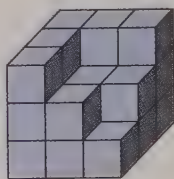
Determine this relationship so that you can answer the following question: How many diagonals does a prism have whose base is a regular polygon with n sides? [Your instructor may or may not give you hints for this problem. If not, I suggest looking at the 4 Steps for Problem Solving on the inside front cover of the *Explorations* volume.]

9. Consider a prism whose base is a regular n -gon—that is, a regular polygon with n sides. How many vertices would such a prism have? How many faces? How many edges? You may want to start with a triangular prism, square prism, pentagonal prism, and so on, and look for patterns.
10. Consider a pyramid whose base is a regular n -gon—that is, a regular polygon with n sides. How many vertices would such a pyramid have? How many faces? How many edges?
11. a. Describe the relationship between the number of vertices of an n -gon prism and an n -gon pyramid.
b. Describe the relationship between the number of faces of an n -gon prism and an n -gon pyramid.
c. Describe the relationship between the number of edges of an n -gon prism and an n -gon pyramid.
12. At the center of every tissue of toilet paper is a cardboard cylinder. Find and examine one of these cylinders. You can see a curved line running along the face of the cylinder.
a. If you cut the cylinder along this line, what would the unfolded shape look like? Predict the shape and explain your reasoning.
b. Why do you think these cylinders are manufactured this way instead of having a vertical cut?
13. Is there one geometric shape that describes *all* the sides of (right) pyramids? If there is, name it and justify your answer. If there is more than one shape, describe the shapes and justify your response.

14. Look at the figures below. For each, (a) describe how to make it, and then (b) describe another way.



a.



b.

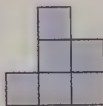
15. Below are the top, front, and side views of a building created with only cubes.

- Without making the building, predict how many cubes it will have. Explain your prediction.
- Predict the left-hand view. Explain your prediction.
- Insert numbers into the top view showing how high the building will be at each spot.
- Draw a picture of the building using the dot paper at the end of the *Explorations* volume.

a. Front



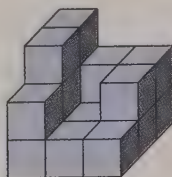
Right-side



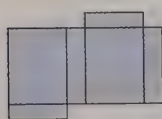
Top



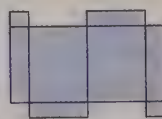
16. Look at the shape shown below. Let's say you wanted to describe that figure to someone on the phone so that that person could make the shape. What might you say?



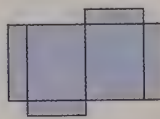
17. Which of the nets below is a possible net for a cereal box?



a.

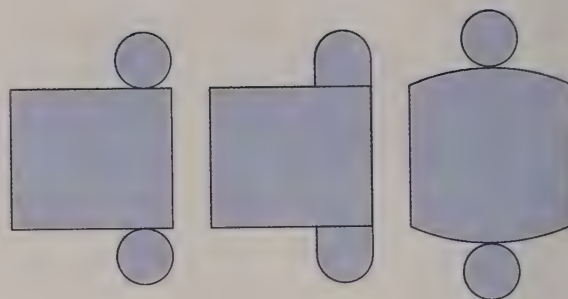


b.

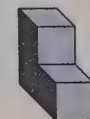
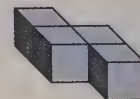
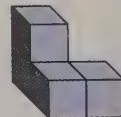


c.

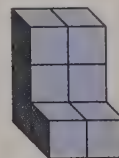
18. Which of the nets following is a possible net for an oatmeal box (which has the shape of a cylinder)?



19. As a child, I was given a puzzle called the Soma cubes, which consists of six different pieces constructed from four cubes and one piece constructed from three cubes. At the top of the next column are diagrams of the seven pieces. In order to do this problem, you need to find a set or make a set.

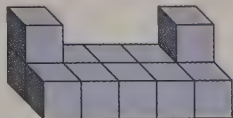


- a. The figure below can be made from two pieces. Describe any thoughts that helped you solve this problem. Figure out a way to communicate your solution. That is, the reader needs to know not only which pieces but also how they were put together.

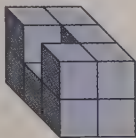


- b. The figure following can be made from three pieces. Describe any thoughts that helped you solve this problem. Figure out a way to communicate your

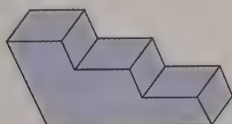
solution. That is, the reader needs to know not only which pieces but also how they were put together.



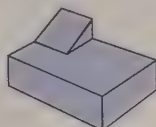
- c. The figure that follows can be made from three pieces. Describe any thoughts that helped you solve this problem. Figure out a way to communicate your solution. That is, the reader needs to know not only which pieces but also how they were put together.



20. Below are drawings of several polyhedra. Draw the top, front, and side views for the polyhedra.

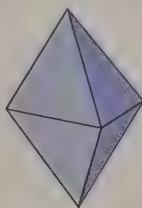


a.



b.

21. How would you make the octahedral dice shown below?



22. Using paperboard and an exacto knife, make an oblique triangular prism.

Part 1:

- Describe your attempts (my first one did not work either).
- Describe what you learned during the construction.
- Describe the relationships among the three lateral faces.

Part 2:

- Predict how to make an oblique rectangular prism with stiff paper. Describe the relationships among the four faces. Explain your prediction.
- Make an oblique rectangular prism. If your prediction was accurate, great. If your prediction was not accurate, describe where your prediction was inaccurate and explain the reasoning behind the part(s) of your prediction that were inaccurate.
- Draw a net for an oblique rectangular prism.

Part 3:

- Predict the relationship among the five lateral faces of an oblique pentagonal prism.
- Predict the relationship among the five lateral faces of an oblique hexagonal prism.

CHAPTER SUMMARY

- Shapes are found everywhere, and virtually all shapes have functional and/or aesthetic value.
- Each geometric shape represents the common characteristics of a set of objects.
- Every shape has multiple attributes, which means that we can classify any set of shapes in multiple ways. Recognizing and understanding these attributes helps us to understand the shape better.
- Classifying leads us to deeper understanding of mathematical structure, which leads to greater mathematical power.
- Looking for and recognizing relationships within and between shapes also leads to understanding of mathematical structure.
- Important spatial thinking abilities include eye-motor coordination, figure-ground perception, perceptual constancy, visual discrimination, and visual memory.
- The van Hiele levels of geometric thinking help us to understand the development of understanding of geometric ideas.

8. Our current knowledge of geometry took thousands of years to develop and is the result of intuitive thinking, inductive thinking, and deductive thinking.
9. The Greeks were the first people to develop mathematical systems, that is, a coherent system of mathematical ideas.
10. Geometric ideas are related to sets; for example, a line segment is a set of points.
11. Geometric ideas are related to functions; for example, the sum of the interior angles of a polygon $= (n - 2)180$.

Basic Concepts

Section 8.1 Basic Concepts of Geometry

hand-eye coordination 443
 figure-ground perception 443
 perceptual constancy 443
 visual discrimination 443
 visual memory 443

Point, line, plane, on, and between are undefined terms. 447

Sets of points may be collinear or noncollinear. 448

Sets of points may be coplanar or not; lines may be coplanar or not. 449

$\longleftrightarrow$ ———
 line 449 line segment 449

$\longrightarrow$ •
 ray 449 endpoint 449

intersect 450

Relationships between lines: perpendicular;
 parallel, concurrent, skew 450

Angles have vertices and sides. 450

Angles have an interior and an exterior. 451

Naming angles

with one letter, one number, three letters 451

Measuring angles

protractor 451 degree 451

Kinds of angles

right 453	acute 453
obtuse 453	straight 453
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vertical 454	

Section 8.2 Two-Dimensional Figures

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 Any simple closed curve partitions the plane into 3 disjoint regions. 460

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congruent 462	

Classifying triangles

By length of sides: equilateral, isosceles, scalene 465

By size of angle: right triangle, obtuse triangle, acute triangle 465

Special line segments in triangles

angle bisector 467	median 467
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The sum of the measures of the angles of a convex polygon $= 180(n - 2)$. 473

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Section 8.3 Three-Dimensional Figures

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Parts

face 480	edge 480
vertex 480	

Classifying

convex 480	concave 480
------------	-------------

Polyhedra

prism 480	box 481
base 480	cube 481
lateral face 480	
dihedral 481	
right prism 481	
oblique prism 481	

Pyramid

pyramid 481
apex 481

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-----------------	-----------------

octahedron 482
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Cylinders, Cones, Spheres

cylinder 483
 oblique cylinder 484
 right cylinder 484
 right cone 484
 oblique cone 484
 sphere 484
 center 484

apex 484

Figures to accompany Investigation 8.3, p. 463

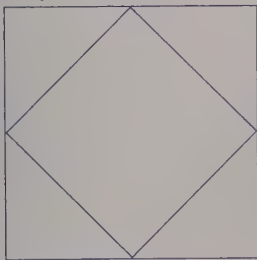


FIGURE 8.76

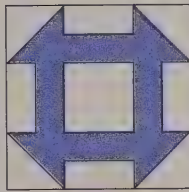


FIGURE 8.77

Geometry as Transforming Shapes

- ◆ 9.1 Congruence Transformations
- ◆ 9.2 Symmetry and Tessellations
- ◆ 9.3 Similarity

In Chapter 8, we focused on understanding shapes—attributes of shapes, different kinds of shapes, and relationships among shapes. In Chapter 9, we are going to look at what happens when we transform shapes; the knowledge we gain from these explorations will help us better understand the patterns that emerge when shapes are put together, as in quilts, floor patterns, art, and so on. In both chapters, there are certain themes that run through our explorations: the notions of deconstructing, which leads to awareness of multiple attributes; classifying, which leads to understanding of structures; and looking for patterns and relationships, which leads to generalizations and, again, to understanding of structures.

Look at the pictures in Figures 9.1 to 9.3. As you do, recall Investigation 8.3, where you drew the churn dash. Different people will see different things. What do *you* see? Note your ideas before reading on. If possible, check your observations with another student. 

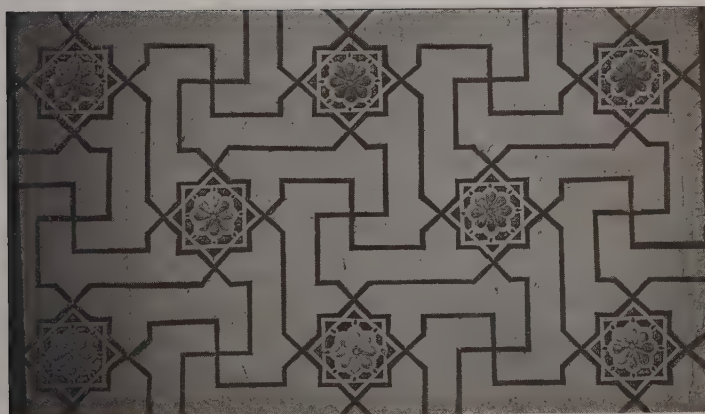


FIGURE 9.1

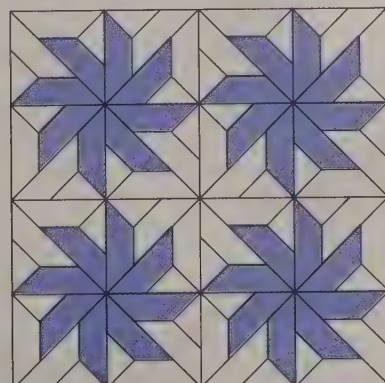


FIGURE 9.2



Source: M.C. Escher's ceiling at Philips International. © 2000 Cordon Art B.V.—Baarn—Holland. All Rights Reserved. Photo courtesy Philips International B.V.

FIGURE 9.3

Figure 9.1 shows an Islamic design composed entirely of two shapes: an 8-pointed star and a 15-sided polygon. What do you notice about the relationships between the 8-pointed stars? about the relationships between the 15-gons? If we focus on the 8-pointed stars, each star can be moved to the position of a nearby star by sliding it—horizontally, vertically, or diagonally. Each 15-gon can be moved to the position of a nearby 15-gon by turning it.

Figure 9.2 illustrates one of many parquet floor designs. Most people see pinwheels composed of 8 congruent isosceles trapezoids. Some people see these embedded in squares. Other people see congruent pinwheels of a different color. Others see white pinwheels composed of 4 white isosceles triangles. Each dark pinwheel can be moved to the position of a nearby light pinwheel by sliding it. Each trapezoid in a pinwheel can be moved to the position of an adjacent trapezoid by a turn.

In Figure 9.3, focus on the butterflies. Though the butterflies are in different positions and of different sizes, they all have the same basic shape. This is true for each of the animals pictured. The arrangement of the animals is not random but intentional. If you focus only on the butterflies, what do you notice? From one perspective, we can say that every butterfly has a twin, and each butterfly can be moved from its position to its twin's position by flipping, turning, or sliding or by a combination of these moves.

Transformations

These three figures illustrate several mathematical transformations. However, before we get into the mathematical aspect of transformations, what do you think when you hear this word? 

Some of the more interesting examples from my students include the transformation of a frog into a prince in fairy tales and the transformation of a

working-class girl into a high-society lady, as seen in *My Fair Lady* with Audrey Hepburn and in *Pretty Woman* with Julia Roberts. Virtually all examples of transformation give a sense of movement and change. In mathematics, there are many kinds of transformations. Figure 9.4 illustrates several. Look at the various transformations of the letter P. How would you describe the transformation? Write your thoughts before reading on. . . .

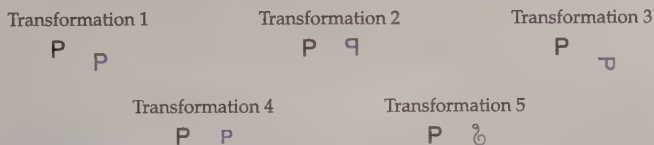


FIGURE 9.4

Most people find the first four transformations fairly straightforward but find the last one very difficult. Rather than give the answer at this point, let me ask a more refined question: What is the same and what is different about each figure and its image after the transformation? Think about it, and then read on. . . .

Table 9.1 below shows the informal names for each of the transformations, the formal names, what is changed, and what is unchanged.

TABLE 9.1

Informal name	Formal name	What is changed?	What is unchanged?
1. Slide	Translation	Position	Size, shape
2. Flip	Reflection	Position	Size, shape
3. Turn	Rotation	Position	Size, shape
4. Shrink	Similarity transformation	Position and size	Shape
5. Distortion	Topological transformation	Position, size, and shape	Neighborhood

In a moment, we will zoom in on the first three transformations. In Section 3, we will focus on transformations in which a figure is reduced or enlarged so that the shape is similar but the size has changed. This set of transformations is called the set of similarity transformations. You may recall studying similar triangles in high school.

What about that last transformation? The last case, which students often call a distortion, is called a **topological transformation** by mathematicians. Topology is one of the newer fields of mathematics. In some ways, it is more complex than other fields you have studied, and yet some topological ideas are readily grasped by elementary school children, including networks and a class of problems that mathematicians call “coloring” problems! If we go back to Section 8.2, the classification of curves connects to topology. That is, any two simple closed curves are topological transformations of each other. When mathematicians seek to determine whether two shapes are topologically equivalent, they look to see whether the “behavior” of the shapes is similar in all

neighborhoods. In the case of transformation 5 in Figure 9.4, if you think of the shapes as being made of a special string that you can twist, stretch and shrink, and think of the letter P as a flag, you could turn the flag into the figure at the right by twisting and stretching the line segment that makes the flag pole and by “smooshing” the flag into a smaller circular shape.

One final example, in three dimensions, that most readers relate to is the notion of morphing, which has appeared in a number of television shows and movies. When someone morphs, her or his initial appearance and morphed image are topologically equivalent. In fact, the person you are now is topologically equivalent to yourself as a newborn baby. These two images help to illustrate the notion that neighborhoods are unchanged—that is, you still have two nostrils, you still have one arm (not five) coming out from each shoulder, and so on. Simply because we cannot study every topic, we will not discuss topological transformations further in this book. However, the interested reader can do more explorations on the web page for this book, which also includes links to web sites for lesson plans in topology for elementary schools.

In elementary school, we want students to explore these important ideas of congruence and similarity in different ways, so that when they investigate these ideas more formally in high school, they will have had concrete *and* substantive interactions with them. Although we will investigate transformations at a more sophisticated level than do elementary school children, these concepts are important ones, and they are ones that most children enjoy exploring.

► “Explorations of flips, slides, turns, stretchers, and shrinkers will illuminate the concepts of congruence and similarity” (*Curriculum Standards*, p. 114).

SECTION

9.1

CONGRUENCE TRANSFORMATIONS

WHAT DO YOU THINK?

- How are translations, reflections, and rotations related?
- How might you describe the translation, reflection, or rotation of a three-dimensional figure in space?
- How are the operations translation, reflection, and rotation like the operations addition, subtraction, multiplication, and division?

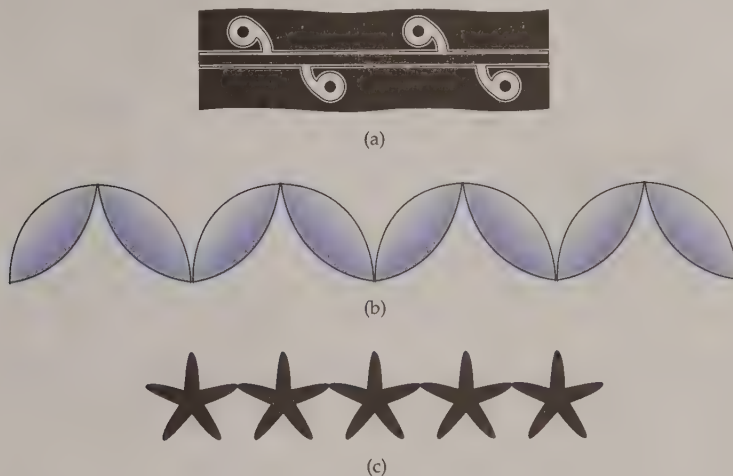
At this point, you have an informal understanding of slides, flips, and turns. In the next few pages, we will examine these transformations more precisely. This precision is needed in order to create aesthetically pleasing designs, such as those found in art, quilts, tiles, or wall coverings. This precision is also needed in order to understand naturally occurring structures, both those that are visible to the eye (honeycombs, snowflakes, and nautilus shells) and those at the molecular level. The following investigations will all be driven by this question: How can we describe what we have to do to move an object from here to there on a plane? If you read these investigations actively, you will probably get stuck at times, frustrated at times, and excited at times. It is important to keep two things in mind. First, much of the knowledge presented in this book represents hundreds and sometimes thousands of years of investigation. Second, the struggle for precision has often led scientists and mathematicians to see new patterns, unanticipated connections, and new properties.

Translations

When you see the word *translation*, you probably think of translating from one language to another: The translator replaces English words and phrases with Spanish words and phrases, for example, but the meaning of the phrase remains the same. In geometry, a translation involves moving an object along a straight line and not turning it.

Each of the designs in Figure 9.5 shows translations. Figure 9.5(a) is from the Native American Yuchi tribe and depicts storm clouds. Figure 9.5(b) shows a quilt border called Orange Peel. Figure 9.5(c) shows a papercutting, which is

a common elementary school activity. As noted above, a translation is a special kind of congruence transformation. In each of the designs, we have congruent figures in the design, and we can imagine picking up part of the figure, moving it in a straight line, and then putting it back down on top of another congruent part of the figure.



Source: (a) Le Roy H. Appleton, *American Indian Design and Decoration* (New York: Dover Publications, 1971), Plate 15 (in insert). (b) Jinny Beyer, *The Quilter's Album of Blocks & Borders* (Delaplane, VA: EPM Publications, Inc., 1986), p. 185.

FIGURE 9.5

Do you see the translations in these designs? Based on what you see, how would you describe translations? The following investigation looks at translations in some depth.

INVESTIGATION 9.1 Understanding Translations

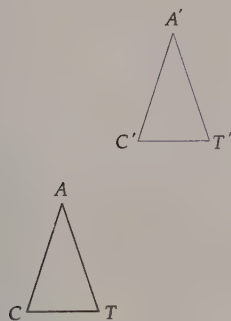


FIGURE 9.6

You may have already worked with translations in *Explorations*. In order to make intricate designs, we often need to be precise in describing the translation. For example, let's say you were talking on the phone to a friend and you wanted to describe the translation of the triangle CAT in Figure 9.6 from its original location to its new location. The original triangle is CAT ; its translation image is $C'A'T'$. How would you do this? You may assume that you and your friend have any resources you need: scissors, ruler, compass, graph paper, etc. Think and then read on. . . .

Note: There are many valid descriptions.

DISCUSSION

One common response is to say that we slide the triangle from one location to the other. The mathematical word for slide is *translation*, and we talk about a figure and its **image**. There are many different ways to describe a translation. Each of these descriptions represents different perspectives and preferences concerning how to communicate what has happened to triangle CAT . Some of the perspectives are easier to describe if we draw the two triangles on graph

paper (see Figure 9.7). What they all have in common is the realization that when we translate a figure, every point on the figure moves the same distance and in the same direction.

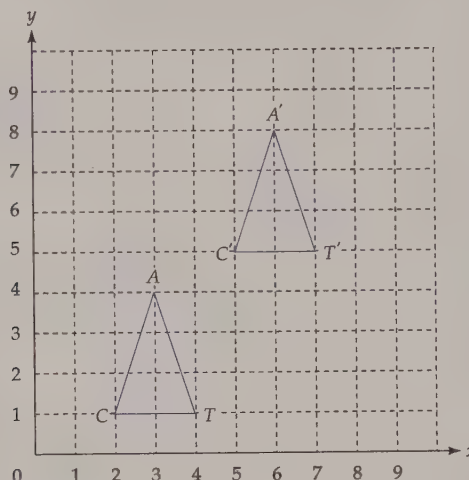


FIGURE 9.7

DESCRIPTION 1: Use “taxicab language”

Just as we could ask a cabbie to “go three blocks east and four blocks north,” we could say that each point has been moved 3 units to the right and 4 units up. For example, copy triangle CAT onto another sheet of paper and cut it out. Then place it on the paper above and move the whole triangle, first 3 units to the right and then 4 units toward the top of the paper.

DESCRIPTION 2: Invent new notation

We can use notation to express this same idea more succinctly:

$$(x, y) \rightarrow (x + 3, y + 4)$$

If we understand the idea of translation, then this notation gives the directions “go 3 units to the right and 4 units toward the top of the page” very succinctly.

DESCRIPTION 3: Specify the distance and the angle

We could say that the figure has been moved a distance of 5 units at about a 54-degree angle from the x -axis (see Figure 9.8). That is, C and C' are 5 units apart; the distance between any point on triangle CAT and the corresponding point on triangle $C'A'T'$ is 5 units. Similarly, the angle formed by the rays CC' and CT is equal to approximately 54 degrees. I encourage you to check this for yourself with a ruler and a protractor.

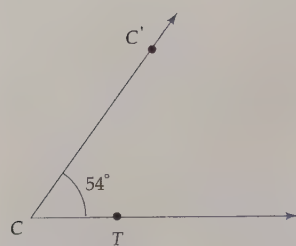


FIGURE 9.8

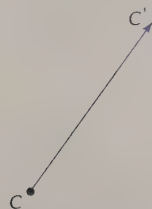


FIGURE 9.9

DESCRIPTION 4: Use vectors

Another alternative is to use a vector. By definition, a **vector** has a length and a direction. For example, the instructions for the translation could be shown by drawing one vector, as in Figure 9.9.

We will formally define **translation** as a transformation on a plane determined by moving each point in the figure the same distance in the same direction.

Properties of Translations

Now let us determine some of the properties of translations. We know from Chapter 8 that two points determine a line. Therefore, if we connect each vertex in the triangle TAR to its image, $T'A'R'$, we have three line segments: TT' , AA' , and RR' (see Figure 9.10). What relationship do you notice among these line segments?

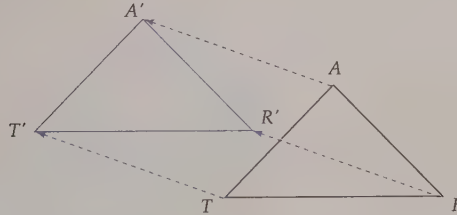


FIGURE 9.10

The three line segments are all congruent (same length) and all parallel (same direction). How might this make the task of drawing the translation image easier? Think and then read on. . . .

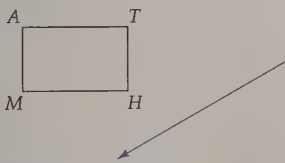


FIGURE 9.11

Translate the rectangle in Figure 9.11, using the translation arrow shown. We can draw four lines, each going through a vertex, that are parallel to the translation arrow. If we copy the length of the arrow using a compass, we can quickly mark off the same lengths on the lines to determine the vertices of $M'A'T'H'$ (see Figure 9.12).

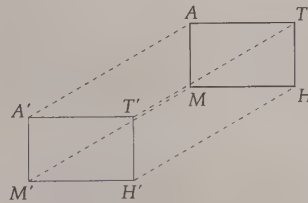


FIGURE 9.12

Reflections

Like translations, reflections may be accomplished in a variety of ways. (See Exploration 9.4 for more work with reflections.) One of the simplest is by folding paper. Trace the triangle and the line below from Figure 9.13 onto a blank sheet of paper, and then fold the paper at the line.

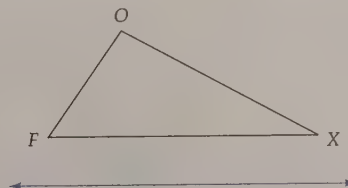


FIGURE 9.13

Now trace over the figure. This is like making a carbon copy of the figure. When you fold the paper on the line, each point on the triangle coincides with its image on the reflected triangle. Do you think we need to trace the whole figure? What do you think is the minimum tracing needed?

If you trace the points F , O , and X , you will have the points F' , O' , and X' , and we now need connect only these three points because we know that three noncollinear points determine a triangle.

INVESTIGATION 9.2 Understanding Reflections

Figure 9.14 shows a trapezoid that has been reflected (flipped) in three different ways. In each case, bold lines denote the original figure and dotted lines denote the flipped image.

- Find the line of reflection in each case.
- Can you discover a rule that would allow us to find the line of reflection in all cases? Stop, think, and write. 

DISCUSSION

You can just read on and find the rule, or you can experience the kind of mathematical thinking that so many men and women and children have found very exciting.

A You may find it helpful to start by tracing these figures on tracing paper and finding the line (by folding the paper) that makes the original figure coincide with its reflected image. Do you see what is happening? Try making figures of your own on paper; fold the paper and then trace the reflection image of the figure you drew. Do this several times. Play the “what-if” and “is it possible” games: What if the line of reflection went through the figure? Is it possible to have the line of reflection go right through the figure? Islamic artists, M. C. Escher, and many others have discovered so much by playing these two games.



FIGURE 9.14

Paper folding is one of many ways to draw reflections. What other ways can you think of?

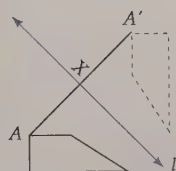


FIGURE 9.15

B *Properties of reflections.* What is true for all reflections is that if we connect any point on the original figure with the corresponding point on the reflected figure, the line of reflection is the perpendicular bisector of that line segment. See Figure 9.15. That is, $A'X = XA$, and AA' and line l are perpendicular.

This realization leads to a more formal definition of reflection. A **reflection** is a transformation that maps a figure so that a line, the **line of reflection**, is the perpendicular bisector of every line segment joining a point on the figure and the corresponding point on the reflected figure.

INVESTIGATION 9.3 Understanding Rotations

This investigation uncovers ways in which we can describe rotations. In both of the diagrams in Figure 9.16, a triangle has been rotated. Can you describe how we can get each triangle from its original position to its new position? In order to describe any rotation precisely, what do we need? Stop, think, and then read on. . .



FIGURE 9.16

DISCUSSION

As with the concept of reflection, you might want to explore this yourself. You can do this by drawing a figure on a piece of paper and then tracing it on another piece of paper. Line up the two pieces of paper so that the top figure lies on the bottom figure, place the tip of your pencil or pen on the paper at some point, and then rotate the bottom sheet of paper. Draw the rotation image by tracing over the image you see from the bottom sheet. This is easier if the top sheet is a sheet of tracing paper, but it works with regular blank paper, too.

In order to rotate any figure (in a plane), we need to select a *center of rotation* (that is, the point that does not move), and we need to decide how much to rotate the figure. “How much” is determined by specifying an angle.

In the case of the first rotation in Figure 9.16, the center of rotation is the point that is the bottom right vertex of the triangle, and the degree of rotation is 90 degrees clockwise. If you trace the figure, place your pencil on the point, and turn the paper 90 degrees, the figure will now be in the new position.

In the second case, the location of the center of rotation is not immediately obvious. If you take any two corresponding points on the two triangles and connect them, the center of rotation will be on the perpendicular bisector of this line segment.

Formally, we say that a **rotation** is a transformation on a plane determined by holding one point fixed and rotating the plane (in our case, the paper) about this point by a certain number of degrees in a certain direction. The fixed point is called the **center of rotation**.

These three transformations—translation, reflection, and rotation—are known as congruence transformations because the images are congruent to the original figure.

Before we go on, it is important to note one important attribute common to each of these three transformations. Recall that we found that when we translate a figure, each point on the figure is translated the same distance. Similarly, when we reflect a figure across a line, any point and its image are the same distance from the line of reflection. What do you think is equidistant with rotations?

When we rotate a figure, any point and its image are the same distance from the center of rotation. Thus, the sense of “equidistance” is something all

congruence transformations share. As you may have seen in Exploration 9.7 (Tessellations), and as you will see in the explorations in the next section, many art designs involve one or more of these transformations.

Let us now examine some of the relationships among these three transformations.

Combining Slides, Flips, and Turns

Now that we understand these three basic transformations of the plane, let us examine combinations of these transformations. If you have been playing the “what-if” game, you may have already thought about this and even investigated it yourself. For example, what if you reflect a figure and then rotate it? As with so many aspects of mathematics, you might expect there to be patterns in the combinations.

Let us do one brief activity that will help you to understand a property of combinations of transformations. Trace Figure 9.17 on a sheet of paper and translate the figure 2 inches in a direction directly above its original position. The translation vector is shown at the left. Then reflect the translated image across the line l . What do you get?

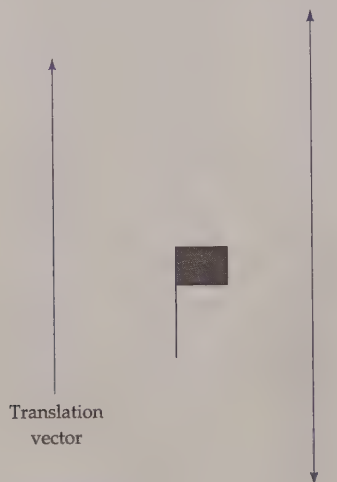


FIGURE 9.17

What if you reverse the order—that is, first reflect the flag over the line and then move it 2 inches north? Will the flag wind up in the same space? In other words, are these operations commutative? First predict the answer and note your justification; then check it out.

It turns out that a translation followed by a reflection is equivalent to a reflection followed by a translation, but only if the translation vector and the reflection line are parallel. We call any combination of transformations a **composite transformation**.

There is a special name for the composite transformation of a translation followed by a reflection with the condition that the translation vector and the line of reflection are parallel (that is, point in the same direction). We call such a composition a **glide reflection**. I recall finding this transformation a bit elusive in terms of *really* understanding it, so let's take a minute to examine this one. Figure 9.18(a) shows a flag. Figure 9.18(b) shows this flag being translated to position 2. Figure 9.18(c) shows the flag in position 2 being reflected across a

horizontal line (which is parallel to the translation vector). The transformation of the flag from position 1 to position 3 is a glide reflection. Note that the translation vector does not need to be horizontal—it can be in any direction. But whatever the direction of the translation vector, the reflection line must be in the same direction.

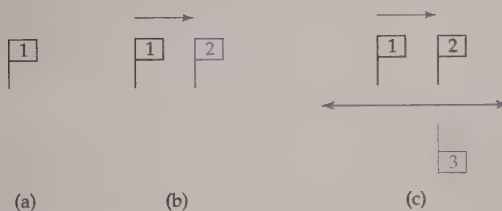


FIGURE 9.18

Let me give you an example of a translation followed by a reflection that is *not* a glide reflection. Figure 9.19(a) shows a flag. Figure 9.19(b) shows this flag being translated to position 2. Figure 9.19(c) shows the flag in position 2 being reflected across a vertical line that is *not* parallel to the translation vector. The transformation of the flag from position 1 to position 3 is *not* a glide reflection, because the translation vector and the reflection line are not parallel.

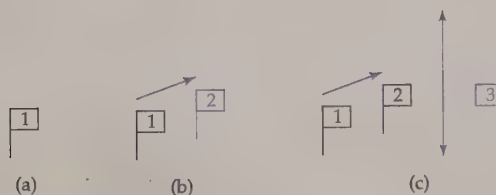


FIGURE 9.19

Probably the most famous example of glide reflection, and one that has helped many students get the sense of glide reflections is a diagram of footprints on the sand (see Figure 9.20).

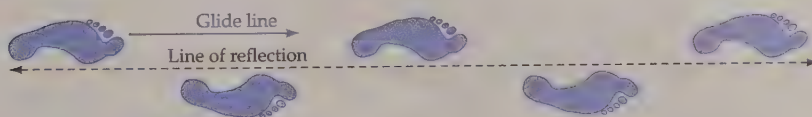


FIGURE 9.20

You can see the reflection (the left and right feet), and you know (from walking on a beach on the ocean or on a lake) the feeling of gliding along the beach. Also, you can connect your experience to the mathematical piece—that the line of reflection and the glide line are parallel (in the same direction), which is a *requirement* for all glide reflections.

Look at Figure 9.21, which is taken from a woodcut by M. C. Escher. Do you see the glide reflections? How would you explain them?

Any black bird can be mapped onto any white bird by translating the bird and then reflecting the translated image across a vertical line. Of course, to do it exactly, we would need to specify the translation vector (direction and distance) and the reflection line (direction and distance from the translated image of the black bird).



Source: M.C. Escher's "Study of Regular Division of the Plane with Birds." © 2000 Cordon Art B.V.—Baarn—Holland. All Rights Reserved.

FIGURE 9.21

At this point, there are lots of possible “what-if” and “is it possible” questions. Which ones come to your mind? Think before reading on. . . . 

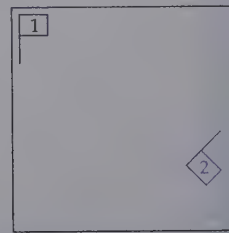
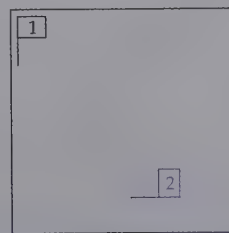
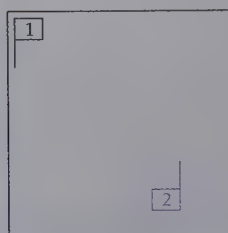
Such questions include the following:

1. Does it matter which one you do first when you do a composite transformation?
2. How many different congruence transformations are there?
3. Is there a least number of steps that will enable you to map one figure onto a congruent image?

Rather than answer these questions directly, let us use the following investigation to examine them.

INVESTIGATION 9.4 Connecting Transformations

In each of the three pictures below, describe how you would get the flag in position 1 to position 2. Then read on. . . . 



DISCUSSION

The curious reader may have realized that there is not one right answer for any of these! For example, in the first picture, you could get from the first to the second position by first reflecting the flag across a vertical line and then reflecting the image across a horizontal line. Or you could reverse the order, doing the horizontal reflection first and the vertical reflection second. Or you could get from the first to the second position by a single rotation. Can you find the center of rotation?

Let us use this example to answer the first question above. In this case, it did not matter which reflection (vertical or horizontal) we did first. This leads most people to conclude that the operation of reflection is commutative. This is connected to our discovery, in Chapter 3, of the commutative properties of addition and multiplication. However, if you conclude that there is a commutative property of reflection, you are wrong! Two reflections produce the same image *only* if the lines intersect at right angles. If the two reflection lines are parallel, we do not get the same image when we reflect across line m and then across line n as we do when we reflect across n and then m . Thus the operation of reflection is commutative only if the lines of reflection intersect at right angles. This is just one of many examples in mathematics where we must think carefully before generalizing!

In the second picture, you could translate the flag to the right, then rotate the image 90 degrees clockwise, and then flip the image across a horizontal line. Or you could rotate the figure 90 degrees clockwise and then do a glide reflection by translating the image so that it is directly above the flag in position b and then reflecting this image. Or you could move from the first to the second position by a single reflection. (Can you find the line of reflection?)

The third situation is more complicated. You could reflect the flag across a horizontal or vertical line, then rotate the image the appropriate amount, and then translate this image the appropriate amount. However, there is a glide reflection that enables us to move the flag from position 1 to position 2.

From these few examples and Exploration 9.4 (Developing Reflection Sense), you have some experiential basis for understanding two amazing mathematical theorems, which represent much hard work on the part of many mathematicians and which have clear applications for people designing patterns, whether they be for clothes, quilts, or more theoretical problems. The first theorem states that the set of congruence transformations is closed with respect to composition. Another way of stating this is that you can replace any combination of any two congruence transformations with a single transformation. For example, a 47-degree clockwise rotation followed by a reflection across a specified line can be done in one step as one of the four congruence transformations.

Let me state the other theorem in a slightly unorthodox way. Imagine any figure on a plane. Now move that figure to any other position on the plane in any orientation. No matter how complicated the shape and no matter how complicated the relationship between the two positions, we will always be able to map the figure from the first to the second position, in one step, with one of the four congruence transformations!

It is beyond the scope of this book to go into more detail about other properties of operations on shapes, but I did want to give you a glimpse of the deeper mathematical structures here. At this point, let us summarize some important connections between operations on numbers and operations on shapes. In both cases, there are many important subsets. In both cases, there are many operations that we can perform. In both cases, we can make tables for those operations (Exploration 3.7, Exploration 9.10). In both cases, we can learn a lot

by taking apart (decomposing) numbers and shapes in various ways. For example, we can take apart 45 as $40 + 5$ or as $9 \cdot 5$. In Chapter 8, you found that we can take apart any polygon into a number of triangles. Finally, just as we talked about properties of operations with numbers (such as closure, commutativity, associativity, identity, and inverse), we encounter the same properties when we do operations on shapes. Table 9.2 summarizes these connections between numbers and shapes, with examples for each.

TABLE 9.2

Subsets	Numbers: N, W, I, Q, R natural numbers, whole numbers, integers, rational numbers, real numbers	Shapes: Triangles, quadrilaterals, regular polygons, concave, etc.
Operations	$+$, $-$, $\times$, $\div$, exponentiation, averaging	Translation, reflection, rotation, glide reflection, similarity transformation, topological transformation
Tables and patterns in tables	Addition and multiplication tables	Multiplication tables for symmetries of various shapes
Decompose/compose	Numbers can be decomposed additively and multiplicatively.	Any polygon can be decomposed into triangles.
Properties	Closure, commutativity associativity, identity, inverse	Closure, commutativity, associativity, identity, inverse

Congruence

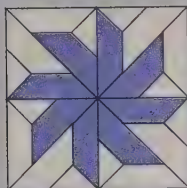
In Chapter 8 we discussed and worked with different concepts of congruence. At the most basic level, we can say that two figures are congruent if they coincide when we superimpose one over the other—in other words, when we move one figure (by translation, reflection, rotation, or glide reflection) so that its image coincides with the other figure. Now that we know that we can move any figure to any position through some combination of transformations (the theorem just cited), we can use this theorem to give a more general definition of congruence:

Two figures are **congruent** iff there is a translation, reflection, rotation, or glide reflection that maps one figure onto the other.

INVESTIGATION 9.5 Transformations and Art

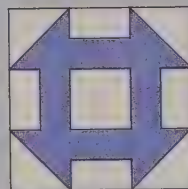
Transformations have a lot to do with the patterns we see in tiles, quilts, and other art, especially Islamic art and the work of M. C. Escher. Look at the two patterns in Figure 9.22, which we have seen before. Do you see

translations, reflections, and/or rotations in these figures? Write your thoughts and then read on. . . . 



Parquet

FIGURE 9.22



Churn Dash

DISCUSSION

In the parquet pattern, we can see that each light trapezoid can be mapped onto a dark trapezoid through a translation along a diagonal line. Each triangle can also be mapped onto another triangle through a 90-degree rotation. Furthermore, if we rotate the figure 90 degrees (clockwise or counterclockwise), the whole figure maps onto itself. If you don't see this, make a copy of the figure, superimpose the figure on the figure in the book, and rotate it.

From another perspective, you can see that the "pinwheel" part of the parquet pattern can be generated by taking only one-quarter of the pinwheel (that is, two adjacent trapezoids) and rotating that portion 90 degrees three times (see Figure 9.23).



FIGURE 9.23

In the churn dash pattern, each of the white triangles can be mapped onto each of the other white triangles across a line of reflection. If we rotate the churn dash 90 degrees, with the center of rotation being the center of the figure, it maps onto itself. Furthermore, the churn dash, like the parquet pattern, can be generated by taking one-quarter of the figure and rotating it 90 degrees (clockwise or counterclockwise) three times (see Figure 9.24) or by reflecting it across a horizontal line and then reflecting the original figure and its image together across a vertical line.



FIGURE 9.24



FIGURE 9.25

Actually, an even smaller piece of the figure (half of the piece of the churn dash in Figure 9.24) will generate the whole churn dash design (see Figure 9.25). Don't just take my word for it. Do you see how? . . . 

If we reflect Figure 9.25 across the diagonal dotted line, we get Figure 9.24. One fun way to verify that Figure 9.25 will generate the churn dash is to tape two small mirrors together and place the two mirrors along the dotted lines of Figure 9.25. Like magic, you will see the whole churn dash! From one perspective, we can call this region the unit of the churn dash; from another perspective, mathematicians call this region a **fundamental domain** or **fundamental region**—that is, a region that under some combination of transformations will produce the whole pattern. This piece of the churn dash is the smallest part of the figure that will generate the whole figure. This looking for the smallest piece of a figure or a design that will generate the whole figure or design occurs in many parts of mathematics, not just with shapes; it is called finding the fundamental region of a pattern. I hope you can see that there might be some value in analyzing a pattern to find the fundamental region of the pattern (you might also think of this as the unit of the pattern). This notion of fundamen-

tal region is actually a little piece of a bigger idea, called the “fundamental region of a group acting on a set,” that occurs in several different branches of mathematics.

Summary

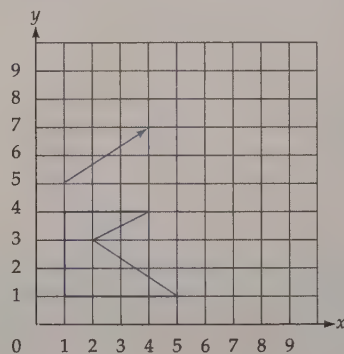
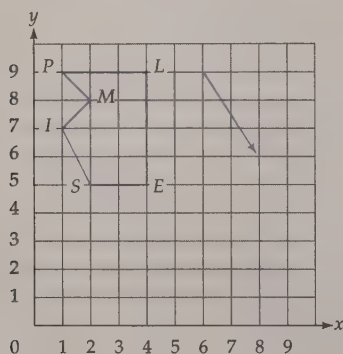
In this section, we have examined four fundamental ways in which we can transform a geometric figure: translation, reflection, rotation, and glide reflection. We have uncovered connections among these transformations. For example, two specific reflections is equivalent to a specific rotation. These connections have led to larger generalizations. We have discovered that there are many similarities between numbers and shapes: They have important subsets, we do operations on them, we can make tables for the operations, we can decompose numbers and shapes, and there are important properties that help us better understand our operations on numbers and shapes.

There is one more connection: All of these operations (addition, subtraction, multiplication, division, translation, reflection, and rotation) are functions. Do you see why? In each case, we can specify the operation by matching any input with a unique output.

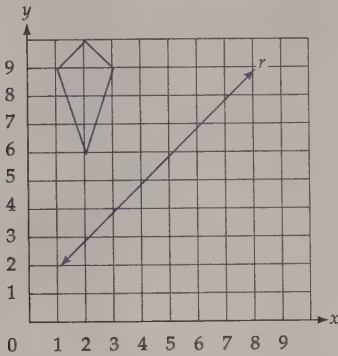
These investigations of geometric transformations of two-dimensional figures in a plane can be extended. What if we examined transformations of geometric figures into different planes? What if we examined transformations of three-dimensional objects? These investigations occupy the attention of many mathematicians and have applications in other fields. Scientists in many fields are benefiting from a deeper mathematical analysis of three-dimensional tessellations. For example, the way in which atoms are packed helps to determine the properties of a compound.

EXERCISES 9.1

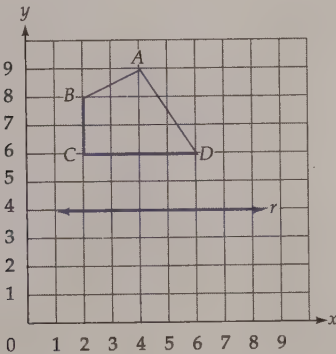
- Below are a figure and a translation vector. Determine the translation. Explain at least two ways to describe the translation other than providing the vector.
- Below are a figure and a translation vector. Without using pencil or pen, determine whether the image overlaps the original figure or not. Explain your reasoning.



3. Find the image of the kite reflected across line r .



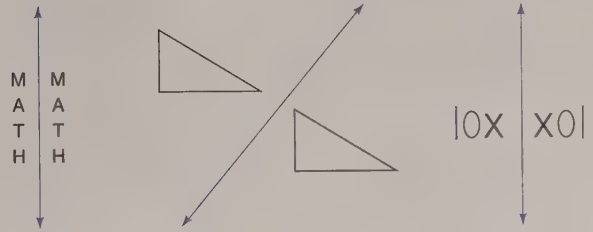
4. The figure below shows quadrilateral $ABCD$ and a line of reflection. Determine the coordinates of A' , B' , C' , and D' using only reasoning (that is, without folding). Explain your reasoning.



5. Below are a figure and its image. Without folding, sketch the approximate location of the line of reflection. Explain your reasoning.



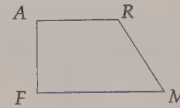
6. Without actually doing the reflection, predict the cases in which the following two images are reflections of each other across the line. Explain your reasoning.



7. Determine the rotation image of triangle MAT if the triangle is rotated 90 degrees clockwise about point A .



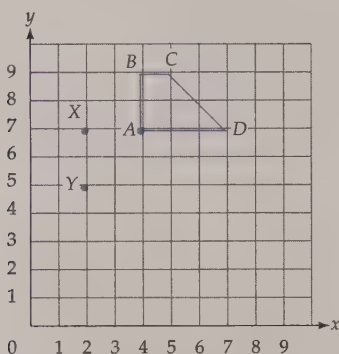
8. Determine the rotation image of trapezoid $FARM$ if the figure is rotated 90 degrees counterclockwise about point F .



9. Determine the rotation image of the figure below if it is rotated 180 degrees clockwise about point A . Explain your reasoning.



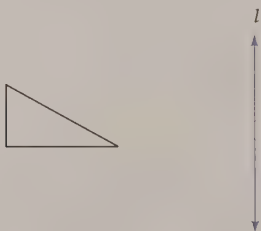
10. a. Determine the coordinates of the vertices of the rotation image of trapezoid $ABCD$ following if it is rotated 90 degrees clockwise about vertex A . Explain your reasoning.
 b. Determine the coordinates of the vertices of the rotation image of trapezoid $ABCD$ following if it is rotated 90 degrees clockwise about point X . Explain your reasoning.
 c. Determine the coordinates of the vertices of the rotation image of trapezoid $ABCD$ following if it is rotated 90 degrees clockwise about point Y . Explain your reasoning.



Exercises 11 and 12 involve various composite transformations. In each case:

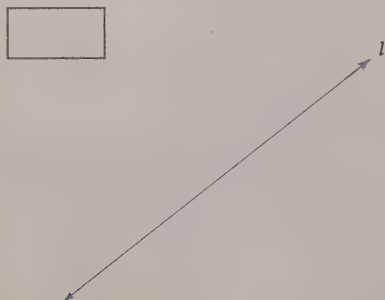
- First predict the relative location and sketch the image of the figure after the transformations. Explain your reasoning.
- Perform the two transformations. If your prediction was correct, great. If not, explain the error in your reasoning. Explain also your plans for “correcting” that error.
- Predict whether the doubly transformed image will be the same if you do the transformations in the opposite order. Then do the two transformations. If your prediction was correct, great. If not, explain the error in your reasoning. Explain also your plans for “correcting” that error.

11. First transformation: translation $\longrightarrow$
Second transformation: reflect the figure across line l .

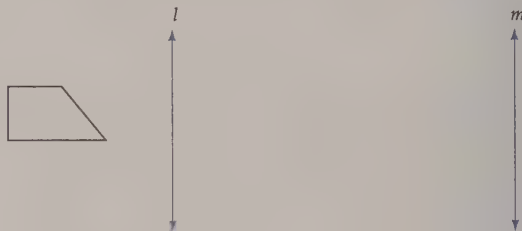


12.

- First transformation: translation $\downarrow$
Second transformation: reflection across line l



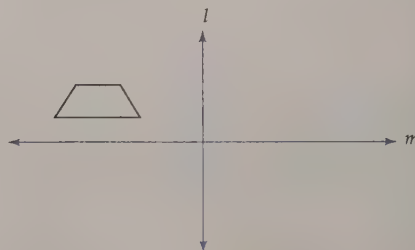
- Under what circumstances will a glide reflection (translation then reflection) be commutative? That is, when will the translation followed by a reflection produce an image that coincides with the reflection followed by the translation? Explain your reasoning.
- Determine the location of the image of the trapezoid below after it is reflected across line l and its reflection image is then reflected across line m .
 - Predict whether the reflection of the trapezoid across line m and then across line l will give you the same image. Explain your reasoning.
 - Test your conjecture, and correct it if necessary.



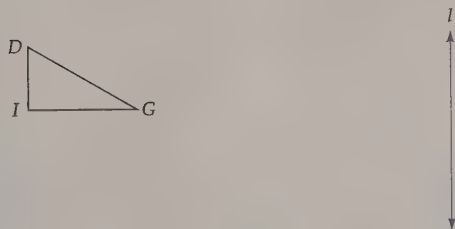
- Determine the location of the image of the rectangle below after it is reflected across line l and its reflection image is then reflected across line m .
 - Predict whether the reflection of the rectangle across line m and then across line l will give you the same image. Explain your reasoning.
 - Test your conjecture, and correct it if necessary.



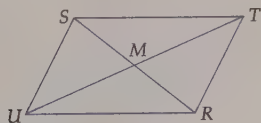
- Determine the location of the image of the trapezoid after it is reflected across line l and its reflection image is then reflected across line m .
 - Predict whether the reflection of the trapezoid across line m and then across line l will give you the same image. Explain your reasoning.
 - Test your conjecture, and correct it if necessary.



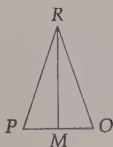
17. a. Rotate $\triangle DIG$ 90° clockwise about point D . Then reflect the rotated image across line l .
 b. Is there another transformation or composition mapping that will produce the same effect?



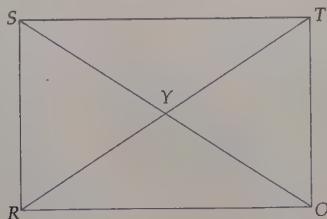
18. Consider parallelogram $STRU$, whose diagonals intersect at M .
 a. Describe a transformation that would map $\overline{ST}$ onto $\overline{RU}$.
 b. Describe a transformation that would map $\angle SRU$ onto $\angle TSR$.
 c. If we reflect $\overline{UM}$ across line $\overleftrightarrow{SR}$, will the image be $\overline{MT}$? If so, explain why. If not, explain why not.



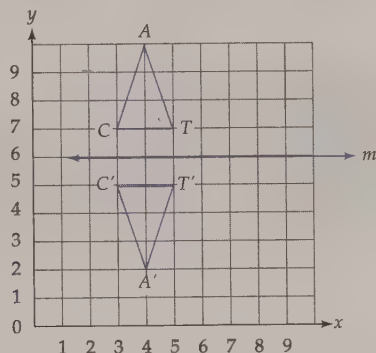
19. Consider isosceles triangle PRO , where M is the midpoint of PR .
 a. Describe a transformation that would map $\overline{PR}$ onto $\overline{OR}$.
 b. What transformation would map $\triangle PMR$ onto $\triangle OMR$?



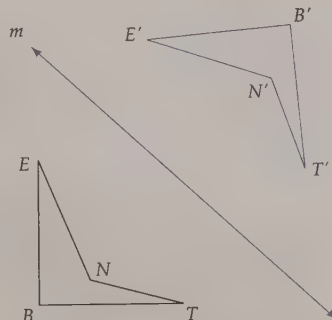
20. a. Describe a single or composite transformation that will map $\triangle SRO$ onto $\triangle OTS$.
 b. Describe a single or composite transformation that will map $\triangle SRO$ onto $\triangle RST$.
 c. Describe a single or composite transformation that will map $\triangle SRO$ onto $\triangle TOR$.



21. The figure below shows triangle CAT , which has been reflected across line m .
 a. What relationships do you see between the coordinates of the vertices in $\triangle CAT$ and those of its reflection image $\triangle C'A'T'$?
 b. What generalizations can you make from this exercise? You may want to gather more evidence on your own—that is, to make and reflect different figures.



22. Consider the concave quadrilateral $BENT$ and its image $B'E'N'T'$. Under what circumstances will $B'E'N'T'$ be the reflection of $BENT$ across line m ?



23. Look back to Figure 9.1. Describe any translations, reflections, rotations, and glide reflections within the figure. For translations, supply a vector or other means of communicating the translation; for reflections, describe the line(s) of reflection; for rotations, describe the angle of rotation.

[illegible]

- a. Describe translations, reflections, and rotations within this table. For example, the first half of the 2s row (2, 4, 6, 8, 0) can be translated (5 boxes horizontally). Without transformation language, we would say the first half of the 2s row is the same as the second half of the 2s row. For the purposes of this exercise, consider only the location of the digit, not its spatial orientation.
- b. Make a multiplication table for base 12 in which only the ones digit of the product is given. Describe the transformations that can map one row or column onto another row or column. What similarities and differences do you see between the two tables?
30. The object of the game below is to move the arrow from the top-left corner to the bottom-right corner, using transformations. The only restriction is that you cannot go back to a square once occupied.
- a. Describe how to move the arrow from the top-left corner to the bottom-right corner in as few moves as possible.

- b. Describe how to move the arrow from the top-left corner to the bottom-right corner in as few moves as possible, with the additional restriction that each move has to be to a square that has at least one point in common with the square in which the figure resides.



SECTION

9.2

SYMMETRY AND TESSELLATIONS

WHAT DO YOU THINK?

- Does a figure have to be closed to have reflection symmetry? Why or why not?
- Does symmetry have practical applications?

*"Symmetry, as wide or as narrow as you may define its meaning, is one idea by which humanity through the ages has tried to comprehend and create order, beauty, and perfection."*¹

The four congruence transformations that we explored in Section 9.1 lead to four symmetries. Our discussion of symmetry will evolve from our examination of symmetries of shapes to the examination of symmetries of patterns, which are more complicated but also more interesting to most people. Before presenting a definition of symmetry, let us explore this idea, which is not new to you.

Symmetry pervades our lives, and many people find symmetrical figures appealing. We see symmetry in both natural objects and human-made objects—snowflakes, starfish, flags, logos, quilts, and many kinds of art, to name but a few. Look at Figure 9.26. Each of its elements is a well-known symbol or figure. For each of them, write what it is about that figure that is visually appealing or that attracts your attention. Describe the symmetry within each figure. Then read on. . .



Chinese
Yin - Yang
"Unity of
opposites"



Canadian
flag



Butterfly



Snowflake

FIGURE 9.26

I think it is very possible that the first mathematical acts of human beings had to do not with numbers but with symmetry. As we saw in Chapter 2, many

¹ Hermann Weyl, *Symmetry* (Princeton, NJ: Princeton University Press, 1952), p. 5.

cultures survived even into the twentieth century without numbers or with very primitive number systems. However, the archaeological records of virtually every culture show symmetry in some of that culture's artifacts: its baskets, its pottery, the artifacts created for celebrations and rituals. I can think of few religious symbols that do not have symmetries within them.

Symmetry is both a mathematical and an aesthetic concept, and it is a fundamental organizing principle in both nature and human culture. The analysis of symmetry (looking for attributes, classifying, and so on) enables us to understand better the organization of patterns (either natural or human-made).

Symmetry abounds at the microscopic level. For example, many chemical molecules are symmetric, as are many viruses. In fact, the most common shape that viruses take is the icosahedron (remember Chapter 8?); it characterizes the viruses that cause herpes, chicken pox, and warts.

Symmetry abounds at the macroscopic level as well. To cite just a few examples, volcanoes are cone-shaped, stars are spherical, honeycombs are hexagonal, and galaxies tend to be spiral or elliptical; all of these shapes are distinguished by their symmetry. "Any understanding of nature must include an understanding of these patterns."² For example, the hexagonal shape of the honeycomb is the solution to the problem of how to get the maximum storage capacity from the minimum amount of building materials!

Symmetry also adds to our understanding of patterns made by humans. For example, understanding symmetry can make it easier to make many very exotic looking prints on skirts and quilts and can also increase our appreciation of their beauty (as we see the multiple symmetries and transformation relationships of various shapes within the design).

Let us then investigate this aspect of mathematics more deeply, to better understand what we mean by the term *symmetry* and what kinds of symmetries there are. We will investigate four kinds of symmetry: translation, reflection, rotation, and glide reflection.

Let us begin with a definition of **symmetry**: a transformation that places the object directly on top of itself.

Reflection and Rotation Symmetry

On the basis of our work with reflections in the previous section, what do you think *reflection symmetry* means? That is, a reflection transformation involves moving a figure somewhere else on the plane, whereas a reflection symmetry involves moving a figure so that its image is directly on top of itself. How would you explain reflection symmetry in your own words? Can you create a figure that you think will have reflection symmetry? Remember that when the learner is actively connecting what he or she already knows to the new ideas, the knowledge is much more likely to be owned than rented. Therefore, really try to answer these questions before reading on. . . . 

A figure has reflection symmetry if there is a line, called the **line of symmetry**, that can be drawn through the figure such that when we fold the paper on the line, the part of the figure on one side will lie directly on top of the other part.

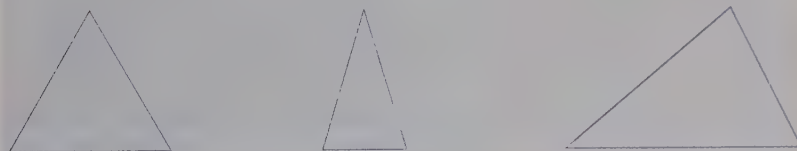
Now let us take a look at rotation symmetry. Thinking of the definition of reflection symmetry and what you know about a rotation transformation, how would you describe rotation symmetry? Can you create a figure that you think will have rotation symmetry? Try to do so before reading on. . . . 

²Ian Stewart, *Nature's Numbers* (New York: HarperCollins, 1995), p. 83. Excerpt from *Nature's Numbers* by Ian Stewart. Copyright © 1995 by Ian Stewart. Basic Books/Perseus Books Group.

Imagine tracing a figure and putting the traced image directly on top of the figure. Informally, we say that the figure has rotation symmetry if we can pick up the tracing, turn it by some amount less than a full turn, and place it down so that it lies directly on the original figure.

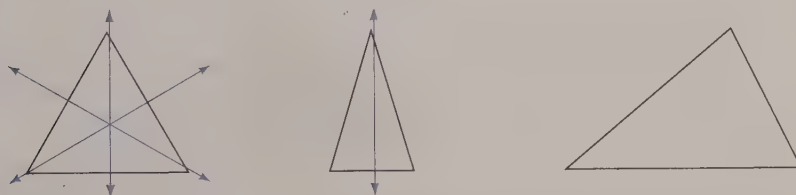
INVESTIGATION 9.6 Reflection and Rotation Symmetry in Triangles

Many of the geometric shapes we investigated in the previous chapter have reflection and/or rotation symmetry. Describe the reflection and rotation symmetries you see in the equilateral triangle, the isosceles triangle, and the scalene triangles below. Then read on. . .



DISCUSSION

We find that the equilateral triangle has three lines of symmetry, the isosceles triangle has one line, and the scalene triangle has none.



With respect to rotation symmetry, only the equilateral triangle has rotation symmetry. How would you describe that symmetry? Think before reading on. . .

Finding the Amount of Rotation in Rotation Symmetry

Some people think of rotational symmetry in terms of how much we have to turn the figure in order for its image to lie directly on the figure. If we turn the equilateral triangle $\frac{1}{3}$ of a whole turn (the center of rotation being the center of the triangle), the image will fit on top of the triangle. If we turn it another $\frac{1}{3}$ turn, the image will fit on top of the triangle again. If we turn it another $\frac{1}{3}$ turn, then we are back to its original position. Thus we can say that the equilateral triangle has $\frac{1}{3}$ -turn symmetry.

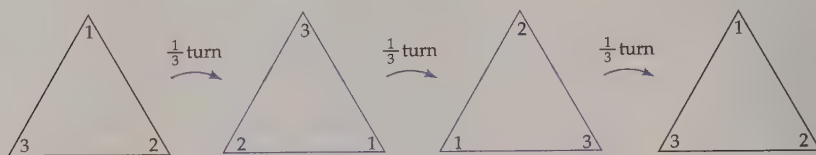
Other people prefer to specify rotation symmetry by the number of degrees needed to make the rotated image fit on top of the figure. In this case, if we rotate the equilateral triangle 120 degrees, the image will fit on top of the figure. If we rotate the equilateral triangle another 120 degrees, the image will fit on top of the figure again. If we rotate the equilateral triangle another 120 degrees, the figure will be back to its original position.

LANGUAGE

Some readers may be wondering why I didn't say that the equilateral triangle has $\frac{1}{3}$ -turn rotation symmetry. Actually, I could have said so, but the word *rotation* is unnecessary.

Some of my students are very puzzled at this point and ask, "How did you figure out it is 120 degrees?" Did you also wonder this? Could you explain why, as though to a ten-year-old (who will probably ask you the question someday)? If not, think before reading on. . . .

Other students find the following illustration useful. The triangle at the left in the figure below is in its original position, with the vertices labeled 1, 2, and 3, respectively. Imagine turning the triangle clockwise until the image is in the same position as the original triangle. The second figure represents the new position. Rotate the triangle again until the image is in the same position as the original triangle. The third figure represents the new position. Rotate the triangle again until the image is in the same position as the original triangle. Because it took three rotations to get back to the original position, each rotation has to equal $360/3 = 120$ degrees.



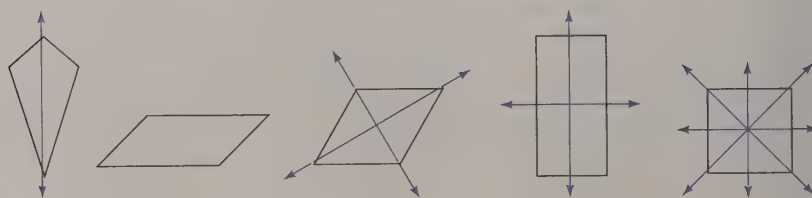
INVESTIGATION 9.7 Reflection and Rotation Symmetry in Quadrilaterals

What about quadrilaterals? Determine the reflection and rotation symmetry of the kite, parallelogram, rhombus, rectangle, and square. I encourage you to use whatever means you need in order to understand this investigation thoroughly. You might trace the figures on a blank sheet of paper, on tracing paper, or on an overhead transparency. Work on this before reading on. . . .



DISCUSSION

Kites have one line of symmetry, parallelograms have no lines of symmetry, rhombi have two, rectangles have two, and squares have four. The most common mistake my students make is to think that parallelograms have reflection symmetry. If you got this one wrong, make several parallelograms on a blank sheet of paper (tracing paper is even better), fold the parallelogram on either diagonal, and hold the paper up to the light. You will clearly see that the parallelogram does not fold onto itself in either case.



With respect to rotation symmetry, a kite has no rotation symmetry, a parallelogram has $\frac{1}{2}$ -turn (or 180-degree rotation) symmetry, a rhombus has $\frac{1}{2}$ -turn (or 180-degree rotation) symmetry, a rectangle has $\frac{1}{2}$ -turn (or 180-degree rotation) symmetry, and a square has $\frac{1}{4}$ -turn (or 90-degree) rotation symmetry.

As you may have already observed, many figures have 180-degree rotation symmetry but no other rotational symmetry. A figure has 180-degree rotation symmetry if it looks the same when turned halfway around. Because this particular kind of rotation symmetry is so common, it has its own special name: **point symmetry**. These three terms are synonymous: $\frac{1}{2}$ -turn symmetry, 180-degree rotation symmetry, and point symmetry.

One last note on rotation symmetry. When I ask my students to tell me whether or not a figure has rotation symmetry, many of them will say that a figure has 360-degree rotation symmetry. What does it mean to say that a figure has 360-degree rotation symmetry? Think and read on. . . . 

To say that a figure has 360-degree rotation symmetry means that if we turn it completely around, it will look the same. However, this is true for *any* figure. Because any figure has 360-degree rotation symmetry, we do not use this term when describing the rotation symmetries of a figure.



MATHEMATICS

When we say that a rectangle has two lines of symmetry, it is important to note that we mean that subset of rectangles that are not squares. Technically, squares are rectangles, so technically, some rectangles (those that are also squares) have four lines of symmetry. This technical definition of certain quadrilaterals both baffles and annoys many students. For example, we also say that a trapezoid has at least one pair of parallel sides, a definition that allows parallelograms to be trapezoids. However, there are reasons for this way of defining certain quadrilaterals. One of the reasons is that because all squares are rectangles, whatever we prove for rectangles holds true automatically for squares.

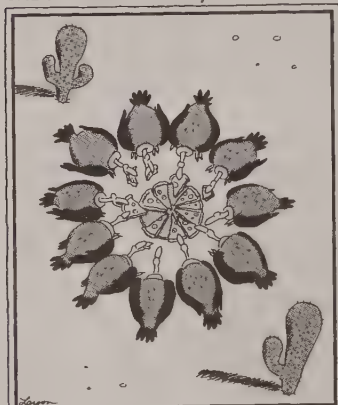
In Section 8.2, we created a family tree for quadrilaterals. By definition, each descendant had the same properties as its immediate ancestor plus at least one new characteristic, thus creating a new class of shapes. When we looked at properties of the diagonals of quadrilaterals, we found that the same family tree applied. That is, every descendant had at least the same properties for its diagonals as its ancestor. As you can see from this exercise, every quadrilateral has at least as many reflection and rotation symmetries as any quadrilateral above it in the “family tree.”

INVESTIGATION 9.8 Reflection and Rotation Symmetry in Other Figures

 The accompanying cartoon is amusing *and* presents us with a great problem. Assuming that the vultures are all congruent and that each vulture grabbed one piece of pizza right in the middle, describe the rotation and reflection symmetries in this idealized diagram. Then read on. . . . 

THE FAR SIDE

By GARY LARSON



Perspectives in nature we rarely enjoy

Source: THE FAR SIDE © 1990 FAR-
WORKS, INC. Used by permission. All
Rights Reserved.

DISCUSSION

This problem gives many students fits. Even when they count the number of vultures, the number 11 is just not a number that makes it easy to describe the symmetries. This is a good spot to try to sell you on a problem-solving strategy that is underutilized by students: making a simpler problem and using deductive or inductive reasoning to build up to the complexity of the given problem. Below are regular polygons with 3, 4, 5, 6, 7, and 8 sides. First draw the lines of symmetry for each figure. What do you see? . . .



Many students realize that the lines of symmetry in the figures with 3, 5, and 7 sides (that is, with an odd number of sides) are somehow “different” from those in the figures with an even number of sides. How would you describe this difference? . . .

When the polygon has an odd number of sides, the lines of symmetry all connect a vertex of the polygon to the middle of the opposite side. When the polygon has an even number of sides, half of the lines of symmetry connect two opposite vertices and half of the lines of symmetry connect the midpoints of two opposite sides. However, in all cases, the number of lines of symmetry is equal to the number of sides in the figure. Thus it seems reasonable to conclude that the original figure has 11 lines of symmetry, and you should be able to draw all 11 lines. Now, what about the rotation symmetry?

If you connect the discussions in the previous investigations to this problem, you should be able to deduce that because there are 11 vultures, this figure has 11-fold symmetry, or that the angle of rotation is $360/11$.

INVESTIGATION 9.9 Letters of the Alphabet and Symmetry

Examine the letters of the alphabet. Which letters have rotation symmetry? What kind(s) of rotation symmetry? Which letters have reflection symmetry? What kind(s) of reflection symmetry? Work on this and then read on. . .

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

DISCUSSION

When we examine the letters in terms of reflection symmetry, we find that there are three kinds of lines of reflection: vertical, horizontal, and diagonal.

The following letters have **vertical line symmetry**: A, H, I, M, O, T, U, V, W, X, Y. I think it is interesting that over half of the letters with vertical line symmetry occur at the end of the alphabet.

The following letters have **horizontal line symmetry**: B, C, D, E, H, I, K, O, X. Here we have the opposite phenomenon: four consecutive letters near the beginning of the alphabet.

Three letters have **diagonal line symmetry** (depending on how they are drawn): O, Q, and X.

The following letters have point symmetry: H, I, N, O, S, X, Z.

Note: The symmetry of some letters depends on how they are drawn. For example, B and K do not always have horizontal line symmetry and Q only has diagonal line symmetry if it is made as a circle (instead of oval-shaped) and if the line segment is on a diameter of the circle.

As we noted in Chapter 1, representation is a new process standard. Above, I represented the solution to the question in a list form. In Figure 9.27 are two Venn diagrams. The first shows just the letters that have line symmetry, and the second shows all the letters. Do these diagrams change or add to your understanding of which letters have what kind of symmetry?

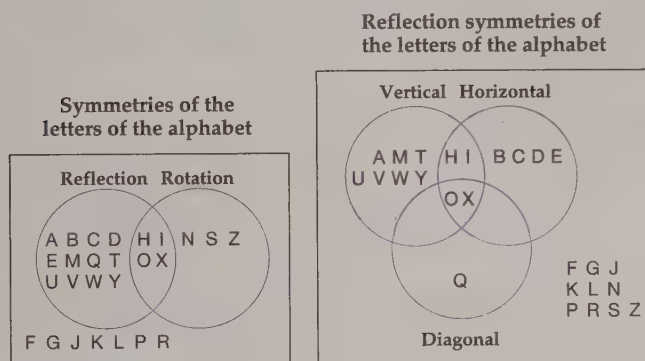


FIGURE 9.27

What Are Patterns?

We have examined the symmetries of several shapes. We are now going to examine symmetries of patterns. Your instructor may have asked you to bring one or more examples of patterns to class. We find patterns in our clothing, on quilts, on ceilings, on floors, on walls, and on the borders of many objects—in short, all over the place. To define the word *pattern* is very difficult, so I will pose a more user-friendly question: Think of a variety of patterns and write down what they all have in common. That is, what are some characteristics (attributes) of all patterns? Then read on. . .



MATHEMATICS

The careful reader may have realized that we also speak of numerical patterns. Recall our work in Chapters 1 and 2. Do those numerical patterns have repetition? Is there a unit for those patterns? Is there some organization of those numbers? Or do we have to have two definitions—one for shape patterns and one for number patterns? What do you think?

One characteristic of all patterns is repetition. A closer examination of repetition leads to a second, related characteristic: There is some *unit* of repetition—that part of the pattern that generates the pattern. Finally, in most patterns that are interesting or functional, there is some *organization*: in other words, the repetition is not random.

This analysis of patterns works fine for certain kinds of patterns, such as checkerboard, herringbone, and quilt patterns. However, this definition always provokes a response in some of my students who consider the definition very carefully. They ask questions like this: “What about patterns you see on a sand dune, or patterns of light on the surface of a lake, or patterns you can see when you ride a canoe on a pond and there is no wind, or the patterns of ripples of sand along the seashore. Aren’t these also patterns?” What a wonderful question!! Although these examples don’t fit this crisp characterization we have described, there is some sense of repetition and organization in each case. The last thing I want to do is perpetuate the belief many students bring to mathematics class that mathematics is fundamentally “different” from the real world. Thus, although we will focus on that subset of patterns where we can



MATHEMATICS

Just as our crisp characterization of the concept of pattern left out a lot of interesting patterns and was restricted to “perfect” patterns, so too did traditional geometry “leave out” a lot of shapes that aren’t perfect or nearly perfect—shapes like clouds, smoke, ocean currents, and weather patterns. A whole new field of geometry has been born since I graduated from college in the 1970s. This field, called fractal geometry, came into being in part because classical Euclidean geometry is very limited in its ability to help us to understand these nonperfect shapes.

find the unit and analyze the organization, I don’t want to exclude this larger sense of patterns.

We are almost ready to examine the symmetries of patterns. However, before we do so, we need to revisit a concept that was presented in Chapter 8—infinity. Recall our discussion in Chapter 8 of the concept of a plane. That is, a mathematical plane is a flat surface that extends infinitely in all directions. Obviously, all examples of planes in a book must be finite, and thus any diagram of a plane is actually a piece (technically, a subset) of a plane. Similarly, mathematical patterns are theoretically infinite. Thus, when you see patterns below and are asked to translate, rotate, and reflect them, you need to realize that, theoretically, there are no edges to the pattern.

Symmetries and Patterns

We have now laid the groundwork for understanding and describing symmetries of patterns. Recall our definition of symmetry: a transformation that places the object directly on top of itself. Thus a pattern has **translation symmetry** if we can lift the pattern, translate it some distance and in some direction, and set it down on top of itself. A pattern has **reflection symmetry** if we can find a line such that if we reflect the pattern across that line, the pattern will fit directly on top of itself. A pattern has **rotation symmetry** if we can find a point such that if we turn the pattern around that point, the pattern will fit directly on top of itself. A pattern has **glide reflection symmetry** if we can find a translation and a line such that if we translate the pattern some distance and in some direction, and then reflect the pattern across a line that is parallel to the translation vector, the pattern will fit directly on top of itself.

Let us now zoom in on the subset of patterns whose characteristics can be clearly described. When an artist or a manufacturer wants to create a new design, the possibilities are virtually limitless. However, as soon as we replace *design* by *pattern*, the designers are limited by the mathematical laws that underlie the formation of patterns. Let us see what this means with one kind of pattern.

INVESTIGATION 9.10 Symmetries of Strip Patterns

We will begin by looking at the mathematically simplest kinds of patterns, which are often called border or strip patterns because the motif (unit) is repeated in only one direction. With your understanding of translations, reflections, rotations, and glide reflections, determine the translation, reflection, rotation, and glide reflection symmetry for each strip pattern below.

- A. What kind(s) of symmetry does the following strip pattern have? Think before reading on. . . 



DISCUSSION

This pattern has translation symmetry because we can translate this pattern so that we can place it on top of itself. The translation vector is parallel to the bottom of the page, and the length of the vector is equal to the distance between any two shapes.

One way to verify this symmetry is to trace the flags on a blank sheet of paper or on an overhead transparency. You can then see that you can move the pattern over 1 unit and it fits onto itself. Remember that, just as the plane is considered to be infinite, extending in all directions, so too are patterns considered to be infinite. Thus, although my figure and your tracing will have a left-most and a right-most flag, the “pattern” is considered to have no ends.

- B.** What kind(s) of symmetry does the following strip pattern have? Think before reading on. . . . 



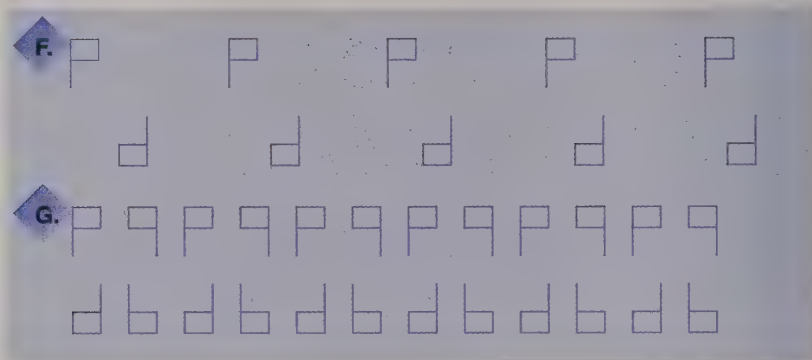
DISCUSSION

This pattern has translation symmetry. This pattern has two vertical lines of symmetry. There are actually two “different” sets of vertical lines that will enable you to put the figure onto itself if you trace this pattern. One of the lines has the flags facing the line, whereas the other line has two flags pointing away from the lines. If you don’t see the vertical lines, turn to page 544, where the symmetries for these strips are shown.

Although the learning why there are two “different” sets of vertical lines of symmetry for this pattern is not one of the big ideas of this section, coming to understand and appreciate mathematical language is one of the big ideas of the book. Thus, let us take just a few moments to pursue why we say there are two different vertical lines of symmetry. Actually, mathematicians say not that the two lines of symmetry are different but rather that they are inequivalent. Two lines of symmetry are equivalent if a symmetry of the pattern can move one reflection line to the other. For example, if you draw one vertical reflection line on the pattern above and then trace the figure and draw another vertical reflection line on the top sheet, these two lines of symmetry are equivalent if you can move the figure so that when the one mirror line is placed on top of the other, the figure fits on top of itself.

- C.** What kind(s) of symmetry does each strip pattern have? Think before reading on. . . . 





DISCUSSION

C. This pattern has translation symmetry. This pattern has horizontal line symmetry. If you trace this pattern and reflect the pattern across a horizontal line, it will fit on top of itself.

D. This pattern has translation symmetry. This pattern has both vertical and horizontal line symmetry.

E. This pattern has translation symmetry. This pattern has glide symmetry. If you reflect the pattern across a horizontal line and then move the pattern in the horizontal direction, it will fit on top of itself.

F. This pattern has translation symmetry. This pattern has $\frac{1}{2}$ -turn symmetry or 180-degree rotation symmetry or point symmetry.

G. This pattern has translation symmetry. This pattern has vertical line symmetry and glide symmetry. If you reflect the pattern across a vertical line, it will fit on top of itself. If you reflect the pattern across a horizontal line and then move the pattern in the horizontal direction, it will fit on top of itself.

You can look on page 545 to see the actual lines of reflection, centers of rotation, and glide vectors.

The Seven Symmetries of Strip Patterns

I find it amazing that any strip or border pattern will fit into one of these seven symmetry types illustrated in A through G above. That is, just as the four congruence transformations are complete in that no more are needed to describe how to map any two congruent figures in a plane onto each other, these seven symmetry types are all that are needed to describe any border pattern.

Let us use a table and introduce some notation to increase the number of readers who understand this. First, you will notice that every one of the strips had translation symmetry. Therefore, we are technically looking at that subset of strip patterns that have translation symmetry. Actually, you will be hard pressed to find a quilt or building or other strip that does not have translational symmetry. As Ian Stewart put it, "Something in the human mind is attracted to symmetry."³

A few words about notation in the far right column of Table 9.3: We will use the letter *m* to indicate a line symmetry (*m* representing mirror), the letter *g* to represent a glide symmetry, the number 2 to indicate twofold turn symmetry, and the letter *l* to indicate a lack of line symmetry.

³ *Nature's Numbers*, p. 73. Excerpt from *Nature's Numbers* by Ian Stewart. Copyright © 1995 by Ian Stewart. Basic Books/Perseus Books Group.

TABLE 9.3

Pattern	TYPE OF SYMMETRY				
	Vertical line	Horizontal line	$\frac{1}{2}$ -turn	Glide	Notation to describe
A					ll
B	Yes				ml
C		Yes			lm
D	Yes	Yes	Yes		mm
E				Yes	lg
F			Yes		l2
G	Yes		Yes	Yes	mg

Some active readers look at this table and argue that there should be more than seven types of symmetry. For example, what about gm? What about m2? These are wonderful questions. It will be left as an exercise to explain why those kinds of symmetry are not found in mathematics textbooks.

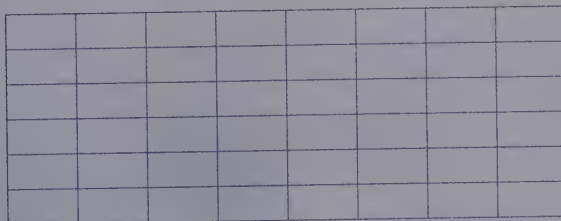
Wallpaper Patterns

Let us examine another kind of planar pattern called wallpaper patterns, ones in which the motif (unit of repetition) can be repeated in two directions. It is important to note that the term *wallpaper pattern* refers to any pattern that can fill the plane if extended. Thus the three patterns below, often seen with bricks, are called wallpaper patterns by mathematicians.

Extending patterns in two dimensions leads to more complexity; consider, for example, all the wallpaper patterns that have been created—thousands, perhaps millions of different ones. However, when we classify them by the kinds of symmetry they possess, any wallpaper pattern falls into one of exactly 17 categories. I find that fascinating! It is a relatively recent discovery, though this issue puzzled mathematicians and scientists for centuries. Interestingly enough, the same 17 categories can be used to classify all crystals. Examining each of these 17 categories is beyond the scope of this book, but the interested reader can search for wallpaper patterns on the web. You will find many descriptions.

INVESTIGATION 9.11 Analyzing Brick Patterns

A. What kind of symmetries does this simple brick pattern have? Think before reading on. . . 



DISCUSSION

It might not surprise you to find that this figure has lots of symmetry!

It has translation symmetry. In fact, the figure can be translated horizontally, vertically, diagonally down to the right, and diagonally up to the right. Thus this figure has four different lines of translation symmetry.

This figure has two vertical lines of symmetry. *Note:* Technically, you could say it has infinitely many vertical lines of symmetry. Recalling the discussion of inequivalence in the previous investigation, we say that this pattern has two inequivalent vertical lines of symmetry.

Similarly, the figure has two inequivalent lines of horizontal symmetry.

The pattern has $\frac{1}{2}$ -turn rotation symmetry, and there are three inequivalent centers of rotation. That is, there are three different centers of rotation that cannot be placed on top of one another by any symmetry of the pattern. See page 521 to see the three centers of rotation. (The reader who finds the notion of inequivalent reflection lines and centers of rotation difficult to understand completely should not be discouraged: These are relatively advanced ideas.)

Finally, when looking for glide symmetry, we find that there are no glide lines that are not already mirror lines. Thus we say that this figure has no non-trivial glide symmetry.

- B.** What about this slightly more complex brick pattern? What symmetries does it have?

**DISCUSSION**

This pattern has translation symmetry. Like the previous pattern, this one can be translated horizontally, vertically, diagonally down to the right, and diagonally up to the right. Thus this figure has four different lines of translational symmetry. However, if you draw the translation vectors, you will find that they are not identical to the four in part A. You can check your thinking on this page.

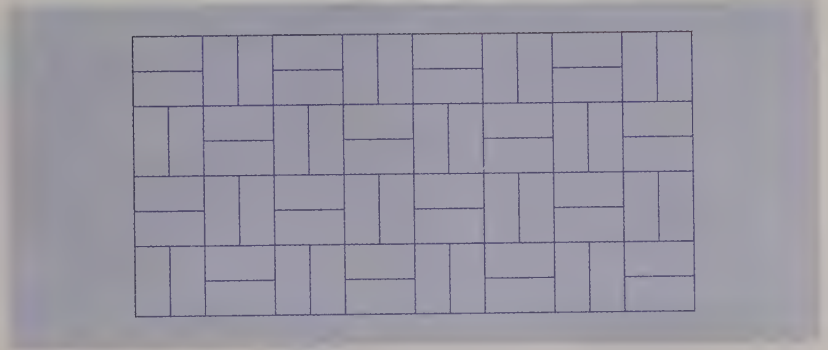
This figure has one line of vertical symmetry.

The figure also has one line of horizontal symmetry.

The pattern has $\frac{1}{2}$ -turn rotation symmetry, and there are three inequivalent centers of rotation. That is, there are three different kinds of points on which you can place your pen, rotate the figure 180 degrees, and have the pattern fit on top of itself.

Finally, this figure has one nontrivial glide symmetry. Did you find it? See this page to check your thinking.

- C.** Finally, what about this more complex brick pattern? What symmetries does it have?



DISCUSSION

This pattern has translation symmetry. Like the two preceding patterns, this pattern can be translated diagonally down to the right and diagonally up to the right, though the four translation vectors are not identical to either of the four above.

This figure has one line of vertical symmetry.

Similarly, the figure has one line of horizontal symmetry.

The pattern has $\frac{1}{2}$ -turn rotation symmetry, and there are three inequivalent centers of rotation.

Finally, this figure has two inequivalent glide lines. Can you specify the reflection and slide lines? See page 522 to check your thinking.

Symmetry Breaking

Our discussion of symmetry would be incomplete if we did not also talk about symmetry breaking. Although there is indeed “something in the human mind that is attracted to symmetry” and symmetry is one important factor in most people’s sense of beauty, at the same time, perfect symmetry is repetitive and predictable. I will not attempt a rigorous definition of symmetry breaking but, rather, will give some examples and descriptions. From one perspective, symmetry breaking occurs when we take a pattern that has “lots” of symmetry and do something to reduce the symmetry. Consider Figure 9.28(a). This figure has four lines of symmetry and fourfold symmetry. If we add two colors [Figure 9.28(b)], we still have fourfold symmetry but now we have no lines of symmetry. Thus, we have broken the symmetry. One basic quilt design can be modified in many ways that break the symmetry. If we color in the center [Figure 9.28(c)], we now have only twofold symmetry and no lines of symmetry. Finally, we can redraw the lines in the middle to give a pinwheel effect. This design now has fourfold symmetry.

These figures illustrate another way of describing symmetry breaking: A symmetry is expected, but that expectation is not met. If you look closely at

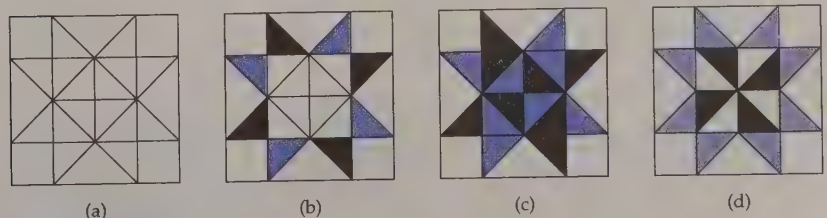


FIGURE 9.28

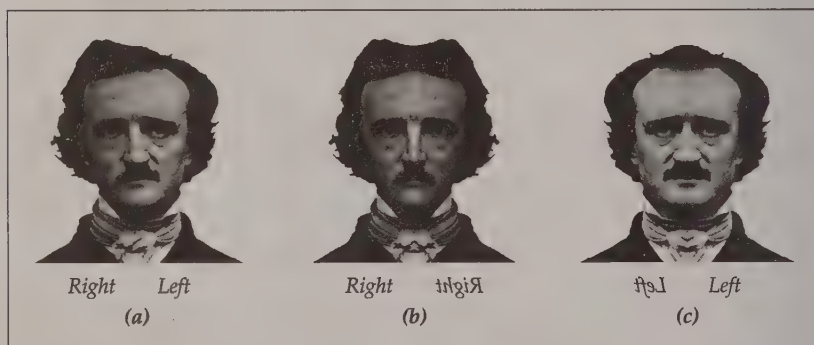


BEYOND THE CLASSROOM

The notion that symmetry is a mathematical concept as well as an aesthetic one was brought home to me several years ago. I was asked to facilitate a four-day workshop for college professors who were invited to Dartmouth College to get a taste of several different courses designed to bridge mathematics and other disciplines—courses that would be more appealing as general education mathematics courses than the traditional offerings. In my workshop, there were 24 college professors: about half were Ph.D. mathematicians and half were art professors. My job was to keep them from killing each other. Seriously, part of my job was to help them recognize what they had in common. One of the many things they had in common was a love of patterns and a love of breaking patterns.

Oriental carpets, which often have very intricate designs and plenty of symmetry, you will see that there is a playfulness with symmetry that results in many intriguing patterns within the larger, more obvious patterns. “In art, too, it seems that the approximation of symmetry, rather than its precision, teases the mind as it pleases the eye.”⁴

In nature, symmetry is always imperfect, just as you will find in the next chapter that measurements, by definition, are always approximations. However, just as a mathematical model of a problem will nicely represent the important or “big” ideas of a problem or phenomenon, while at the same time idealizing or simplifying other parts, so too do physicists, chemists, biologists, and other scientists often speak in terms of symmetries in their work as though the symmetries were perfect. Did you every wonder why our faces are not perfectly symmetric but rather “almost” symmetric. The figure below shows a photograph of Edgar Allen Poe. The middle figure shows what he would look like if you reflected his right side, and the figure at the right shows what he would look like if you reflected his left side.



MATHEMATICS

Where else have you seen tessellations in this book? Figures 9.1 and 9.2 are tessellations, and the three brick designs in Investigation 9.11 are tessellations.

Tessellations

For thousands of years, people have chosen to cover surfaces (for example, floors, walls, tables, beds, or roads) with patterns. We will use transformational geometry, including our knowledge of symmetries of patterns to examine one kind of covering called tessellations. Look at the pictures in Figure 9.29, all of which are examples of tessellations. From these pictures, what do you think *tessellation* means? That is, a pattern that tessellates is one in which ...



LANGUAGE

The word *tessellate* comes from the Latin word *tessella*, which referred to small tiles that the Romans used for pavement and to decorate buildings.

In preparing for this book, I read more than 10 authors’ descriptions of tessellation. It was interesting to note that none of the definitions were identical, unlike the definitions of, say, isosceles triangle! However, all of the definitions had two elements in common: There can be no gaps, and there can be no overlap. We will say that a figure or a combination of figures *tessellates* the plane if a regular repetition of the figure or figures covers the plane so that there are no gaps and no overlapping of figures. Look at the four designs in Figure 9.29 on the next page. Do you see that all of these designs qualify as tessellations? However, only certain geometric figures and combinations of figures will tessellate, and mathematics holds the clue to why.

⁴<http://forum.swarthmore.edu/geometry/rugs/symmetry/breaking.html>.

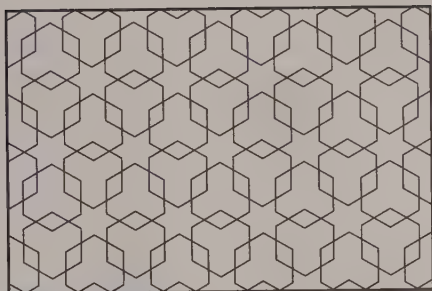
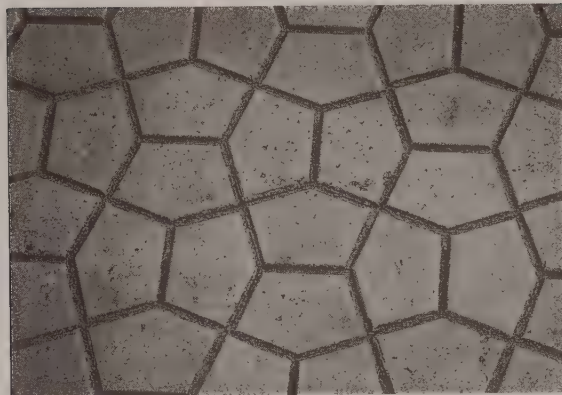
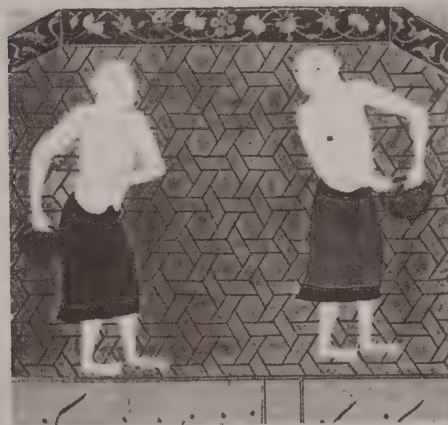


FIGURE 9.29



One difficulty that people commonly encounter when first making sense of this definition has to do with the borders. When discussing tessellations, we will not be concerned with whether the borders are smooth or straight. For example, if we were covering a floor with square tiles, unless both the length and the width of the floor were equal to a multiple of the length of the tile, the person laying the floor would have to cut the square to fit the edge of the room. Thus, when we say that a pattern tessellates, we are talking not about the edges of the design, but about the design itself.

We will investigate some basic ideas related to two-dimensional tessellations here. Exploration 9.7 carries these ideas further.

INVESTIGATION 9.12 Which Triangles Tessellate?

One starting point for investigating tessellations is to ask, which figures will tessellate? We will begin with the triangle (see Figure 9.30). What triangles tessellate? Just a reminder that you will learn more if you really do think before reading on. . . .



Note your first thoughts as predictions. Then make copies of some triangles and test your ideas. One simple way to do so is to fold a piece of paper in half, then in half again, and then in half again. Now if you make a triangle and then cut the figure, you will have 8 copies of the triangle.



FIGURE 9.30

DISCUSSION

Many students are surprised to find that all triangles tessellate! Did you predict this? If not, make some copies of different triangles and confirm this fact for yourself. Then come back to the text. Now that you realize that all triangles tessellate, can you justify this—that is, explain why? Work on this question before reading on. . .

Let us begin with a scalene triangle, shown in Figure 9.31(a). I have labeled the three angles of the triangle. If we rotate the triangle 180 degrees (or reflect it vertically and then horizontally), we can join the triangle and its image together [see Figure 9.31(b)]. The combining of the triangles creates a parallelogram. If we take this parallelogram and translate it, we now have four triangles [see Figure 9.31(c)]. Figure 9.32 shows that we can extend this pattern in all directions infinitely.

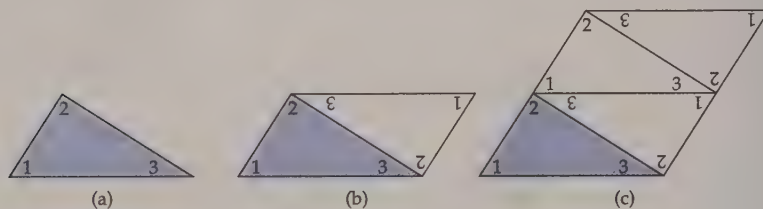


FIGURE 9.31

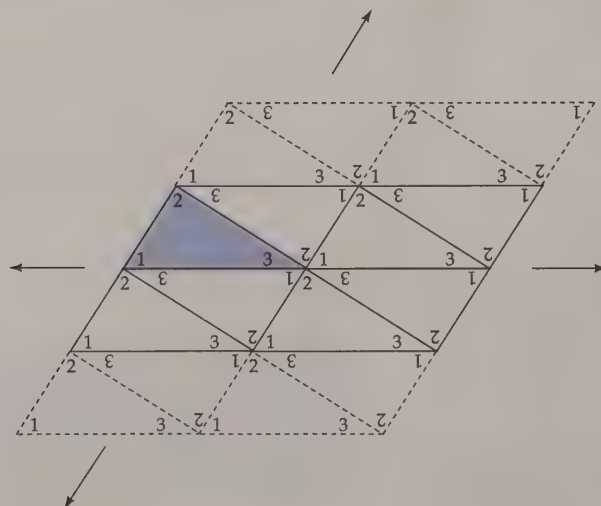


FIGURE 9.32



LEARNING

Recall the discussion of the van Hiele levels in Chapter 8. Where is your level of understanding of the justification that all triangles tessellate?

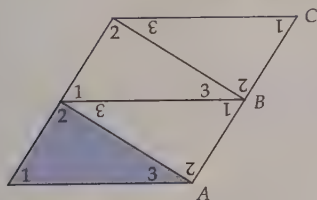


FIGURE 9.33

Our eyes tell us that it “looks as if” the triangle tessellates. We will now take some time to confirm what our eyes tell us. Again, this is not a formal geometry course, so we will give not a two-column proof but a more informal justification. Figure 9.33 below repeats Figure 9.31(c), but with some vertices labeled to make communication easier. We can see that we have angles $\angle 1$, $\angle 3$, and $\angle 2$ meeting at vertex B .

We know from Chapter 8 that the sum of the angles of any triangle is 180 degrees, and therefore $\angle ABC = 180$. However, this means that $\overleftrightarrow{AC}$ is a straight line. Looking back at Figure 9.31(d), we can see that there will be six angles at *any* vertex ($\angle 1$, $\angle 2$, $\angle 3$, $\angle 1$, $\angle 2$, and $\angle 3$), and the sum of these six angles is 360. As you might be thinking, 360 is an important number; that is, one complete revolution about a point is 360 degrees. Thus we can repeat this triangle in such a way that there are no gaps or overlap, and so the triangle tessellates.

In a similar fashion, we can show that any quadrilateral also tessellates the plane. Does this surprise you? If so, make a number of different quadrilaterals to convince yourself that, indeed, *any* quadrilateral does tessellate. Justification of this will be left as an exercise.

INVESTIGATION 9.13 Which Regular Polygons Tessellate?

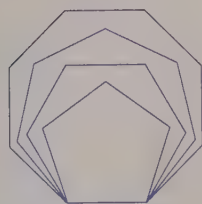


FIGURE 9.34

Another common starting point for understanding tessellations is to examine that subset of all polygons called regular polygons—that is, polygons in which all sides are the same length and all angles have the same measure. Which regular polygons will tessellate? Work on this before reading on. You might want to use Figure 9.34 as a template to make copies, or you might choose to apply your understanding of the properties of regular polygons. In either case, do some thinking on your own before reading on. . . .

DISCUSSION

It turns out that only three regular polygons tessellate: the equilateral triangle, the square, and the regular hexagon. (Note that an equilateral triangle is a regular triangle and that a square is a regular quadrilateral.) The reason has to do with the sum of the angles.

Whether or not you concluded that these three are the only regular polygons that tessellate, take a few minutes to see whether you can justify this conclusion—that is, explain why it is true. Then read on. . . .

If you recall from Chapter 8, the sum of the angles of any polygon = $(n - 2)180$, where n = the number of sides in the polygon. From this formula, we can find the measure of each interior angle of regular polygons, shown in Table 9.4. Can you see how these numbers help us to understand why only three regular polygons tessellate? Think before reading on. . . .

Table 9.5 helps us to understand why the equilateral triangle, square, and regular hexagon tessellate, but no other regular polygons will. If we place three pentagons at a common vertex, the sum of those angles is 324. This is not quite 360; however, there is not enough room to place a fourth pentagon at that vertex (see Figure 9.35). Similarly, if we place two octagons at a common

TABLE 9.4

Figure	Sum of angles (degrees)	Measure of each interior angle (degrees)
Equilateral triangle	180	60
Square	360	90
Regular pentagon	540	108
Regular hexagon	720	120
Regular octagon	1080	135

TABLE 9.5

Figure	Number of angles at common vertex	Sum of those angles (degrees)
Equilateral triangle	6	360
Square	4	360
Regular pentagon	3	324
Regular pentagon	4	432
Regular hexagon	3	360
Regular octagon	2	270
Regular octagon	3	405



FIGURE 9.35

vertex, the sum of the two angles is 270, and there is not enough room to place a third octagon at that vertex.

If you are playing the “what-if” game in your mind, you may have realized that $270 + 90 = 360$ and conjectured that a combination of octagons and squares *will* tessellate. This is true, as Figure 9.36 illustrates. There are many combinations of regular polygons that will tessellate. Mathematicians have names for certain subsets. One that we will consider here is the **semiregular tessellation**—a tessellation of two or more regular polygons that are arranged so that the same polygons appear in the same order around each vertex point. The tessellation in Figure 9.36(a) is a semiregular tessellation because the two figures are a square and a regular octagon. The tessellation in Figure 9.36(b) is not a semiregular tessellation because the two figures are a square and a non-regular octagon. As you can see, by varying the lengths of the sides of the octagons (that is, the ratio of the noncongruent sides), we can change the appearance of the tessellation.

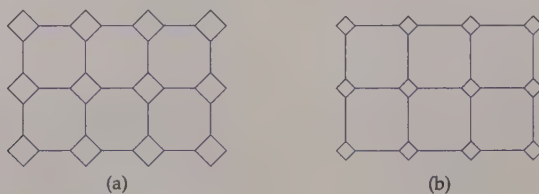


FIGURE 9.36

Exploration 9.7 offers many interesting ways to extend your understanding of tessellations.

INVESTIGATION 9.14 Generating Pictures Through Transformations

Let us apply our understanding of translations, reflections, rotations, and glide reflections to see how a tessellation figure is composed. Figure 9.37 shows one of my favorite tessellation figures, one that appears in many Islamic designs. Let's say this was a jigsaw puzzle and all the pieces were congruent. How would you put the puzzle together? I have numbered some of the pieces to make communication simpler. Make your own plan before reading on. . . .

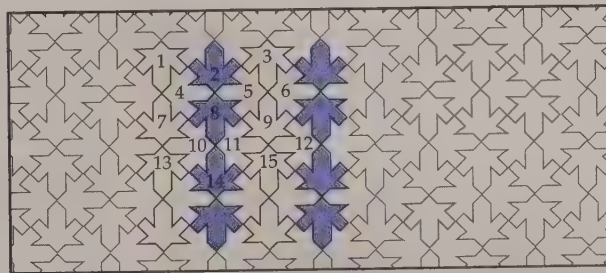


MATHEMATICS

Recall our discussion of unit in Chapter 5. We will also discuss unit in the next chapter when we discuss measurement. One of the goals of this course is for you to see that unit is not just "inch, foot, ounce, etc." What do you think "unit" means?

DISCUSSION

We have learned that if we have two congruent figures in the same plane, they can be mapped onto each other by some combination of translation, reflection, rotation, and glide reflection. In this pattern, each of the figures has line symmetry. As we look for patterns within this tessellation, we notice that each figure has many reflection neighbors and many rotation neighbors. For example, figures 1 and 7 are reflection images of each other; so are figures 2 and 8, figures 3 and 9, and figures 4 and 5, to name but a few. Similarly, each of these pairs are also rotation images of each other. If we consider figures 2, 4, 5, and 8 as a unit, we can generate the whole pattern by translating this unit in a diagonal direction. If you turn the paper 45 degrees, it is much easier to see that these four figures can easily generate the whole pattern!



Source: From *Introduction to Tessellations* by Dale Seymour and Jill Britton. © 1989 by Dale Seymour Publications, an imprint of Pearson Learning. Used by permission.

FIGURE 9.37

Symmetry for Three-Dimensional Objects

Thus far we have examined symmetry with two-dimensional figures. However, we see symmetry in many three-dimensional figures. Buildings often have symmetry; think of the Taj Mahal or the U.S. Capitol building. Everyday objects such as forks, nuts and bolts, and windows have symmetry. Many natural forms, including starfish, honeycombs, pyrite, and crystals, have symmetry.

Let us briefly examine symmetry of three-dimensional objects. We will focus on reflection and rotation symmetry.

When we defined reflection symmetry with two-dimensional figures, we used a line. Therefore, it should make sense that when we define reflection symmetry with three-dimensional figures, we will use a plane. Because we



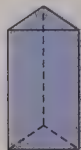
have added one more dimension to the figure (two to three), we add one more dimension to the instrument of reflection (one to two).

If there is a plane that can be drawn through a three-dimensional figure so that one half of the figure is a mirror image of the other, then the figure has reflection symmetry. The plane is called the **plane of symmetry**.

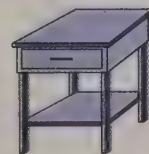
Similarly, when we defined rotation symmetry with two-dimensional figures, we used a point (a one-dimensional object). We define rotation symmetry with three-dimensional figures using a line (a two-dimensional object).

If there is a line that can be drawn, around which a three-dimensional figure can be rotated so that it coincides with itself, then the figure has rotation symmetry. The line about which the figure is rotated is called the **axis of symmetry**.

Figure 9.38 shows two three-dimensional objects that have reflection and/or rotation symmetry. In each case, determine and try to describe the plane(s) of symmetry and the axis (or axes) of symmetry. Then read on. . . 



Equilateral prism



Nightstand

FIGURE 9.38

The prism has 120-degree rotation symmetry about its axis, the line connecting the centers of the two bases. The prism has one plane of symmetry, a horizontal plane. The nightstand has one plane of symmetry, a vertical plane.

Summary

We have found that each of the transformations from Section 9.1 leads to a type of symmetry—translation, reflection, rotation, and glide reflection. One of the more important goals is for you to see symmetry as a movement—as a verb,

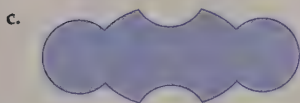
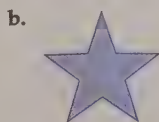
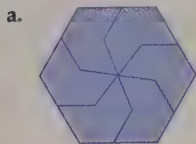
rather than a noun. When we see symmetry in an object, we do so by moving that object or parts of that object in our minds. Perhaps that is why symmetry is so appealing: It requires activeness on the part of our minds to appreciate it. When we find that an object has symmetry, we are using the notion of congruence; that is, we have found that we can move the figure (or part of the figure) and superimpose the figure on itself. We have seen that symmetry has many applications in the world of art (quilts, mosaics), in everyday life (wallpaper and tiles), and in science (symmetry at the atomic level). We have seen that symmetry helps us to understand better how and why tessellations work.

Finally, we have introduced the notion of three-dimensional symmetry, which has many important applications in everyday life and in science. I have talked with scientists about the usefulness of symmetry in their fields. A chemist told me that molecules of similar symmetry will give similar spectra and that advances in our mathematical understanding of symmetry have helped organic chemists to understand the structure of organic compounds. At a molecular level, the atomic structure of various elements and compounds determines how they will be packed together. As Marjorie Senechal has written, "if all we learn about symmetry is to identify it, we miss the whole point. Symmetry is an effect, not a cause."⁵

⁵ Marjorie Senechal, *On the Shoulders of Giants* p. 153. Reprinted with permission from *On the Shoulders of Giants: New Approaches to Numeracy*. Copyright © 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, D.C.

EXERCISES 9.2

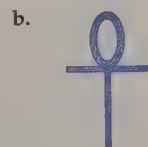
1. Describe the rotation and reflection symmetries in the figures below.



2. Describe the rotation and reflection symmetries in the figures below.



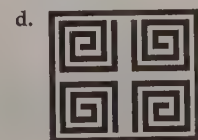
Maple leaf
Canada



"Life"
Ancient Egypt



"Unity is Strength"
Ghana

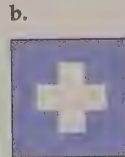


Inca design

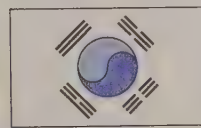
3. Describe the rotation and reflection symmetries in the flags below.



Barbados



Switzerland



Korea

4. Describe the rotation and reflection symmetries in the logos below.



Chevrolet

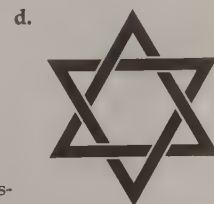


Volkswagen

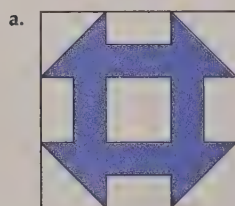
Sources: (a) Courtesy of Chevrolet Motor Division, General Motors Corporation. (b) Volkswagen logo courtesy of Volkswagen of America



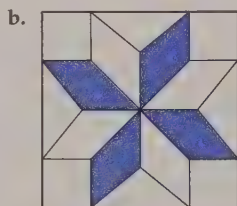
Source: The Octagon logo is a Registered trademark of The Chase Manhattan Corporation and is used here with its expressed permission.



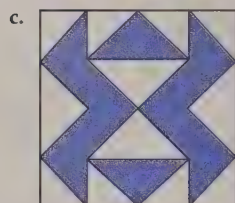
5. Describe the rotation and reflection symmetries in the quilt patterns below.



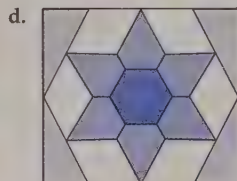
Churn Dash



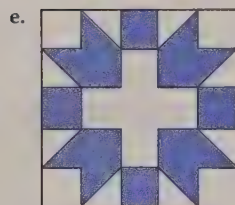
Eight Point Star



Fool's Puzzle



Diamonds in the Sky



Hearth and Home

6. a. Describe the rotation and reflection symmetries of the tangram pieces.
b. Make a design that has symmetry with some or all of the tangram pieces. Explain the symmetry.
7. Classify the pentominoes by the kind(s) of symmetry that they have. Explain the different subsets.
8. Describe the rotation and reflection symmetries of the ten digits in base 10.
9. Recall the family tree for the properties of quadrilaterals in Chapter 8. When we look at the symmetries of the quadrilaterals in the family tree, will that same family tree still hold true? That is, does each descendant quadrilateral have the same symmetries as its ancestor plus at least one more symmetry? Justify your answer.
10. Tell whether the statements below are true or false. If a statement is true, explain why. If it is false, provide a counterexample.
- If a figure has rotation symmetry, it must also have reflection symmetry.
 - If a figure has point symmetry, it must also have rotation symmetry.
 - If a figure does not have point symmetry, it cannot have reflection symmetry.

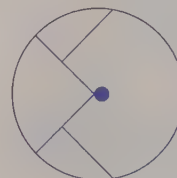
11. Create a figure (not one copied from this book) that has the symmetry described below. In each case, describe your thought processes, including ideas that did not pan out.
- 60-degree rotation symmetry
 - 120-degree rotation symmetry
 - 90-degree rotation symmetry
 - Two lines of symmetry
 - Three lines of symmetry
 - At least one line of symmetry and some kind of rotation symmetry
 - Reflection symmetry but not rotation symmetry
 - Rotation symmetry and exactly one line of reflection
12. a. Complete the figure following so that it has point symmetry but no other rotation symmetry.



- b. Complete the figure following so that it has two lines of symmetry.



- c. Complete the figure following so that it has two lines of symmetry.



- d. Complete the figure following so that it has point symmetry and no reflection symmetry.

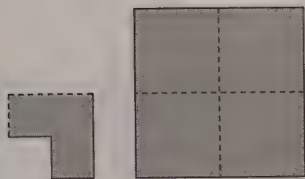


13. In each part, the square piece of paper shown at the right is folded along the dashed lines and then cut as shown at the left. Predict what the paper will look like when unfolded.

a.



b.



14. Sketch where you would cut the folded paper shown at the left in order to make the shape shown at the right when you unfold the paper. The dashed line represents the fold line.

a.



b.



c.

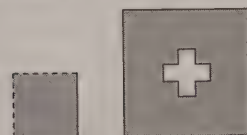


15. Sketch where you would cut the folded paper shown at the left in order to make the shape shown at the right when you unfold the paper. The dashed lines represent the two fold lines.

a.



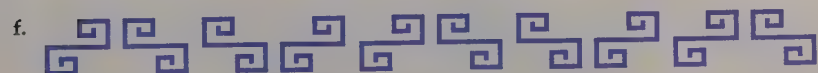
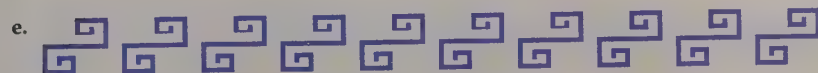
b.



c.



16. Describe the symmetries of the following strip patterns.



17. Describe the symmetries in the following pictures, taken at the Taj Mahal.

a.



b.



c.



d.



18. In each case, make your own original strip pattern with the symmetry indicated, and justify that it does indeed have this symmetry.

- a. ll b. lm c. ml d. mm e. $l2$ f. mg g. lg

19. Make the following strip pattern using grip paper or using a computer graphics program. Explain your solution path.



20. Describe the rotation and reflection symmetries in the Navajo designs as they are drawn below. Then, imagining them to extend to the right and left indefinitely, describe any translation symmetries.

a.



b.



c.



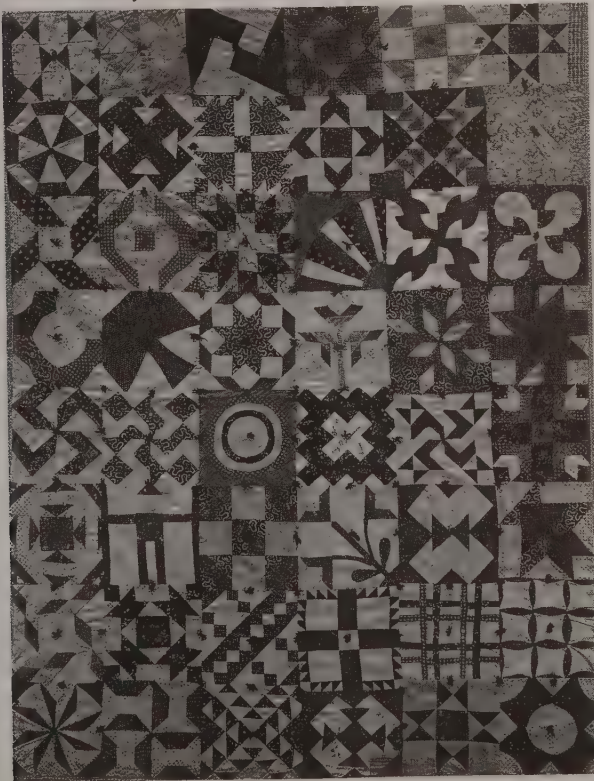
21. Make the designs using computer software. Describe how you made the designs.

22. a. Describe the symmetries of the quilt below at the left. Ignore the border.
 b. Describe the symmetries of the quilt below at the right. Ignore the border.



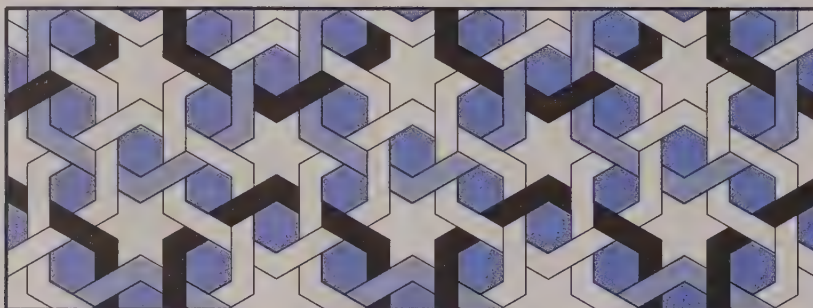
Source: Indian Pine Quilt, quiltmaker unidentified, embroidered initials BB, Maine, 1880–1890, 86" × 82". Cotton, with cotton embroidery. Collection of THE MUSEUM OF AMERICAN FOLK ART, New York, Gift of Cyril Irwin Nelson in memory of his grandparents, Guerdon Stearns and Elinor Irwin (Chase) Holden, and in honor of his parents, Cyril Arthur and Elise Macy Nelson. 1982.22.1

- c. Describe the symmetries of the quilt blocks selected by your instructor from the Sampler quilt below.



23. Look at the pattern below.

- Find a shape and an image of that shape after a translation. Describe the shape and specify the translation.
- Find a shape and an image of that shape after a reflection. Describe the shape and specify the reflection.
- Find a shape and an image of that shape after a rotation. Describe the shape and specify the rotation.
- Find a shape and an image of that shape after a glide reflection. Describe the shape and specify the glide reflection.



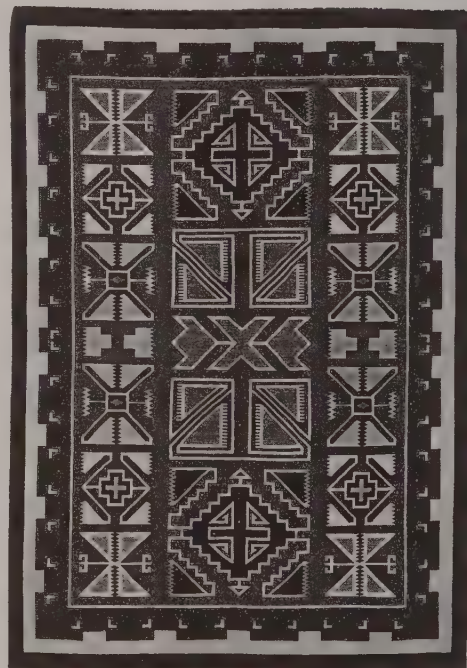
Source: From *Introduction to Tessellations* by Dale Seymour and Jill Britton. © 1989 by Dale Seymour Publications, an imprint of Pearson Learning. Used by permission.

24. Follow the directions for Exercise 23.

a.

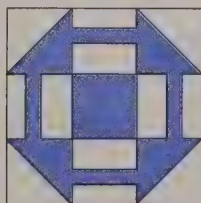


b.



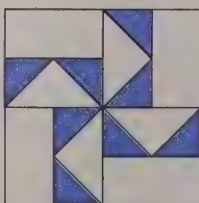
25. Give instructions for making the patterns below as though you were talking to someone on the phone. Use symmetries when possible.

a.



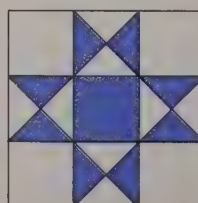
Shoo Fly

b.



Amish Design

c.



Star

26. Design a flag that has
- Rotation symmetry
 - Reflection symmetry
 - Rotation symmetry but no reflection symmetry
 - Reflection symmetry but no rotation symmetry
27. Examine a deck of cards and separate them into subsets that have the same kinds of symmetry.
28. The word "HIDE" has one (horizontal) line of symmetry. This means that if we flip the word over, it will still spell HIDE. Find another word that has horizontal symmetry.
29. The word "WITH" has one (vertical) line of symmetry if we write it vertically. Find another word that has vertical symmetry.

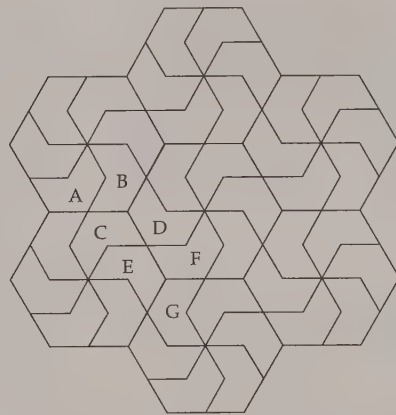
W
I
T
H

30. Find a two-dimensional or three-dimensional figure on campus or from your home that has some symmetries. Sketch the figure and describe the symmetries.
31. Find examples of children's toys that have symmetries.
32. Describe the symmetries in a nut and explain why it might have these symmetries.

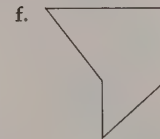
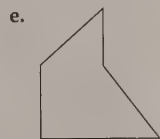
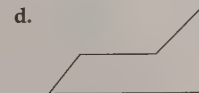
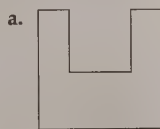


33. Look at the tessellation patterns in Figure 9.29. Describe the symmetries of each of the patterns.
- The design based on the letter C.
 - The Japanese pavement design
 - The Islamic design made from hexagons
 - The Islamic dancers
34. Select one pair of figures from Investigation 9.14.
- Describe a translation that will map that pair onto another pair.
 - Describe a reflection that will map that pair onto another pair.
 - Describe a rotation that will map that pair onto another pair.
 - Describe a glide reflection that will map that pair onto another pair.
35. Which of the 12 pentominoes will tessellate?
36. You have learned that a regular hexagon will tessellate. Below are two conjectures. There are three parts to what you turn in: your response: true or false; your justification of your response; your work—that is, the hexagons that you made up and tested to see whether they tessellated or not.
- If a hexagon has line or rotation symmetry, then it must tessellate.
 - All convex hexagons will tessellate.

37. Look at the tessellation below, a hexagon that has been decomposed into six congruent hexagons called chevrons.
- Can you identify two chevrons in which the one is a translation of another? Describe the translation.
 - Can you identify two chevrons in which the one is a reflection of another? Describe the reflection.
 - Can you identify two chevrons in which the one is a rotation of another? Describe the rotation.
 - Can you identify two chevrons in which the one is a glide reflection of another? Describe the glide reflection.



38. Which of the following figures tessellate? If you believe a figure tessellates, you need to show why. If you believe a figure does not tessellate, you need to explain why not.





SECTION

9.3

SIMILARITY

WHAT DO YOU THINK?

- What does it mean to say that two figures or two objects are similar?
- Are an object and its shadow similar? Why or why not?

CHILDREN

You have used the concept “similar” since you were young. “Similarity seems to be a very fundamental concept. Preschoolers understand that miniature animals, doll clothes, and play houses are all small versions of familiar things. The fact that even such young children know what these tiny objects are supposed to represent shows that they intuitively understand change of scale. Building and taking apart scale models of towers, bridges, houses, shapes of any kind give the child—of any age—a firm grasp of this idea.”⁶

In Section 9.1, we explored several transformations that let us change the position of an object but preserve its size and shape. That is, all of the images were congruent to the original figure. In this section, we will investigate a transformation in which the size of the figure is changed but the shape is not; we will call these transformations **similarity transformations**.

In mathematics, the term *similar* has a very specific meaning. I believe that if you participate in the development of the idea, rather than just having the mathematical definition presented to you, you will come to own the concept and the related properties. Therefore, consider this opening question: What does it mean to say that two figures or two objects are similar? Can you think of real-life objects that are similar? Think and then read on. . . .



A response that many students articulate in one way or another is that two objects that are similar are different in size but look the same.

One of the more common everyday uses of *similar* is in photocopying, when enlarging or reducing a figure. Also, children (and adults) build scale models, which are mathematically similar to what they represent (see Figure 9.39).



FIGURE 9.39

⁶Marjorie Senechal, *On the Shoulders of Giants* pp. 141–142. Reprinted with permission from *On the Shoulders of Giants: New Approaches to Numeracy*. Copyright © 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, D.C.

INVESTIGATION 9.15 Developing an Understanding of Similarity in Mathematics

The purpose of this investigation and Explorations 9.11 and 9.12 is to help you construct an understanding of similarity that goes deeper than “one more definition or formula to memorize.” As you do the investigations and explorations, the notion of similarity as a transformation—that is, as a verb—is critical. One of the most common applications of similarity beyond the classroom involves enlarging or shrinking a figure (as in photocopying) or an object (as in scale models).

Let us begin our investigation of similarity by using the notion of *enlarging* and *reducing* on a copy machine. When a photocopier enlarges or reduces a diagram or picture, the enlargement or reduction is similar to the original. Connecting to our statement that similarity is a transformation, each of the figures in the left column of Figure 9.40 has been transformed by a similarity mapping to the figure on the right. Most people would agree that the members of each pair are similar.

Can you go beyond “same shape” to develop a definition for similar figures? It may help to think about commonalities among all four pairs of figures. Write down your thoughts or conjectures before reading on. . . .



FIGURE 9.40



FIGURE 9.41

DISCUSSION

As in our investigation of transformations and symmetry, a “hands-on” approach can be quite instructive. Trace the figures and cut them out or make a photocopy of the page and cut out the eight figures. Place the original on top of the enlargement, so that two sides of the smaller figure lie on top of the corresponding sides of the larger figure, as shown in Figure 9.41.

Now what do you notice? Does this activity match your definition, does it contradict it, or does it give you some new ideas? If one of the latter is true, revise your definition of similarity. Then read on. . . 

Angles In Chapter 8, we defined two polygons as being congruent iff all corresponding angles were congruent and all corresponding sides were congruent. What relationships do you notice between angles and sides of the similar figures that might enable us to define two polygons being similar? Write down your present hypothesis before reading on. . . 

It can be easily determined that in each of the four cases in Figure 9.40, the corresponding angles of the similar figures are equal. If you worked with the pattern blocks in Exploration 9.11, or with the reptiles in Exploration 9.13, you may have come to the same conclusion. You can confirm this relationship either with a protractor or by placing the smaller figure on top of the larger one and verifying by inspection.

Sides What about the sides? Obviously, the sides are not congruent. What relationship do they have? Think about this before reading on. . . 

One observation is that if both figures are aligned as shown, then corresponding sides are parallel. Another possible observation involves the lengths. Measure the lengths of corresponding sides. Do you notice anything? Do this before reading on. . . 

It turns out that the lengths of corresponding sides are proportional. What does that mean? Can you explain that statement to someone who doesn't understand it? If you don't understand it yourself, go back and review the discussion of proportion in Chapter 6 before reading on. . . 

When we say that the lengths of corresponding sides are proportional, this means that if we find the ratio of the lengths of two corresponding sides, the ratio of the lengths of every other pair of corresponding sides will be the same number.

These observations lead us to a definition of similarity:

Two polygons are **similar** iff corresponding angles are congruent and corresponding sides are proportional.

We use the symbol $\sim$ to denote similarity. For example, in Figure 9.42, $\triangle ABC \sim \triangle XYZ$.

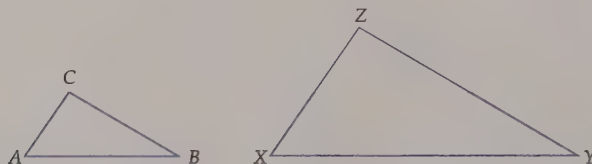


FIGURE 9.42

INVESTIGATION 9.16 Making Similar Drawings

The notion of enlarging and shrinking animals and humans has fascinated people for ages. *Gulliver's Travels* by Jonathan Swift is an ageless

classic. Movies dealing with this include *The Incredible Shrinking Man* and *Honey, I Shrunk the Kids*. Knowing that a figure can be blown up and shrunk by making corresponding angles equal and corresponding sides proportional is fine if you have a simple polygon, but this process can be very cumbersome if you have a more complex figure. Consider the two pictures in Figure 9.43 (a simple sailboat and a shark) and do a 100 percent enlargement—that is, make the base of the new sailboat twice as long as the one in Figure 9.43 and the length of the new shark twice as long as the shark in Figure 9.43.

How might you apply our definition of similarity to draw larger pictures of each of these figures? Work on this yourself before reading on.



FIGURE 9.43

DISCUSSION

There are several possibilities. We will investigate two, both involving graph paper. What if we placed these figures over a grid—either traced them onto the graph paper or placed a sheet of transparent graph paper over the figures (see Figure 9.44)? How would this help?

Focus on the figure We have two choices. First, we can draw the similar but bigger figure on the same kind of graph paper but make each line twice as long. For example, the bottom of the sailboat is 7 units long in Figure 9.44, and so it would be 14 units long in the new picture. This works fine with horizontal and vertical lines, but what about the diagonal lines on the sailboat and the curved lines on the shark?

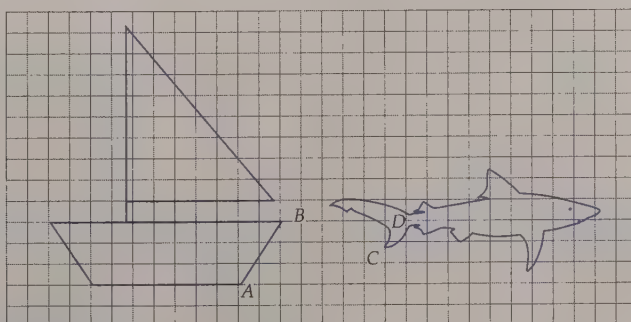


FIGURE 9.44

For example, we could measure the length of the diagonal line going up from the bottom of the sailboat, points *A* and *B*, and we could measure the

length of the curve from one point of the shark to another point, marked C and D in Figure 9.44. But how would that help? Think about it, because the answer involves applying our understanding of similarity. Then read on. . . 

Because the angles must be the same in similar figures, we can measure the angle and the distance. We keep the angle the same but make the distance twice as long. We can accomplish the same goal, in a simpler fashion, however, by connecting the idea of slope. That is, to get from A to B , we go 2 units to the right and 3 units up. If we increase these figures by 100 percent (go 4 units to the right and 6 units up), we will have the same angle on our bigger sailboat, and the distance will be 100 percent longer. Check it out! We use the same idea for the shark.

Focus on the paper Another strategy would be to use a piece of graph paper with the distance between the lines twice as long. Then we could use the same number when counting on the bigger grid; for example, to get from A to B on the sailboat, we would move 2 units to the right and 3 units up. We can copy the shark grid by grid also. We are essentially checking to see that each grid is “similar,” and this can be done in more than one way. For example, look at the top left-hand square in both pictures in Figure 9.45. In both cases, we start at the top left-hand corner, and we meet the right side of the square about one-fourth of the way down. To check our work, we use proportional reasoning. How do we do this?

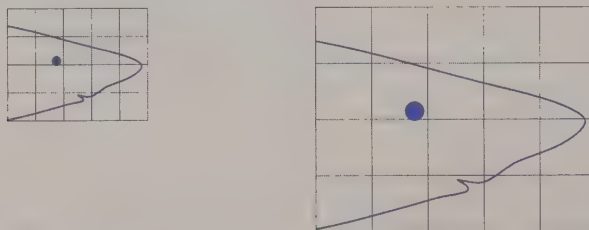


FIGURE 9.45



FIGURE 9.46

The line that we draw divides the square into two regions. We visually check to see if the ratio of the larger to the smaller region is the same in both cases. Figure 9.46 shows a case in which the artist would decide that it “wasn’t right” and would do it over.

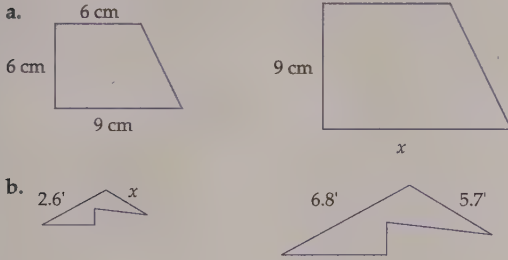
Summary

In this section, we have investigated the idea of similarity so that your thinking about this concept could move up the van Hiele scale—that is, beyond “similarity means they look alike.” We have also looked at how some ways to draw similar figures connect to the concept of similarity. In the explorations, you will find other ways to look at the idea of similarity.

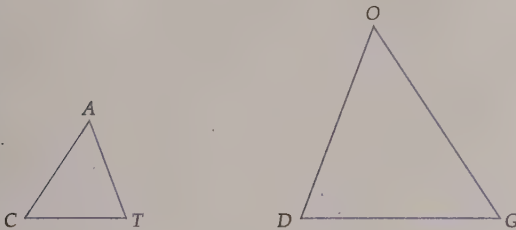
In this chapter, we have investigated only a small subset of the field of transformational geometry. Another field, called topology, examines transformations in which one shape can be turned into another shape by stretching, shrinking, bending, and/or twisting, but without making any cuts or tears in the figure. Thus, a square and a triangle are topologically equivalent. Recall the research showing that young children are not able to copy a square and a triangle accurately. Mathematically, we say that young children intuitively understand topological mappings before they understand congruence mappings. Other applications of topology include understanding networks and maps.

EXERCISES 9.3

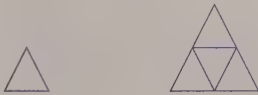
1. In each case below, the two polygons are similar. Find the length of the side labeled x .



2. The two triangles below are similar, but they have gotten mixed up. Determine the corresponding parts and explain how you did so.



3. The figure at the right below has been created from four triangles that are congruent to the triangle at the left. Are the two triangles similar? Explain why or why not.



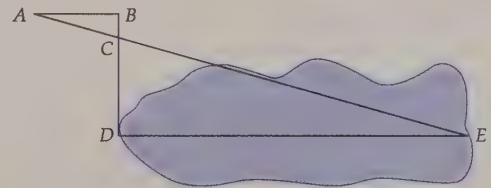
4. For each of the questions below, answer yes or no. In either case, justify your response.
- Are all isosceles triangles similar?
 - Are all equilateral triangles similar?
 - Are all squares similar?
 - Are all rhombuses similar?
 - Are congruent polygons similar?
5. Gerald has a photo that is 8×11 and a frame that is 4×6 .
- Explain to him why we cannot reduce the photo so that it will fit perfectly in the frame.

- Explain to him his options for cutting so that the reduced photo will fit.

6. When we use an overhead projector in a classroom, is the image on the screen similar to the object on the overhead projector?
7. Below is a map of Utah. If the actual distance across the bottom of Utah is 275 miles, determine the scale of the map. That is, 1 inch = x miles, or 1 cm = x miles.



8. The diagram below shows how we can find the distance across a lake using the principle of similar triangles.



- Explain how points C , B , and A would be determined, as though you were talking to someone who understands the idea of similar triangles but has never heard of this method before.
 - Explain why $\triangle ABC \sim \triangle EDC$.
 - If $AB = 324$ feet, $BC = 103$ feet, and $CD = 212$ feet, what is the distance across the pond?
9. Explain how you could use shadows and similarity to find the height of a building.
10. A common standard setting for enlarging on a copy machine is 121 percent. What does this mean?
11. Find a quilt block that contains similar figures. Explain why the figures are similar.
12. Find a tessellation pattern that contains similar figures. Explain why the figures are similar.

CHAPTER SUMMARY

- Both congruence and similarity are kinds of transformations.
- Congruence can be defined in Euclidean terms (the classical definition) and in terms of congruence mappings.
- There are different ways to perform and describe different transformations. Which one(s) we use depends partly on preference and partly on purpose.
- Translations, reflections, rotations, and glide reflections are congruence transformations—the figure is moved, but its shape and size are not changed.

5. The geometric transformations of translation, reflection, rotations, and glide reflection are operations, with many similarities to the operations of addition, subtraction, multiplication, and division.
6. Symmetry has to do with congruence within a figure.
7. The many symmetries in geometric figures include translation, reflection, rotations, and glide reflection symmetry. Many figures have more than one kind of symmetry.

Basic Concepts

Section 9.1 Congruence Transformations

Four transformations

In everyday language: slide 493 flip 493
turn 493 shrinking 493

In mathematical language: similarity 493
translation 496 reflection 498 rotation 499

Expressing translations:

congruence transformation 493
similarity transformation 493
image 495
using taxicab language 496
specifying the distance and the angle 496
vector 496 translation 496
line of reflection 498 reflection 498
center of rotation 499 rotation 499

composite transformation 500

glide reflection 500

congruent 504

Given two congruent figures on the same plane,
one can be mapped onto the other by some
combination of transformations 503

Section 9.2 Symmetry and Tessellations

line of symmetry 512

vertical line symmetry 516

horizontal line symmetry 516

diagonal line symmetry 516

Symmetry for two-dimensional figures

translation symmetry 512

reflection symmetry 512

rotation symmetry 512

point symmetry 515

glide reflection symmetry 518

Symmetry for three-dimensional objects

tessellation 524 tessellate 524

semiregular tessellations 528

plane of symmetry 530

axis of symmetry 530

Section 9.3 Similarity

similar 540

Solutions for Investigation 9.10, pages 518-520

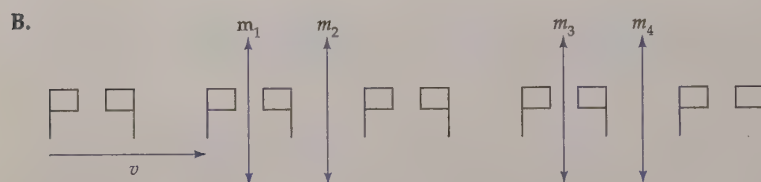
v = translation vector

m = mirror line

$\cdot$ = center of rotation

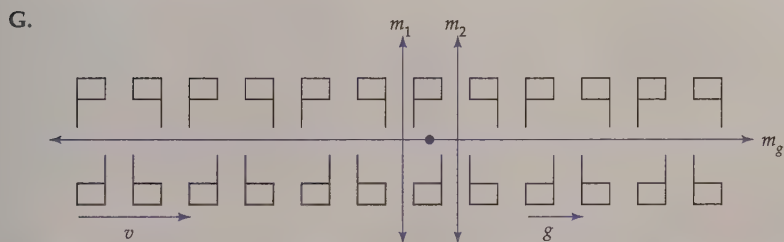
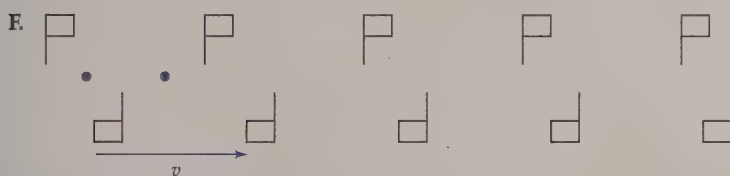
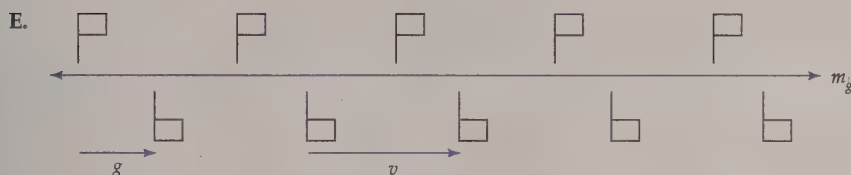
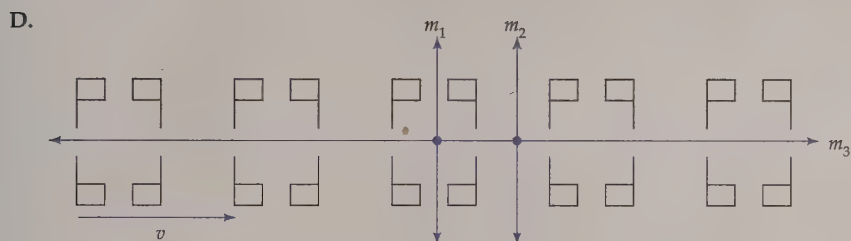
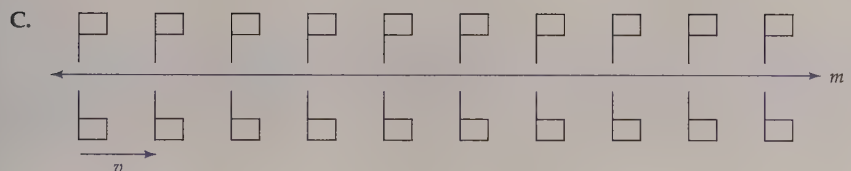
g = glide vector

m_g = mirror line for glide reflection



m_1 and m_3 are equivalent.

m_2 and m_4 are equivalent.



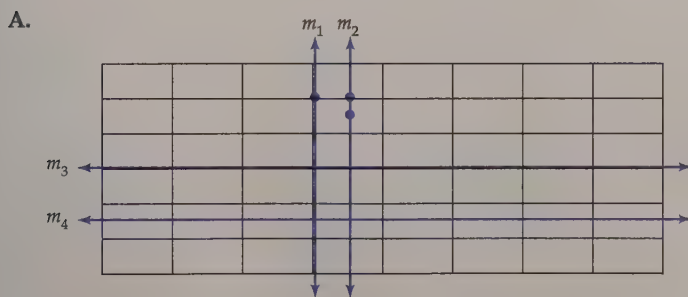
Solutions for Investigation 9.11, pages 521-523

m = mirror line

$\cdot$ = center of rotation

g = glide vector

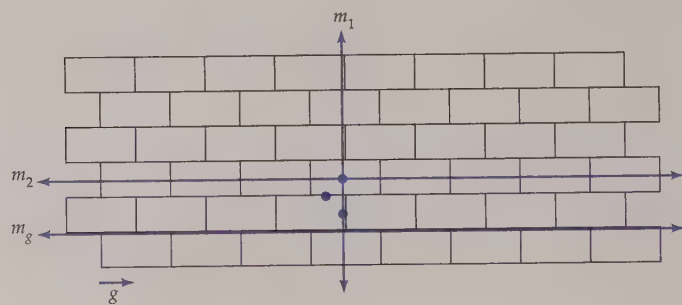
m_g = mirror line for glide reflection



Translation vectors



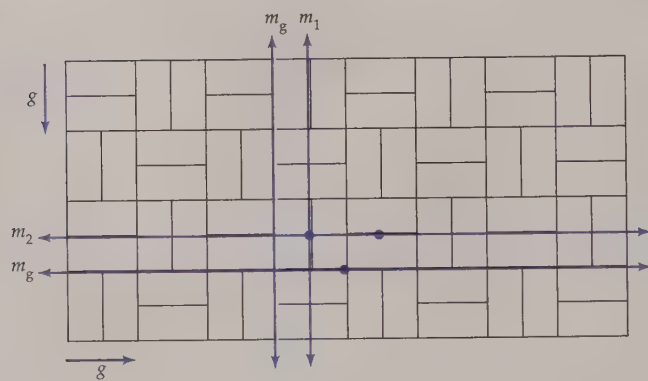
B.



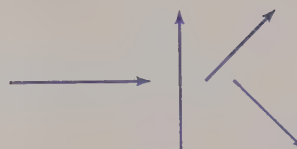
Translation vectors



C.



Translation vectors



Geometry as Measurement

- ◆ 10.1 Systems of Measurement
- ◆ 10.2 Perimeter and Area
- ◆ 10.3 Surface Area and Volume

What do you think of when you hear the term *measurement*? In what ways do you use measurement, and in what ways are you measured? We measure and are measured all the time. When we ask how much or how many, we are often asking a question requiring a direct or indirect measurement:

- How much time will it take?
- How far is it from here to there?
- How much paint do we need to paint the house?

Long before they start school, children are fascinated by measurement, and some of their measurement questions leave parents confused, too:

The child asks, "How long until we get there?" The parent says, "One hour." The child asks, "Is that long?"

Or the parent divides the candy bar into two pieces (measures), and one child says, "His half is bigger than mine."

By the end of this chapter, you will come to understand that measurement is far more than just an examination of the quantities we use in everyday life, such as length, area, volume, or time. You will want to refer to the NCTM measurement standards before you have your children investigate this topic.

In this chapter, we will investigate several important aspects of measurement.

Big Idea	Example/Elaboration
1. Most objects have multiple attributes that are measurable.	Consider a garage that is not attached to a house. We might want to know its perimeter, for a fence; or the surface area of the sides, to paint it; or its volume, to know how much it will hold.
2. There is a process that we use to measure all attributes.	This three-step process is the same whether we are measuring distance or density.
3. The units that we use are arbitrary as opposed to absolute.	All units were invented by people.
4. Virtually all measurements are approximations.	There is no “exact” height of a building. How we report the height depends on the precision that is needed: 12 m, 12.2 m, 12.21 m, 12.214 m, etc.
5. Precision and accuracy are not the same thing.	A measurement can be very precise, but inaccurate or it can be accurate, but not very precise.
6. Different attributes of an object are generally not related in simple ways.	Doubling the area of a yard does not necessarily double the length of the fence around the yard.

SECTION

10.1

SYSTEMS OF MEASUREMENT

WHAT DO YOU THINK?

- What does it mean to measure?
- What are some difficulties we encounter when measuring?
- When do we need our measurements to be precise, and when it is acceptable to have just a “rough” measurement?
- How do we measure heights of mountains?

Visualize yourself measuring different things—for example, how far to your parents’ house, how much paint you would need to buy to paint a house, how much a package weighs. We might want several measures of something. Consider a room. If we wanted to stencil the top of the wall around the room, we would want to know the room’s perimeter. If we wanted to paint the room, we would want to know its surface area. If we wanted to store things in the room, we would want to know its volume. Think of the process of measuring in each of these situations. What is common to all of them? Think and then read on. . . . 

In each case, we must

1. Identify what we want to measure—that is, identify the *attribute* we want to measure.
2. Select or determine the *unit* we will use to tell us how much.
3. Determine the amount we have in terms of the units we have chosen.

Keep these three aspects in mind as you read the text and do the explorations.

Development of Measurement Systems

Measuring is probably one of human beings’ earliest forms of mathematical activity. In the following pages, we will examine the development of the units and systems of measurement that people commonly use in everyday life.

Time One of the first quantities that people measured was time. Many thousands of years ago, people developed ways to determine the number of days in a year. Why do you think they wanted to know this? Think before reading on. . . . 

There are many reasons, such as wanting to know when to celebrate certain rituals, when to hunt, or when to plant. Early divisions of the day were probably marked by three significant times: sunrise, midday, and sunset. Later, midnight became significant too. We say that 1 day = 24 hours, 1 hour = 60 minutes, and 1 minute = 60 seconds. However, this has not always been so. Where do you think 24 and 60 came from? Think and then read on. . . . 

LANGUAGE

The origins of our words for the days of the week can be traced to the Babylonians and correspond to the seven "planets" they had identified: Sun, Moon, Mars, Mercury, Jupiter, Venus, and Saturn. Which ones do you recognize? What might have happened to the names of the other days?

Over 3000 years ago, the Babylonians divided the day into 12 hours and the night into 12 hours. However, the length of an hour depended on the time of year; that is, in the winter, a day hour was shorter than a night hour. At some point, the Greeks decided to divide the entire day into 24 equal parts, but it was not until around 1330 that the hour was standardized—that is, that an hour in January and an hour in June were equal, and an hour in Stockholm and an hour in Venice were equal.

For centuries, people used the sundial to measure the passage of time. The earliest known sundial comes from Egypt and dates from about 1500 B.C. The sundial, of course, has several limitations: It is not useful on cloudy days or at night, and it is not useful for small amounts of time. An early attempt to address this limitation was the water clock. The first water clock came from Egypt and dates from about 1400 B.C., and the first evidence of an hourglass is from ancient Rome in about 300 B.C. Columbus used a sand clock to measure time on his voyage.¹ It was not until the 1700s that the mechanical watch (with springs) was both accurate enough and inexpensive enough to be used outside of scientific experiments. The present definition of a second is the duration of 9,192,631,770 periods of the cesium-133 atom!² Believe it or not, there are actually machines that can count these periods. The ability to measure accurately time intervals of one-millionth of a second is crucial in fields like particle physics.

Length The history of the various units that different peoples have selected to measure length (see Figure 10.1) is quite fascinating, even to those who are not "math people."

The earliest recorded linear unit of measurement was the cubit, the distance from the point of the elbow to the outstretched tip of the middle finger. The word *cubit* is derived from the Latin *cubitus*, meaning "elbow."

It should not be surprising that the foot was one of the units of measurement in ancient times. The Roman writer Plutarch stated that the foot was based on the actual length of Hercules' foot. It is likely that the French foot was originally the actual measurement of the length of King Charlemagne's foot.³ In the tenth century, King Edgar I decreed that the yard would be the distance from the tip of *his* nose to the tip of the middle finger of *his* outstretched arm. The reason behind the decree was a desire to regulate trade in textiles, so that 1 yard of cloth would be approximately the same in all parts of England.

LANGUAGE

In the language of King Henry I (1496), the word for yard literally meant "stick." Thus our term *yardstick* is redundant. This is similar to speaking of the river that flows along the border between Texas and Mexico as the Rio Grande River. In Spanish, Rio Grande literally means "big river."

¹ Daniel Boorstin, *The Discoverers* (New York: Random House, 1983), p. 34.

² H. Arthur Klein, *The World of Measurements* (New York: Simon and Schuster, 1974), p. 163.

³ Terry A. Richardson, *A Guide to Metrics* (Ann Arbor, MI: Prakken Publications, 1978), p. 2.

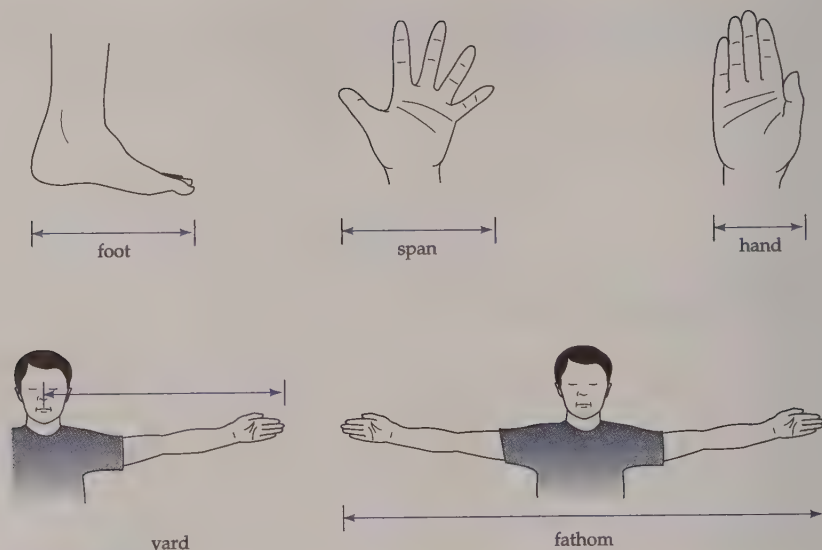


FIGURE 10.1



CHILDREN

Virtually all mathematics textbooks now have young children first measure lengths with nonstandard units. For example, they might ask students to find how long the desk is in terms of paper clips. Why do you think teachers are being encouraged to start with nonstandard units?

There are several reasons for doing this. First, it helps to keep the focus of the students' thinking on the attribute that is being measured and

learned. If you did Exploration 10.2, you may have seen this concept in action. For example, it is much simpler (conceptually) to line up and count paper clips than it is to line up and count units on a ruler. Second, the answers can be more understandable to the children—for example, the length of the soccer field in terms of footsteps or the number of body lengths of one student in the class.

The Romans divided 1 foot into 12 units (*uncia*, from which our word *inch* comes). The Roman pace consisted of 5 feet—that is, the distance of a left step and a right step—and their mile (*milia passuum*) was 1000 paces. Do you see a connection between their mile and our mile? When Alexander the Great conquered the known world, he had professional pacers to measure how far his army had traveled so that their maps would be accurate. These pacers were trained to make each step virtually the same length regardless of terrain: flat, uphill, or downhill.

Weight Two ancient units for weight were the barleycorn and the seed of the carob plant (from which comes the word *carat* for the measure used to weigh gold and diamonds). Over the centuries, three different systems of weights evolved: the avoirdupois system for everyday use, the troy system to weigh precious metals and gems, and the apothecaries' system for very small amounts. However, conversion was confusing. For example, in the troy system, there were 12 ounces in a pound, whereas in the avoirdupois system, there were 16 ounces in a pound!



HISTORY

The familiar nursery rhyme Jack and Jill was written as a protest against King Charles of England's taxation of jack(pots) of liquor. King Charles not only lost his crown, he lost his head! (He was beheaded.)

Volume and capacity Over time, two related systems of measure for what we call volume evolved, depending on whether the thing being measured was dry or wet. For dry volume, some basic units were the ounce, pint, quart, peck, and bushel. For liquid volume, there were a host of units. Thomas Jefferson almost certainly memorized the following ditty, which was commonly used in the eighteenth century to teach schoolchildren the units:

Two mouthfuls are a jigger; two jiggers are a jack; two jacks are a jill; two jills are a cup; two cups are a pint; two pints are a quart; two quarts are a pottle; two pottles are a gallon; two gallons are a pail; two pails are a peck; two pecks are a bushel; two bushels are a strike; two strikes are a coomb; two coombs are a cask; two casks are a barrel; two barrels are a hoghead; two hogheads are a pipe; two pipes are a tun—and there my story is done!⁴

Which measures do you recognize?

Temperature In 1714, a German instrument maker named Gabriel Fahrenheit made the first mercury thermometer. He designated the lowest temperature he could create in the laboratory as 0° and the normal temperature of the body as 98°. On his scale, the freezing point of water is 32° and the boiling point of water is 212°.

Let us now examine the metric system, with an emphasis on understanding its structure.

The Metric System

The idea of a system of measures based on powers of 10, which is what the **metric system** is, was first proposed in 1670 by Gabriel Mouton, from Lyons, France. Over the next 100 years, many scientists and nonscientists made various proposals for a uniform system of measurement. Before the development of the metric system, there were literally hundreds of systems of measurement in Europe alone. What is particularly important is that they were not uniform, so that a bushel in one location was not the same as a bushel in another location. These differences were often used by the rich to exploit the poor, who obviously resented this. One of the first acts of the French government after the French Revolution in 1789 was to develop a uniform system of measurement so that the rich could not cheat the poor. In 1793, the French Academy of Sciences proposed a new metric system for *all* units of measurement.

The United States has not officially adopted the metric system, and its adoption is not imminent. (Our system is called the **U.S. customary system**.) However, metric units are finding their way into everyday life—for example, some gas stations sell gasoline by the liter, some road markers now give distances in kilometers as well as miles, and most soda is now sold by the liter. Furthermore, international travel is so common that many Americans encounter the metric system in their travels (even to Canada and Mexico), and most industries that compete in international markets have gone metric, since most of the rest of the world uses the metric system. Thus, it is helpful to have a rough idea of the metric system.



HISTORY

In 1785, Congress passed Thomas Jefferson's proposal for a decimal system of currency.

On July 4, 1790, Thomas Jefferson presented to Congress a report on weights and measures that included two proposals: the first a complete transformation of our system of weights and measures based on powers of 10, and the second a plan for a partial transformation. George Washington supported Jefferson's efforts. However, the Congress decided to adopt neither plan.

⁴S. Carl Hirsch, *Meter Means Measure: The Story of the Metric System* (New York: Viking Press, 1973), p. 29.

There is one feature of the metric system that it is helpful to know before we examine the different units. Once the French had determined the unit, whether it was meter, liter, or gram, they used Greek prefixes for *multiples* of this unit and Latin prefixes for *fractions* of this unit.

Several of these prefixes show up in familiar words; for example, a millennium is 1000 years, a century is 100 years, and a decade is 10 years.

Table 10.1 shows the metric prefixes and their relationship to the basic metric units. Note that in everyday life, we rarely use *deci-*, *deca-*, or *hecto-*.

TABLE 10.1

Prefix	Milli	Centi	Deci	Deca	Hecto	Kilo
Relationship to the basic unit	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	10	100	1000
Example	milliliter	centimeter				kilogram

Metric Length

The standard unit of length in the metric system is the **meter**, which was defined (most likely after much debate) as one ten-millionth of the length of the line that starts at the equator and goes to the North Pole through Barcelona, Paris, and Dunkirk. In other words, this distance was decreed to be 10 million meters, and the meter was the length that was one ten-millionth of that distance. How they decided upon 10 million and how they knew the exact distance to the North Pole is an interesting story in itself! The most commonly used metric units of length are, from largest to smallest, *kilometer*, *meter*, *centimeter*, and *millimeter*. Most yardsticks and rulers sold in the United States have inches and feet on one side and metric measurements on the other.

Becoming Comfortable with Metric Measurements

What if you heard that someone is 182 centimeters tall and weighs 80 kilograms? Most Americans can easily visualize 6 feet and 176 pounds, the U.S. customary equivalents of the metric measures given in the first sentence, but are less comfortable with measurements given in metric units. We will address this problem by providing some reference measures and conversion ratios.

The diagrams in Figure 10.2 give approximations for a meter, a centimeter, and a millimeter.

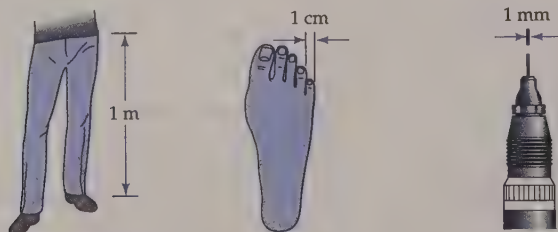


FIGURE 10.2

Some common conversion ratios are 1 meter $\approx$ 39 inches, 1 mile $\approx$ 1.6 kilometers, and 1 inch $\approx$ 2.5 centimeters.

INVESTIGATION 10.1 Developing Metric Sense

Try to do these problems yourself. Then read on. . . 

A. Insert the decimal point in the proper place.

- The diameter of a penny is 19 centimeters.
- The length of a page of notebook paper is 279 centimeters.
- The common adult height of an elephant is about 39 meters.

B. If the speed limit says 90 kilometers per hour, what is the speed in miles per hour?

DISCUSSION

A. The diameter of a penny is 1.9 centimeters.

- The length of a page of notebook paper is 27.9 centimeters.
- The common adult height of an elephant is about 3.9 meters.

B. We can use dimensional analysis (introduced in Chapter 1 and discussed in Chapter 5) to convert this speed to miles per hour. We can also use reasoning to deduce that we need to divide 90 by 1.6.

$$\frac{90 \text{ kilometers}}{\text{hour}} \times \frac{1 \text{ mile}}{1.6 \text{ kilometers}} \approx \frac{56 \text{ miles}}{\text{hour}}$$

Metric Volume

The metric system was designed so that we use the same units to measure volume (dry) and capacity (liquid). The standard metric unit for volume is the **liter**. A liter is approximately 34 ounces, or slightly more than a quart. The liter is also defined to be the volume of a cube whose sides are 10 centimeters. Thus, 1 liter is equivalent to 1000 cubic centimeters, abbreviated as cm^3 (see Figure 10.3).

However, a milliliter is also $1/1000$ of a liter. Therefore, we have the equivalence between liquid and dry measures: $1 \text{ cm}^3 = 1 \text{ ml}$. Most canned goods give the amount in milliliters. If a can contains 240 ml, how many liters is that? Do this before reading on. . . 

Because there are 1000 ml in a liter, we must divide by 1000. Therefore, $240 \text{ ml} = 0.24 \text{ liter}$, about one-fourth of a liter.

The creators of the metric system not only worked to connect liquid and dry measures, but also connected volume and mass. They defined a kilogram to be the mass of 1 liter of pure water at a certain temperature.

Metric Mass

The standard of mass in the metric system is the **gram**. Technically, mass and weight refer to different attributes. **Weight** refers to the force that gravity exerts on an object, and **mass** refers to the amount of matter that makes up the object. If you went to the moon, you would weigh about one-sixth as much as you do on Earth, but your mass would be the same. Because our planet is much larger than the moon, the force of gravity on your body is much greater on Earth. You

 1 cubic centimeter (cm^3)

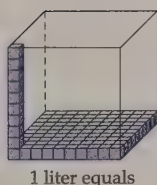


FIGURE 10.3

may recall hearing that because the force of gravity is less on the moon, we could jump six times as high on the moon as we can on Earth.

Some metric units of mass—kilogram, gram, and microgram—are encountered more often than others. One kilogram is approximately equal to 2.2 pounds. Where do you see metric mass in everyday life in the United States? Think before reading on. . . .

The net weight on most canned goods is generally given in both ounces and grams. This is one of the few places where one might see metric units for mass used in the United States at present. Some scales also have weights in metric units.

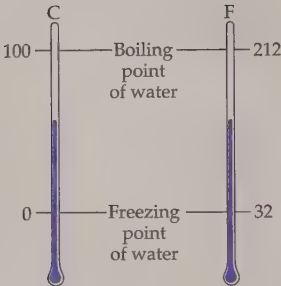


FIGURE 10.4

Metric Temperature

In 1742, a Swedish astronomer named Anders Celsius proposed a modification in the units of measurement that the Fahrenheit system used. He proposed that the reference points be the freezing point of water (0°) and the boiling point of water (100°) (see Figure 10.4). This *Celsius* system was also called the *centigrade* system (that is, 100 grades).

Time and Angles

As mentioned earlier, the French Academy of Sciences recommended that *all* known measures be based on powers of 10. Thus, they recommended changing the calendar so that there would be 10 months in a year, 10 days in a week, and 10 hours in a day. They also proposed that there be 400 degrees in a circle, which would mean that a right angle had 100 degrees. As you know, not all of these proposals survived! When Napoleon came to power, he repealed the law making metric measure compulsory. Although the French continued to use metric units for length, volume, and weight, they (happily) threw out the metric system for time and returned to the more familiar 7-day week, 24-hour day, 60-minute hour, and 60-second minute.

Measurement of Other Quantities

The quantities that we have examined in this section—length, volume, mass, time, and temperature—represent only a small subset of the quantities measured in our world. What other things do we measure? Why do we need to measure them? What are some of the units used to measure those things? What instruments are used? Think about things that we measure and make a table like Table 10.2 for those things you can think of. Do this before reading on. . . .

TABLE 10.2

What do we measure?	Why?	Units	Instruments
Noise	To study noise levels that can cause hearing damage	Decibels	Audiometer

The following is only a partial list: speed (miles per hour), fuel consumption (miles per gallon), light intensity (candela), electric current (amperes), effi-

ciency (production per unit of time), density (mass per unit volume), and infant mortality (deaths per 1000 live births).

Precision

There is another aspect of measurement that requires some attention. Virtually all measurements are approximations. For example, if you ask for the dimensions of a room, you might be told that it is 14 feet by 11 feet. If you ask for more precision—let's say you are buying a rug—you might be told that it is 14 feet 2 inches by 10 feet 11 inches.

The precision of our measurement depends on the reason we are measuring. Sometimes we want to measure amounts with much precision, and sometimes less precision is quite acceptable. For example, when selling cloth, clerks generally measure the length roughly and then add an extra couple of inches. However, if you go to a candy store or buy meat, the amount is generally weighed to the nearest tenth of an ounce, and cylinders in cars need to be accurate to the nearest thousandth of an inch. See also Explorations 10.1, 10.2, and 10.3.

Though many people use the words **precise** and **accurate** synonymously, the two concepts are not identical. A measurement can be very precise (for example, 34.628 meters) but be inaccurate (if the measurer made a mistake). Similarly, a measurement can be not very precise (for example, $16\frac{1}{2}$ feet) but be very accurate (that is, the actual distance is closer to $16\frac{1}{2}$ feet than to 16 feet or 17 feet).

Greatest possible error We can quantify the amount of precision in our measurement, and we use two terms to do so. First, we can determine the **greatest possible error (GPE)**. If you were to measure the width of a sheet of paper to the nearest millimeter, what would you get? Do so before reading on if you have a ruler close at hand. . . . 

It is 21.6 centimeters, or 216 millimeters. That is, my reporting of 216 millimeters means that I believe the true width is closer to 216 millimeters than to 215 millimeters or 217 millimeters. What do you think the greatest possible error would be in this case? That is, if someone were to measure the width of the paper to the nearest tenth of a millimeter, what would be the biggest difference between my measurement and the most precise measurement? Think and then read on. . . . 

The greatest possible error would be 0.5 millimeter—that is, one-half the unit that I used to measure.

Relative error The **relative error** tells us the percent error of the greatest possible error. The following example illustrates this concept. A measurement of 300 centimeters and a measurement of 3 centimeters both have the same greatest possible error, 0.5 centimeter, but 0.5 centimeter is a much bigger part of 3 centimeters than it is of 300 centimeters. What is the relative error in these two cases (3 centimeters and 300 centimeters)? Do this before reading on. . . . 

The relative error in the former case is

$$\frac{0.5 \text{ centimeter}}{3 \text{ centimeters}} \times 100 = 17\%$$

The relative error in the latter case is

$$\frac{0.5 \text{ centimeter}}{300 \text{ centimeters}} \times 100 = 0.2\%$$

Measurement as a Function

At a very basic level, measuring tells how much of something there is. From another perspective, measurement is the act of assigning numbers to amounts. The number refers to a specified unit. Do you realize that the operation of measuring an amount is a function? If you do, describe how it fits. If not, go back and review the discussion of functions in Chapter 2. Then write your thoughts before reading on. . . . 

In order for an operation to qualify as a function, each element of the input set must be mapped onto exactly one element of the output set. As we found in Chapter 2, the two sets need not be sets of numbers. When we measure, we take the amount to be measured (the input element) and determine the number that corresponds to that amount. Though you need not know that measuring represents a functional relationship between two sets, this is simply one more example of the underlying connectedness and relatedness of so many seemingly different mathematical ideas and topics.

Summary

In this section, we have explored what it means to measure an object, and we have realized that many objects have several attributes that can be measured—for example, weight, surface area, and volume. We have also examined where many of the units came from, both U.S. customary and metric. We have noted that many, if not most, of the numbers we use represent measurements. We have learned that most measurements are approximations and that there is a difference between precision and accuracy. Finally, we considered measurement from a functional perspective.

EXERCISES 10.1

- Think about each object below and describe all of the measurable attributes of that object. Next to each property, briefly describe why someone might want to measure that attribute.
 - a pond
 - a garage
 - a house
 - a carpet
 - a banana
 - a tree
- Select the unit below that you would use to express the following. Briefly explain your choice.

Length: millimeters, centimeters, meters, kilometers
 Volume: milliliters, liters
 Mass: milligrams, grams, kilograms

 - Your height
 - Your weight
 - The length of a butterfly
 - How much fluid there is in a drinking cup
 - The mass of a fingernail
 - The volume of a thimble
- Fill in the blanks:
 - 500 m = ____ km
 - 4.5 cm = ____ mm
 - 670 mL = ____ L
 - 3.6 L = ____ mL
 - 450 g = ____ kg
 - 35 kg = ____ g
 - 24 mg = ____ g
- There are about 6 billion people on Earth. If they all lined up and held hands, how long a line would they form?
- The lengths of most older U.S. swimming pools are multiples of yards, generally 25 yards. However, newer pools are 25 or 50 meters long. Suppose an American swimmer is practicing for a 400-meter race. Eight

- lengths of the pool is 400 yards, which is “close” to 400 meters. How close is it? That is, how much shorter is 400 yards than 400 meters? First, predict the amount, using only mental math. Explain how you got your prediction and how you did the operations in your head. Then determine the amount (to the nearest yard).
- Alan and Bill are going to have a 400-meter race, but Alan gets a head start because he is slower. Their best times of the season are 82 seconds and 75 seconds. How much (distance) of a head start should Alan get?
 - Let’s say your estimate of a length was 100 feet and the actual length was 105 feet. Clearly this would not be “as big” a difference as that between an estimate of 10 feet and an actual length of 15 feet, although both estimates were “off” by 5 feet. Compare the accuracy of the estimates in such a way that we can see from the language that the first estimate was much closer than the other.
 - To determine the relationship between miles and kilometers, if we use the conversion 39.37 inches = 1 meter, we find that 1 kilometer is approximately equal to 0.62 mile. What if we use the conversion 2.54 centimeters = 1 inch? How far apart are the answers (1 kilometer = x miles) obtained by using the different conversions?
 - When examining the relationship between kilometers and miles, we now have two conversion formulas: 1 kilometer = 0.62 mile and 1 mile = 1.6 kilometers. What relationship do 0.62 and 1.6 have?
 - The Biblical Goliath was said to be six cubits and one span tall (1 Samuel 17:4). How tall do you think he was in feet and inches? Explain both your reasoning and your calculations.
 - Suppose you were King Henry I (1496), who declared that henceforth, 1 yard would be equal to the distance from his nose to his forefinger. How long would a yard be, based on your body?
 - The historical table below⁵ compares the relative lengths of 1 foot in various countries in 1788.

A Comparison of the American foot with the feet of other Countries.

The American foot being divided into 1000 parts, or into 12 inches, the feet of several other Countries will be as follow.

	Parts	Inch.lin.points
America	— 1000 — — —	12 0 0 dec.
London	— 1000 — — —	12 0 0
Antwerp	— 946 — — —	11 4 1, 32
Bologna	— 1204 — — —	14 5 2, 25
Bremen	— 964 — — —	11 6 4, 89
Cologne	— 954 — — —	11 5 2, 25
Copenhagen	— 965 — — —	11 6 5, 76

PIKE’S “ARITHMETICK,” 1788
Early difficulties with weights and measures.

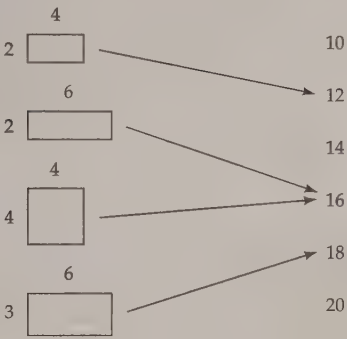
- Explain the meaning of the column entitled “Parts,” as though you were talking to a fifth grader who didn’t understand.
 - Make a graph for the column entitled “Parts.”
 - Describe the variation among the various feet.
 - What do you think the second column means?
 - Are the numbers in the two columns proportional? How did you determine your answer to this question?
- Say the United States did go metric.
 - What would you propose as the size of a standard sheet of paper? Justify your choice.
 - What would you propose as the two standard sizes for soft drinks to replace 12 ounces and 16 ounces?
 - Instead of buying a pound of butter, what would we buy?
 - If you filled up your car’s gas tank and it took 48.3 liters, approximately how many gallons is that?
 - Use mental math and estimation to determine approximately how many gallons of gas this amount is equivalent to.
 - By how much would your conversion be off if you used 1 liter $\approx$ 1 quart instead of the more precise 1 liter $\approx$ 34 ounces conversion?
 - The average human heart pumps about 60 milliliters of blood for each beat.
 - About how many liters of blood does your heart pump each day?
 - Do you think a reasonable degree of precision for this question would be to report the answer to the nearest 100 liters, 10 liters, 1 liter, or 1 milliliter? Explain your reasoning.
 - About how many gallons is this?
 - Suppose we measured the passage of time during the day as the Babylonians did. Compare the length of a daytime hour on the longest day of the year to the length of a daytime hour on the shortest day of the year where you live. Part of this question has to do with resourcefulness, an important trait for teachers to develop. Where can you go to get the information needed to answer this question?
 - Which is greater, an increase of 5° Fahrenheit or an increase of 5° Celsius? Justify your answer.
 - A train is traveling 60 miles per hour and is about to enter a tunnel that is 1200 feet long. How long will it take the train to pass completely through the tunnel? What additional information do you need in order to solve this problem?
 - One way to determine the speed of the current of a river is to measure how long it takes a leaf to pass under a bridge. For example, let’s say it takes a

⁵Louis Karpinski, *The History of Arithmetic* (New York: Russell & Russell, Inc., 1965), p. 159. Reprinted with permission of Scribner, a Division of Simon & Schuster. From *The History of Arithmetic* by Louis Karpinski (Russell & Russell, New York, 1965).

- leaf 9 seconds to pass under a bridge that is 35 feet wide. What is the speed of the current in miles per hour?
- b. How might we determine the speed of the current if we are in a boat?
20. Two cups are half full. Cup A holds coffee. Cup B holds milk. One teaspoon of coffee is taken from cup A to cup B. Then one teaspoon of the mixture is taken from cup B to cup A. The two cups now have the same amount of liquid. Is there more coffee in cup A than milk in cup B, the same amount of coffee in A as there is milk in B, or less coffee in A than milk in B? Explain your reasoning.
21. Let's say that a 12-ounce cup of coffee is 20 percent milk. If you add 6 ounces of coffee, what is the percentage of milk in the new mixture?
22. a. Determine the following dimensions for your body: height, wrist, neck, leg, arm, foot.
b. Determine the following ratios: height:foot, height:leg, neck:wrist, height:arm.
c. Before determining the ratios for the entire class, predict which ratio will have the most variation and which the least variation. Explain your reasoning.
d. Gather the class data. Decide on a way to determine the variation and to compare the variation.
23. Will the ratio of the heights of an adult and a child be equivalent to the ratio of the lengths of their feet? If it will be, explain why you think so. If not, predict which ratio will be greater and why. Gather some data. Do the data support your predictions? Explain.
24. You will need a scale for these problems.
a. Buy several bananas. Compare the price per pound that you paid and the price per pound of what you actually eat. If there are enough data, graph the actual price for the entire class. Before you do so, predict the shape of the distribution: normal, uniform, and so on.
b. Do the same for peanuts.
c. Buy several oranges of two types, one type with a thick skin and one type with a thin skin. Compare the ratios for both kinds of oranges.
25. Before they wash their faces, most people run the tap until the water is hot. How much water is wasted waiting for the water to get hot?
a. With a partner, determine a plan for answering this question.
b. Determine the amount of water (in U.S. customary units) wasted at the place where you live.
c. Describe the relative precision of your answer. For example, if you say 3 quarts, is your answer to the nearest quart?
d. Graph the class data.
e. Let's say the average for the United States is close to the average in this class and that 200 million people in the United States wash their faces every day. How many gallons of water are wasted?
26. You will need a dropper for this problem. Let's say you have a leaky faucet that drips every 6 seconds. How much water is wasted in one day? Explain the assumptions that you made and describe your solution path.
27. Take a sheet of blank paper. Tear it in half. Place one half on top of the other. Tear these two sheets in half. Place one half (two sheets) on top of the other. Assume for the purposes of this problem that you were able to continue this process for a total of 20 times. What would be the height of the stack of paper?
28. Select a trip in your part of the country for which there is not one obvious shortest way.
a. Determine which way is shortest by using a map. Explain your work.
b. Have at least one person try more than one way to determine the actual distance and the actual time it took in each case.
c. Is the shorter way faster?
d. Describe the possible sources of error in part (a).
e. Determine the average speed for each route.
f. On the basis of the data you have, which way would you go, or do you need more data?
29. a. Measure your classroom in meters and then make a scale model of your classroom so that 1 centimeter on your model represents 1 meter.
b. Measure your classroom in feet and inches and then make a scale model of your classroom so that 1 inch on your model represents 1 foot.
c. Are the computations in both problems about the same, or are the computations for the metric-scale map easier? Explain.
30. In *The History of Arithmetic*, Louis Karpinski describes the Babylonian system of measure:
3 lines = 1 sossus
10 sossus = 1 palm
3 palms = 1 small ell (cubit)
5 palms = 1 large ell
6 large ells = 30 palms = 1 large seed
60 palms = 1 gar
60 gar = 1 ush
30 ush = 1 kask
The Babylonian palm was equal to approximately 4 of our inches.
a. Compare the Babylonian kask to a metric length.
b. Describe the connections between their choice of units and their numeration system.
c. The cubic palm (called a ka) was used as a unit of capacity. Compare the cubic palm to the liter.
d. The weight of one ka of water was the unit of weight and was called one mina. Thus, the units of length

and weight were connected. In analyzing the Babylonian system, Louis Karpinski cites the parallels between the ancient Babylonian system and the modern metric system, noting that both systems established a connection between linear and cubic measure and a unit of weight. He further states that the parallel is “one of the most striking [parallels] to be found in the development of scientific ideas.”⁶ Explain Karpinski’s praise for the Babylonian system.

31. Is the relation “the perimeter of” a function? Justify your answer.



SECTION

10.2

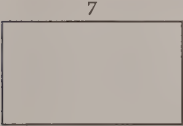
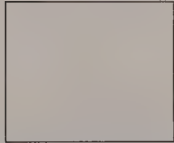
PERIMETER AND AREA

WHAT DO YOU THINK?

- Can two shapes with different perimeters have the same area? Why or why not?
- What does pi mean?

In this section, we will investigate the concepts of perimeter and area, both for geometric figures for which there are formulas and for irregularly shaped objects. It may surprise you that many students do not understand these concepts well. For example, consider the data in Table 10.3 concerning questions reported in the *Results from the Fourth Mathematics Assessment*.⁷

TABLE 10.3

Question	PERCENT CORRECT	
	Grade 3	Grade 7
What is the perimeter of the rectangle below? 	17	46
What is the area of this rectangle? 	20	56
What is the area of this rectangle? 	5	46

⁶Louis Karpinski, *The History of Arithmetic* (New York: Russell & Russell, Inc., 1965), p. 159.
⁷Mary M. Lindquist, ed., *Results from the Fourth Mathematics Assessment* (Reston, VA: NCTM, 1989), p. 40.

The NCTM Standards stress that teachers should not just help students better understand how to perform the procedures for determining perimeters and areas. Rather, if the students develop these formulas through problem-solving and reasoning, if they are required to express their understanding in words also (not just in formulas), and if they can see connections among the formulas, *then* performance on these and more challenging problems will increase dramatically.

Perimeter

Let us begin first with the concept of **perimeter**, essentially the distance around an object. Many practical applications of perimeter involve surrounding an object—for example, fencing a yard or running a baseboard around the base of a room. We will first look at distances around circles.

Circumference and π

When we determine the distances around figures and objects, sometimes the path is not a straight line but rather is a circle. You may remember a formula involving the distance around a circle, or the **circumference**, and that it involves π . You may have constructed a definition of π in Exploration 10.9. Before reading on, take a few moments to think about what π means. That is, the value of π is 3.14 (to two decimal places), but what does π mean? For example, where did the number 3.14 come from? How is this number related to the circumference or diameter or radius of a circle (not the formula, but a description of the relationship)? Write your thoughts before reading on. . . .

In one sense, π is a ratio—that is, π is the ratio of the circumference to the diameter of *any* circle. If we could precisely measure the circumference and diameter of any circle and then divide the circumference of the circle by its diameter, we would *always* get π . If we call the circumference C and the diameter d , we have $\pi = C/d$.

Thus, we have the formulas

$$C = \pi d \quad \text{and} \quad C = 2\pi r$$

In Figure 10.5, C is the center of the circle, $\overline{AC}$, $\overline{BC}$, and $\overline{DC}$ are radii (plural of radius), and $\overline{AD}$ is a diameter.

There is another way to think about π . Imagine placing a string around the circumference of the circle in Figure 10.5 and then straightening out that string. If the diameter of the circle is d , then the length of the string is $3.14d$. That is, if we unwrap the circumference, its length will always be 3.14 times the length of the diameter.

Let us now examine one application of these formulas.

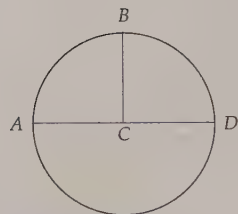


FIGURE 10.5

INVESTIGATION 10.2 What Is the Length of the Arc?

In Figure 10.6, the radius of the circle is 3 centimeters and the measure of angle BOC is 42 degrees. How long is arc BC ? Think about this and then read on. . . .

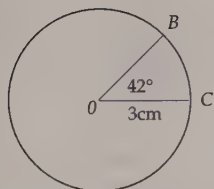


FIGURE 10.6

DISCUSSION**STRATEGY 1: Make a simpler problem**

What if the angle had been 90 degrees? Can you solve that problem? Think before reading on. . . 

If the angle had been 90 degrees, then the arc would simply have been one-fourth of the circumference of the circle. Do you see why? Can you adapt this strategy to solve this problem now? Try to do so yourself before reading on. . . 

Because 42 does not divide 360 evenly, we don't have an easy fraction. However, we can multiply the circumference of the circle by $42/360$.

STRATEGY 2: Use a proportion

We could also solve this problem using a proportion. Can you make the proportion? Try to do so before reading on. . . 

Arc $\widehat{BC}$ is a fraction of the circumference of the circle. Similarly, 42 degrees is a fraction of 360 degrees, the number of degrees in a circle. From the ratio construct of fractions, these two ratios are equal. Why is this?

The circumference of the circle is πd , which equals approximately 18.84 inches, so we can set up the following proportion, because the two part-whole ratios are equal:

$$\frac{\text{arc } \widehat{BC}}{18.84 \text{ inches}} = \frac{42^\circ}{360^\circ}$$

When we solve the equation, we find that the length of arc BC is about 2.2 inches.

Area

Earlier we stated that perimeter questions generally deal with “how much” it takes to surround or go around an object. In order to answer perimeter questions, we have to select an appropriate unit, and thus the answer takes the form of how many of those units. Area questions generally deal with “how much” it takes to *cover* an object—for example, how much fertilizer to cover a lawn, how much material to cover a bed. In order to answer area questions, we have to select an appropriate unit, and thus the answer takes the form of how many of those units. However, the units for perimeter and area are not the same. For example, if we have a 20-foot by 10-foot garden, we say that we need 60 feet of fence to surround the garden, but we would say that the area of the garden is 200 *square* feet. The need for units and the difference between units for perimeter, area, and volume are generally not well understood by students and therefore are worth emphasizing more than once.

Many people remember or recognize the basic formulas for determining areas, but few people understand why they work or where they came from. Let us investigate them for a bit. You may have encountered these ideas in Explorations 10.6 and 10.7. Understanding them will pay dividends when we attempt to solve more complex problems. The simplest case for area involves rectangles and squares.

Just as we have found that average is a number that gives us a sense of the middle of a set of data, and standard deviation is a number that gives us a sense of the variation of a set of data, **area** is a number that gives us a sense of the size

of something—for example, how much surface that object covers or how much material (cloth, plastic, paint, carpet) it would take to cover that surface. If the geometric figure is a square or a rectangle, we can easily determine the area by multiplying the *base* by the *height*.

Base and height Because most students use the more common terms *length* and *width*, let us take a moment to examine what *base* and *height* mean, because they will become important when we investigate other figures for which the terms *length* and *width* are not appropriate.

Any polygon can be rotated so that its bottom side will be parallel to the bottom of the paper. Thus, any side can be taken as the **base** of the polygon. For example, we can rotate the triangle at the left in Figure 10.7 so that any one of its sides is the base. Which one we choose is generally an arbitrary decision. See Figure 10.7.

The height of a polygon is the distance from the side chosen as the base to the point farthest away, measured along a line perpendicular to the base. See Figure 10.7 for an illustration of this notion with triangles.

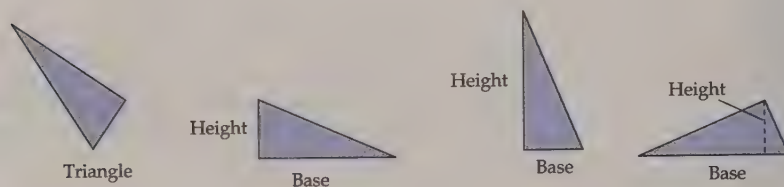


FIGURE 10.7

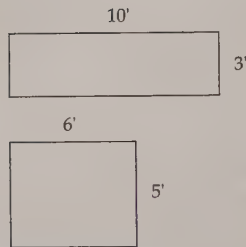


FIGURE 10.8

Thinking about area by looking at rectangles Let us focus on some more subtle aspects of the concept of area.

What does it mean to say that one rectangle (or other figure) covers more area than another? Consider the diagrams in Figure 10.8 of two pieces of plastic sheeting that I use to cover two different woodpiles in my back yard. One piece is 10 feet by 3 feet, and the other is 6 feet by 5 feet.

When I ask young children if the sheets are the same size or if one is bigger, they generally say that the 10-foot by 3-foot sheet is bigger. Older children and adults know that the areas of the sheets are equal, 30 square feet. We can demonstrate this by covering each figure with 30 squares, each of which is 1 foot by 1 foot (see Figure 10.9).

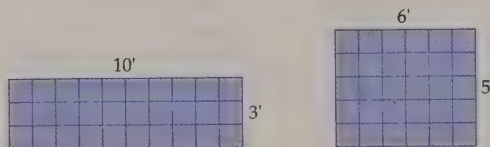


FIGURE 10.9

Although this may seem very simple and straightforward, many people are at least initially stumped when they try to explain what *area* means when the dimensions of a rectangle are not whole numbers—for example, when a rectangle is 3.5 centimeters by 2.5 centimeters. We say that the area of the rectangle is 8.75 square centimeters, because we know that the area of a rectangle is obtained by multiplying its length by its width. Can you explain what that 8.75 means, as if you were talking to a child who understands that in the previous problem, both sheets of plastic have an area of 30 square feet? Do this before reading on. . . .



Figure 10.10 shows one way of explaining what it means to say that the area is 8.75 square centimeters: cutting up the rectangle and reassembling the pieces in a certain way. Do you understand this diagram? Did you explain the meaning of the 8.75 square centimeters in another way that also makes sense? Take some time to see if you can understand the diagrams below before you read my explanation. 

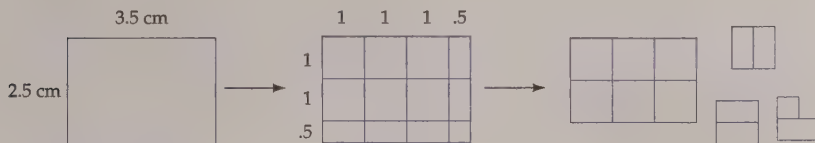


FIGURE 10.10

We can partition the rectangle in a manner similar to the way we used partitioning in the multiplication of fractions and decimals. That is, we have 6 one-centimeter by one-centimeter squares, and we can combine the remaining pieces to make another 2.75 square centimeters (see Figure 10.10).

Area as a function From another perspective, once we specify the unit of measurement, the relationship between any surface and its area is a functional relationship: For every surface, there is a unique number that we associate with its area. Even when the surface is not a rectangle or the dimensions are not whole numbers, the number that represents the area tells us the equivalent number of square units that this object covers.

Let us now turn our attention to understanding some of the more commonly used formulas for determining the area.

Understanding the area formula for parallelograms The formulas for many common figures are striking, both in their simplicity and in their connection to one another. It still surprises me that so few people understand why they work. The formula for determining the area of a parallelogram is connected to the formula for determining the area of a rectangle. The diagram at the right in Figure 10.11 illustrates the connection. Can you see it? How would you explain it?

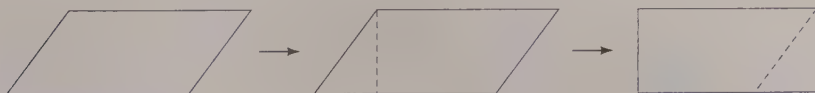


FIGURE 10.11

If we cut the triangle from the parallelogram and reconnect it at the right-hand side, we have transformed the parallelogram into a rectangle. We have not added or subtracted any area, so the areas of the parallelogram and the rectangle must be equal. From Figure 10.11, you can see that the base of the parallelogram is congruent to the base of the rectangle and that the height of the parallelogram is congruent to the height of the rectangle. Because we know that the areas of the two are equal, we can now state that *the area of any parallelogram is equal to the product of its base and its height*; that is, $A = bh$ (see Figure 10.12).

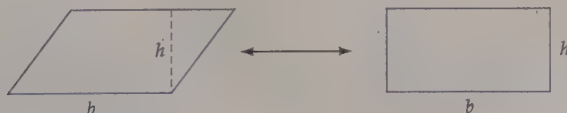


FIGURE 10.12

Understanding the area formula for triangles We will use what we have learned about parallelograms to develop the formula for the area of triangles. Just as we connected the area of any parallelogram to a rectangle, take a few minutes to see if you can adapt this idea to develop formulas for the areas of triangles. Then read on. . . 

Consider triangle MAN (see Figure 10.13). If we make a congruent copy of this triangle and move that triangle into place as shown (that is, by rotating it 180 degrees), we form a parallelogram. Do you see how we can use the formula for parallelograms to determine the formula for triangles now? See if you can do it on your own before reading on. . . 

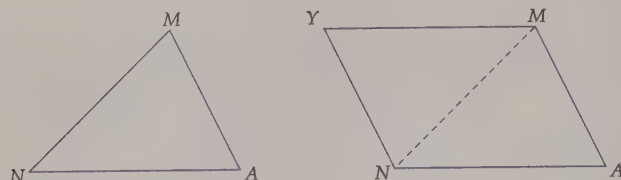


FIGURE 10.13

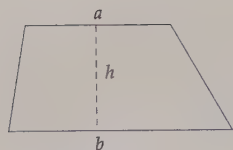


FIGURE 10.14

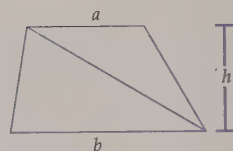


FIGURE 10.15

The base of the parallelogram and the base of the triangle are identical, and so are the heights.

If the area of the parallelogram = base $\cdot$ height, then (since the area of the two triangles is equal to the area of the parallelogram), the area of the triangle = $\frac{1}{2}(\text{base} \cdot \text{height})$, or $A = \frac{1}{2}b \cdot h$.

Understanding the area formula for trapezoids Can you now apply what you have learned about the areas of these figures to determine the formula for the area of a trapezoid (see Figure 10.14)? Try to do so before reading on. . . 

There are several ways to derive the formula for the area of a trapezoid. We will discuss two that connect to our preceding discussions. First, we can draw a diagonal of the trapezoid, which cuts (decomposes) it into two triangles. We know how to find the areas of these triangles: The areas are $\frac{1}{2}ah$ and $\frac{1}{2}bh$. The sum of the areas of the two triangles is equal to the area of the trapezoid, so the area A of the trapezoid is $\frac{1}{2}ah + \frac{1}{2}bh$, or $A = \frac{1}{2}(a + b)h$. See Figure 10.15.

Figure 10.16 shows another method for finding the area of a trapezoid. If we construct a congruent trapezoid and connect it (via a translation and a rotation) to the original trapezoid, we have a parallelogram, and the area of the parallelogram is equal to the product of its base and its height; that is, the area is equal to $(a + b)h$. Because the area of the trapezoid is equal to one-half the area of this parallelogram, the area of the trapezoid = $\frac{1}{2}(a + b)h$.

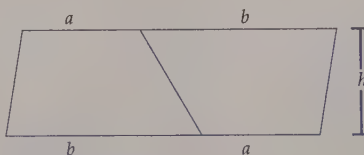


FIGURE 10.16

MATHEMATICS

Did you realize that a trapezoid can be thought of as a triangle with its head cut off? Actually, this is not just a silly thought, because this way of thinking about trapezoids helps us connect the area formula for the trapezoid with the triangle. We could have determined the formula for the area of a triangle by thinking of a triangle as a trapezoid where the top base = 0. In this case, the area is $\frac{1}{2}(b_1 + b_2)h = \frac{1}{2}b_1h$ —that is, $\frac{1}{2}bh$.

Pythagorean Theorem

As we saw in Chapter 8, the relationship between the lengths of the sides of a right triangle was known long before it was proved by Pythagoras. That is, Pythagoras proved that *for any right triangle, the sum of the squares of the lengths of the two sides (say, a and b) is equal to the square of the length of the hypotenuse (say, c); that is, $a^2 + b^2 = c^2$ [see Figure 10.17(a)].*

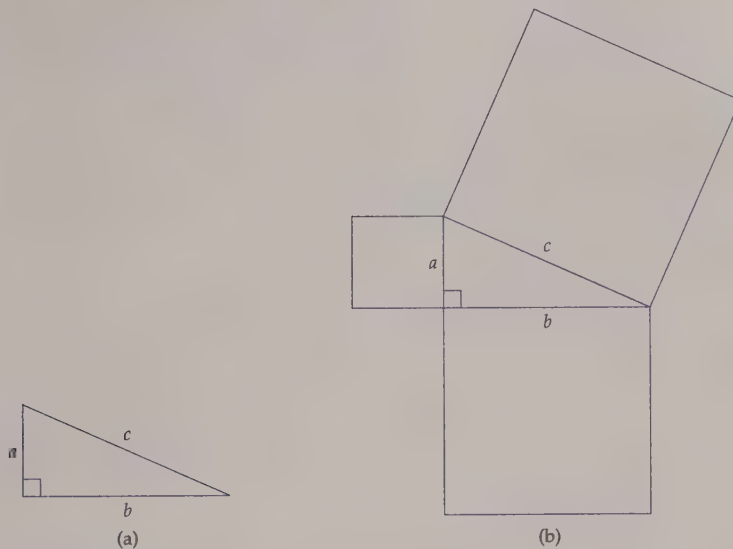


FIGURE 10.17

It is important to note that for the Greeks, this was not an algebraic relationship but rather a geometric relationship [see Figure 10.17(b)]. That is, if you make a square whose sides are congruent to the length of side a , a square whose sides are congruent to the length of side b , and a square whose sides are congruent to the length of the hypotenuse c , then the sum of the areas of the two smaller squares will be equal to the area of the larger square.

At the last count, there were over 370 different proofs of the Pythagorean theorem, including a proof by one of our presidents, James Garfield. The Pythagorean theorem has many practical applications, such as the one below, where it is used to determine the length of something that we cannot measure directly. The need for the Pythagorean theorem arises both in determining lengths of lines and in determining areas.

INVESTIGATION 10.3 Using the Pythagorean Theorem

We can use the Pythagorean theorem to determine the length of a lake. In Figure 10.18, a right triangle has been created in such a way that the length of the lake is part of the length of one side of the triangle. The lengths of AB , BC , and CD have been measured, because they are on land. The triangle has been carefully made so that all lines are straight lines, so that point D lies on side AC , and so that angle C is 90 degrees. What is the length of the lake? Work on this before reading on. . . 

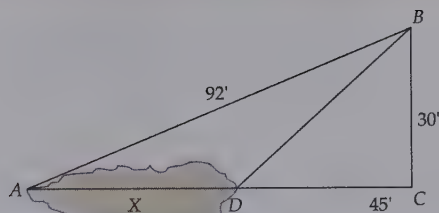


FIGURE 10.18

DISCUSSION

Let us discuss two different strategies and then discuss problem-solving.

STRATEGY 1

Let x be the length of the pond. Using the Pythagorean theorem, we have

$$(x + 45)^2 + 30^2 = 92^2$$

We now have a quadratic equation, which we can solve using the *quadratic formula*:

$$x^2 + 90x + 2025 + 900 = 8464$$

$$x^2 + 90x - 5539 = 0$$

$$x \approx 42 \text{ feet}$$

STRATEGY 2

Looking at the problem from a different perspective, we can find the length of the base of the triangle ($\overline{AC}$) rather easily, and then we can subtract 45 feet from the base to find the length of the lake.

$$AC^2 + BC^2 = AB^2$$

$$AC^2 + 900 = 8484$$

$$AC^2 = 7584$$

$$AC \approx 87 \text{ feet (to the nearest whole foot)}$$

Thus, $AD \approx 42$ feet ($87 - 45$).

This investigation illustrates the idea of examining a problem from different perspectives. The second solution is quicker and simpler, although the first solution is not difficult for someone who is adept at algebra. Often in mathematics problems (and in real-world problems too), a problem that at first glance seems very complex and difficult may, when looked at from another perspective, be much simpler than it first appeared.

MATHEMATICS

The *quadratic formula*, which you may remember from algebra, lets you solve an equation of the form

$$ax^2 + bx + c = 0$$

that is, a quadratic equation. The formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

INVESTIGATION 10.4 Understanding the Area Formula for Circles

The last area formula that we will consider now is that for the area of a circle. The purpose of this investigation is in the realm of “number sense,” helping you to see why $A \approx 3.14r^2$ (and therefore $A = \pi r^2$) makes sense. Exploration 10.10 examines another way to make sense of this area formula. Consider the circle in Figure 10.19, whose radius is r inches. In

Figure 10.19(b), I have circumscribed a square around the circle. What is the area of the square? Determine this and then read on. . . 

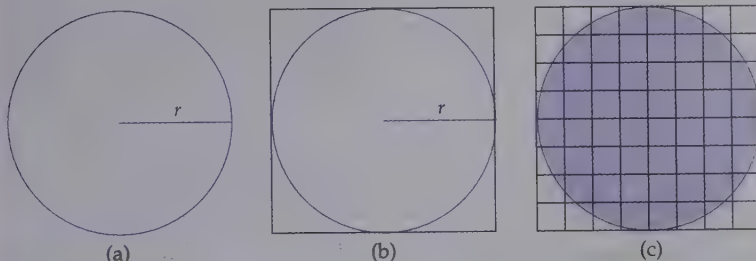


FIGURE 10.19

DISCUSSION

If the radius of the circle is r inches, then the length of each side of the square is $2r$, and thus the area of the square is $(2r)(2r) = 4r^2$.

The area of the circle thus is clearly less than 4 times r^2 . We can use our spatial sense to estimate that the circle covers about $3/4$ as much space as the square and thus approximate the area of the circle as $3r^2$, or we can place a grid over the figure and determine what fraction of the square is covered by the circle. In this case, we get a more accurate estimate of the area of the circle [see Figure 10.19(c)].



HISTORY

Some of the area formulas we have investigated were known by the Egyptians, Babylonians, Chinese, and other peoples thousands of years ago. We know that both the Egyptians and the Babylonians knew not only the formulas for the area of the square and the rectangle, but also the formula for the area of a trapezoid. They knew the formula for finding the area of a right triangle. There were correct and incorrect formulas for the areas of

isosceles triangles. For example, there is an example in the Rhind papyrus (from ancient Egypt) of finding the area of an isosceles triangle by finding half the product of the base and the length of one of the equal sides, instead of the altitude. For the circle, the Babylonians developed the formula $A = C^2/12$ and the Egyptians developed the formula $A = (\frac{8}{9}d)^2$. It will be left as an exercise to see how close their formulas were to the actual formula.

INVESTIGATION 10.5 A 16-Inch Pizza Versus an 8-Inch Pizza

Let's say you are going out to have pizza with several friends. You are thinking of getting one large pizza—let's say its diameter is 16 inches. However, some people are vegetarians, and so you decide to get two little pizzas, each of which is 8 inches in diameter. Suddenly someone asks, "Are we getting the same amount of pizza if we get two little pizzas instead of one large one?" What do you think? Please work on this before reading on. . . 

DISCUSSION

This is generally one of the most amazing investigations in this book because so many people are so surprised. To determine the area, we can use the area formulas: If the large pizza has a diameter of 16 inches, it has a radius of 8 inches. Therefore its area is $\pi(8)^2 = 64\pi \approx 201$ square inches.

If the small pizza has a diameter of 8 inches, it has a radius of 4 inches.

Therefore, its area is $\pi(4)^2 = 16\pi$. Therefore, the area of two small pizzas is 32π , or approximately 100.5 square inches.

Amazing, isn't it? Not only are two 8-inch pizzas not equal to one 16-inch pizza, but it takes *four* 8-inch pizzas to have the same area as *one* 16-inch pizza. Some people still don't believe their eyes. If you find this result amazing, draw a circle to represent the large pizza and then figure out how to place the two smaller pizzas inside the circle.

INVESTIGATION 10.6 How Big Is the Footprint?

The area of most objects cannot be determined by a formula. For example, biologists and environmentalists want to know the total surface area of the leaves of a tree in order to determine the amount of oxygen the tree produces. What if we wanted to measure the area of another irregularly shaped object, like the footprint in Figure 10.20?

Take some time to think about how you might determine the area and to try out your ideas. Then read on. . . . 

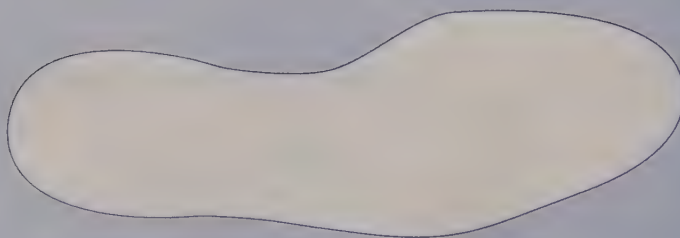


FIGURE 10.20

DISCUSSION

There are many, many possible strategies. Several are briefly described below.

STRATEGY 1: Use graph paper

This is one of the simpler strategies. There are two issues worth pursuing: (1) what to do with the partially filled squares, and (2) the advantages and disadvantages of graph paper with smaller squares.

STRATEGY 2: Draw rectangles and triangles

We can partition the footprint into rectangles and triangles. One advantage of this over the previous strategy is that in this case, one rectangle will account for a large portion of the print. One disadvantage is that there is room for error, depending on how the rectangles and triangles are made. This procedure could also become a bit tedious.

STRATEGY 3: Draw trapezoids

We can partition the footprint into trapezoids. This strategy arises from the realization that trapezoids are easily broken into rectangles and triangles; also,

this method involves less computation than the previous method. I can approximate the area of the footprint with four trapezoids.

STRATEGY 4: Find the perimeter

We can trace the figure with string to get the perimeter of the print, then make a rectangle with the string and compute the area of the rectangle. There is a flaw in the reasoning behind this strategy. If you do not see this flaw, revisit this problem after doing the next investigation.

STRATEGY 5: Weigh it

We can trace this print onto a piece of thick posterboard and cut it out. We can then cut out another piece of posterboard in the shape of a rectangle. The ratio of the area of the footprint to the area of the rectangle will be equal to the ratio of the weight of the footprint to the weight of the rectangle. Therefore, we can use the following proportion:

$$\frac{\text{Area of footprint}}{\text{Area of rectangle}} = \frac{\text{weight of footprint}}{\text{weight of rectangle}}$$

INVESTIGATION 10.7 Making a Fence with Maximum Area

 Joshua has decided to convert part of his back yard into a garden. He knows from his neighbor's experience that he needs to fence in the garden to keep the animals out. At the hardware store, he buys 200 feet of chicken wire. After he gets home and unrolls the wire, he finds to his surprise that there are many different size gardens that he can make with 200 feet of fencing. He decides that he wants to get the largest garden from 200 feet of fencing. If he wants a garden that is rectangular in shape, what are the dimensions of that garden? Work on this before reading on. . . . 

DISCUSSION

STRATEGY 1: Use guess–check–revise

TABLE 10.1

Length	Width	Area	Reasoning
80 feet	20 feet	1600 square feet	If the perimeter is 200, then $l + w = 100$. Reduce the length by 10 and increase the width by 10.
70 feet	30 feet	2100 square feet	This has more area, so reduce the length by 10 and increase the width by 10.
60 feet	40 feet	2400 square feet	This has more area, so reduce the length by 10 and increase the width by 10.
50 feet	50 feet	2500 square feet	Why does this feel like it “should” be the biggest garden?

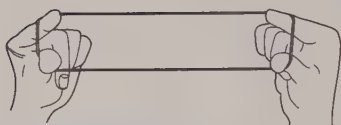


FIGURE 10.21

STRATEGY 2: Reason and make a model

A very tactile way to “feel” this investigation is to cut a piece of string and tie the ends together. Now, using your two thumbs and two forefingers, make as skinny a rectangle as you can (see Figure 10.21).

Now slowly move your thumbs and forefingers apart—that is, decrease the length and increase the width. What seems to be happening to the area? Why must it stop increasing when the length and width are equal?

STRATEGY 3: Be creative and adventurous

What if we made the garden in the shape of a circle? Do this before reading on. . . 

Because the circumference of the circle is 200 feet, the radius of the circle will be $r = C/2\pi \approx 200/(2 \cdot 3.14) = 31.8$ feet. Now that we know the radius, we can find the area:

$$A = \pi r^2 \approx 3.14(31.8)^2 = 3175 \text{ square feet}$$

Compare the area of the circular garden to that of the square garden: The former shape will give Joshua more than 25 percent more area.

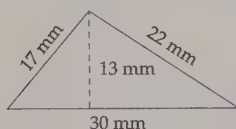
Summary

In this section, we examined some measurement ideas and procedures related to perimeter and area. In one sense, perimeter concerns how much is needed to surround an object, and area refers to how much is needed to cover an object. We have learned that π can be viewed both as how many times a diameter can wrap around a circle and as the ratio between a circle's circumference and its diameter. We have explored area formulas (and why they work) for triangles, for certain quadrilaterals, and for circles. We have explored the Pythagorean theorem and an application of this theorem in problem-solving. We have moved beyond more routine applications of perimeter and area to examine how areas of irregularly shaped figures are determined. Finally, we have found that the relationship between perimeter and area is not a simple one. For example, if we double the area of a square, we do not double its perimeter. Similarly, two objects can have the same perimeter but different areas, and two objects can have the same area but different perimeters.

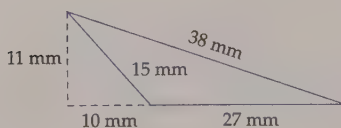
EXERCISES 10.2

1. Determine the area and perimeter of the figures below.

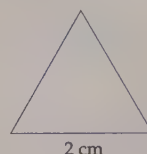
- a. Acute scalene triangle



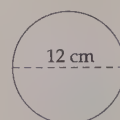
- b. Obtuse scalene triangle



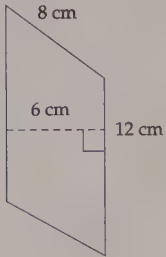
- c. Equilateral triangle



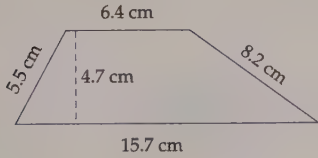
- d. Circle



e. Parallelogram

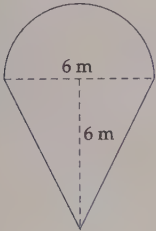


f. Trapezoid

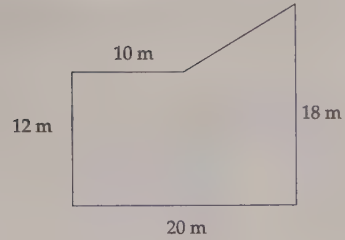


2. Determine the area and perimeter of the figures below.

a.

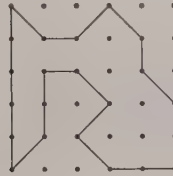


b.

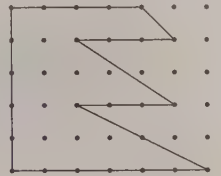


3. Determine the area of each of the following polygons on the geoboard dot paper.

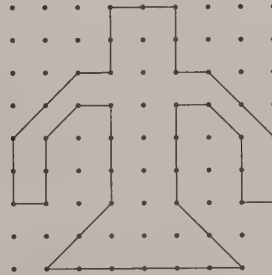
a.



b.

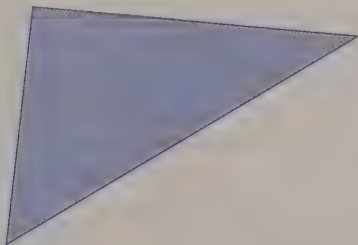


c.

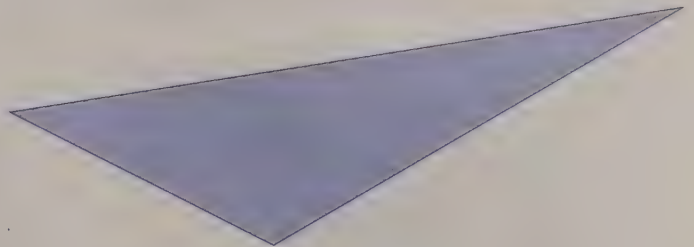


4. Find the area of the triangles below, in square centimeters using a ruler to measure the appropriate lengths.

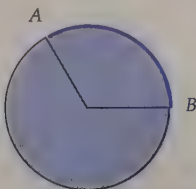
a.



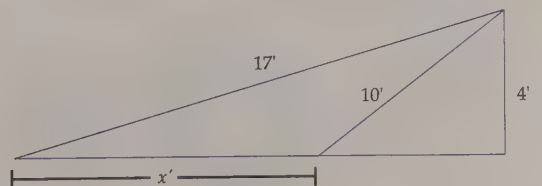
b.



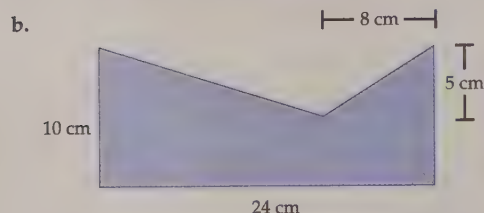
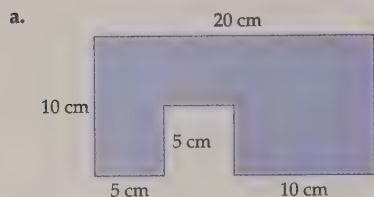
5. Determine the length of the arc $\widehat{AB}$ if the diameter of the circle is 10 feet and the angle is 128 degrees.



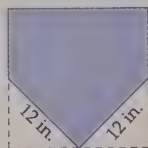
6. Find the length of x , to the nearest tenth of a foot. *Note:* The figure is not drawn to scale.



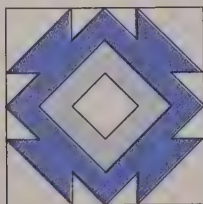
7. Determine the area of each figure below in two different ways.



8. Let's say we have a square whose sides all measure 10 inches. Determine to two decimal places the dimensions of the square that has twice the area of this square.
9. Bernice wants to seed her large back yard with grass. Her back yard is in the shape of a rectangle with dimensions 120 feet by 80 feet. The seed costs \$3.99 for a 1-pound bag, and each bag covers up to 1050 square feet. How much will the seed cost?
10. Kayla wants to spread bark mulch to make a uniform border around a 6-foot by 10-foot shed. She figures that she has enough mulch to cover 40 square feet. How wide should the border be?
11. A farmer has a fence that encloses a square plot with an area of 36 square meters. If the farmer uses this fence to enclose a circular flower garden, what will the area of the garden be?
12. A Little Leaguer asks her mother to make her a home plate for a sandlot baseball game. According to her encyclopedia, an official plate is made from a square by making two 12-inch diagonal cuts as shown below. What is the length of the side of the original square?



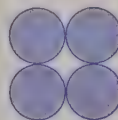
13. What percent of the quilt square below is dark blue?



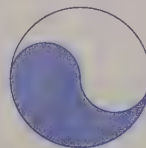
14. An equilateral triangle whose area was 100 square centimeters was reflected to make a congruent equilateral triangle so that the six smaller triangles are all congruent.



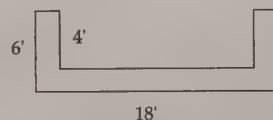
- a. What is the area of each of the little triangles?
- b. What is the area of the hexagon in the middle?
15. If each of the circles has a diameter of 10 meters, what is the area of the region inside the four circles?



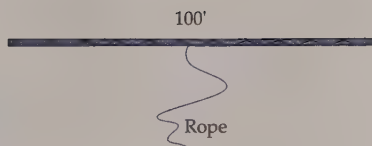
16. The figure below is a common representation of the famous yin-yang relationship, which originated in China.
- a. Can you explain how the figure is made?
- b. If the radius of the circle is r , what is the length of the curved line in the interior of the circle that separates the white region from the black region?



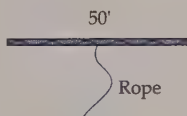
17. A circular flower bed is 6 meters in diameter and has a circular sidewalk 1 meter wide around it. Find the area of the sidewalk in square meters.
18. Sixteen circles (of equal area) are cut out of a square sheet of tin whose sides are 40 centimeters. What is the area of wasted tin if the circles are made as large as possible?
19. How many 2-inch by 3-inch by 8-inch bricks will you need to build a (uniformly wide) brick walk with the shape shown in the figure below? (Lay the bricks so that the largest face is up.)



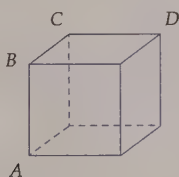
20. a. Suppose a goat was tethered at the middle of a 100-foot-long fence and the length of the rope was 50 feet, as shown in the diagram below. Over how much area could the goat graze? Describe any assumptions you make in order to solve the problem.



- b. What if the point at which the goat was tethered was at the middle of a 50-foot-long fence, as shown in the diagram below? Over how much area could the goat graze? Describe any assumptions you make in order to solve the problem.



21. Suppose a wire that is stretched tightly around the Earth is elongated by 5 meters and then placed back around the Earth. Could a person crawl under the elongated wire?
22. Let us examine three ways to travel from point A to point D on the cube below. Assume that the cube is 1 meter on each side.



- What would the distance be if you went from A to B and then from B to C and then from C to D ?
 - If you had to travel on the surface of the cube but didn't have to travel on edges, what is the shortest distance?
 - What if you could go (in space) directly from A to D ? Now what is the shortest distance?
 - Compare the three distances.
23. How many right triangles with sides 3 inches and 6 inches can be cut from a sheet of paper that is 18 inches by 40 inches?
24. The steps to the entrance of a school are 40 inches high. The school has decided to make the building wheelchair-accessible. If a city ordinance states that the ramp cannot be steeper than 5 degrees, how long must the ramp be? Solve this without using trigonometry.

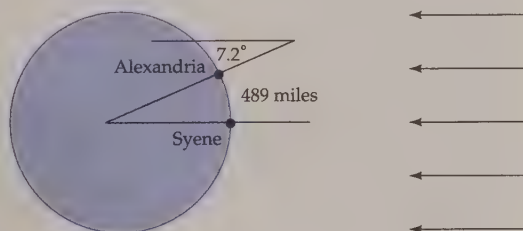


25. Some people have an intuitive sense of whether things will fit or not. For example, let's say you have bought a circular table top with a diameter of 7 feet and a thickness of 3 inches, and your doorway is 78 inches by

30 inches. First, predict whether or not the table top will fit. Then determine whether or not it will fit.



26. Compare the Babylonian formula for the area of the circle, $A = C^2/12$, and the Egyptian formula, $A = (\frac{8}{9}d)^2$. Which is more accurate? Justify your choice.
27. Around 240 B.C., Eratosthenes, a Greek mathematician, estimated the circumference of the Earth. His estimate was off by only 1 percent! Let us see what he did. He knew that on a certain day, the sun would be directly overhead at a town called Syene (now called Aswan). However, the sun would not be exactly overhead in Alexandria, which was 5000 stadia (about 490 miles) north of Syene. Using the figure below, determine how Eratosthenes measured the circumference of the Earth. Explain your reasoning, your assumptions, and your calculations.



28. On a blank sheet of paper, draw from memory:
- A rectangle the size of a dollar bill
 - A rectangle the size of a floppy disk
 - A circle the size of a penny
 - A circle approximating the top view of a soda can
- Before you check your estimates, predict the relative accuracy of your estimates, from 1 to 4, with 1 denoting the drawing that you predict will be "closest" to the actual shape.
 - Determine the actual area of each of your shapes and the actual objects. Describe how you determined which of your measurements was "closest" to the actual measurement.
29. In Investigation 10.6, we discussed several strategies for finding the area of irregularly shaped figures. Describe a situation for which one of the strategies would be preferable to the others. Make a convincing argument that the particular strategy would indeed be more practical and/or more accurate.

30. Describe how you would find the surface area of the top of each of the following:



31. Make a photocopy of a map of your state and determine the area of your state.

- Show and explain your work.
- Justify the precision in your answer—for example, the choice of unit and the number of decimal places, if any.
- Describe and explain your degree of confidence in your result.
- Describe any difficulties you had and how you overcame them.

32. How much would 1 million pennies weigh?

33. Several years ago the Treasury Department designed a new \$100 bill. They printed \$80 billion in new \$100 bills. How large a room would be needed to contain all these \$100 bills?

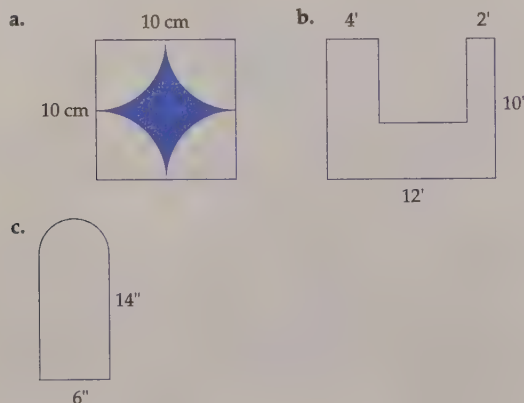
34. One way of buying tennis balls is in a tin that contains three tennis balls. Is the tin taller or bigger around?

35. Draw on geoboard dot paper a polygon whose perimeter is 20 units and whose area is less than 10 square units. Then draw a figure whose perimeter is 20 units and whose area is greater than 20 square units. Describe your solution process. This description should include your reasoning, guesses that didn't work, and what you learned from those guesses.

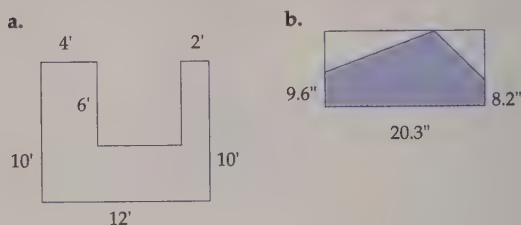
36. Let's say you have two similar polygons and the ratio of their perimeters is 2:1. What is the ratio of corresponding sides? What is the ratio of their areas? Justify your answer.

37. How is the size of a TV measured? If we are comparing a 15-inch TV and a 30-inch TV, what is twice as big about the bigger TV?

38. If possible, determine the perimeter of the figures below. If there is not enough information, explain why.



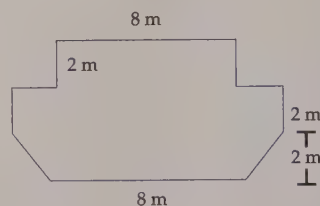
39. If possible, determine the area of the figures below. If there is not enough information, explain why.



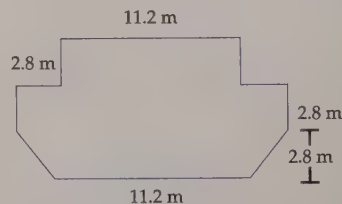
40. For each of the problems below, if there is not enough information given for there to be exactly one answer, explain why and provide one more piece of information that would enable us to determine the solution. If there is enough information, solve the problem.

- The hypotenuse of a right isosceles triangle is 10 inches. Find the area.
- The perimeter of one rectangle is twice the perimeter of another rectangle. The dimensions of the rectangle with the bigger perimeter are 8 inches by 6 inches. Find the dimensions of the smaller rectangle.
- The area of a rectangle is 20 centimeters. Find its perimeter.
- A farmer is fencing his pasture. The fence posts will be 10 feet apart. The pasture is in the shape of a rectangle whose length is double its width. The farmer needs 54 fence posts. What are the dimensions of the pasture?

41. a. Explain what assumptions you need to make in order to determine the area of the figure below. Then determine the area.



- b. If the figure below is "similar" to the figure above, explain how you can apply your work from the previous problem to find the area of the figure below.



42. Anne used a grid to determine the area of an irregularly shaped figure, and she determined that the area was 23 square centimeters. Then she realized that she wanted square inches. How can she convert from square

centimeters to square inches? There are 2.5 centimeters in 1 inch.

43. Adam is also confused. He wonders why, if you take a rectangle and multiply the length of each side by 1.5, the area of the new rectangle isn't 1.5 times as big as the area of the old rectangle. How would you explain this to him?

44. If a circle has a radius of 5 inches, using the formula, we would say that the area of the circle is approximately 78.5 square inches. A fellow student has come to you and asked why we use *square inches* to represent the area of the circle. What does 78.5 square inches mean?

SECTION

10.3

SURFACE AREA AND VOLUME

WHAT DO YOU THINK?

- Is the relationship between surface area and volume a functional relationship? Why or why not?
- If we doubled the volume of a room, would its surface area double too?

In Section 10.2, we examined two attributes of two-dimensional objects: perimeter and area. In this section, we will examine two attributes of three-dimensional objects: surface area and volume. We can think of **surface area** as the area needed to cover all the faces of a three-dimensional object, and we can think of volume as the amount of space contained within that object.

Understanding the Surface Area of Prisms

Very simply, the *surface area of a prism is equal to the sum of the areas of all of its faces*. Thus, to determine the surface area, we need to be able to determine the dimensions of each face. Let us consider a rectangular prism first. Determine the surface area of the rectangular prism in Figure 10.22 and then read on. It may help first to draw a net of that prism, which is shown at the right. 

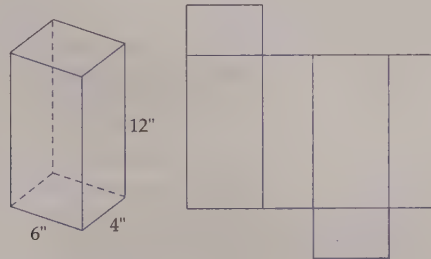


FIGURE 10.22

There are patterns in this prism that make its surface area easier to determine. You can see that the bottom and top bases are congruent; similarly, the front and back faces are congruent, and the right and left faces are congruent. Thus,

$$\text{Surface area of the prism} = 2(24 \text{ in}^2) + 2(72 \text{ in}^2) + 2(48 \text{ in}^2) = 288 \text{ in}^2$$

With a concrete example under our belts, let us now investigate the surface area of prisms whose bases are regular polygons. We will use this thinking to develop the general formulas for the surface areas and volumes of some basic three-dimensional geometric figures. Most readers find it necessary to read this material more than once and to read actively, with pencil in hand. After all, I am presenting you a summary of discoveries that took hundreds of years of effort by some of the brightest people of their time!

What do we know about the different surfaces of prisms that would make the task easier? Look at the three diagrams in Figure 10.23, showing three prisms and their nets. What do you see? Think before reading on. . . . 

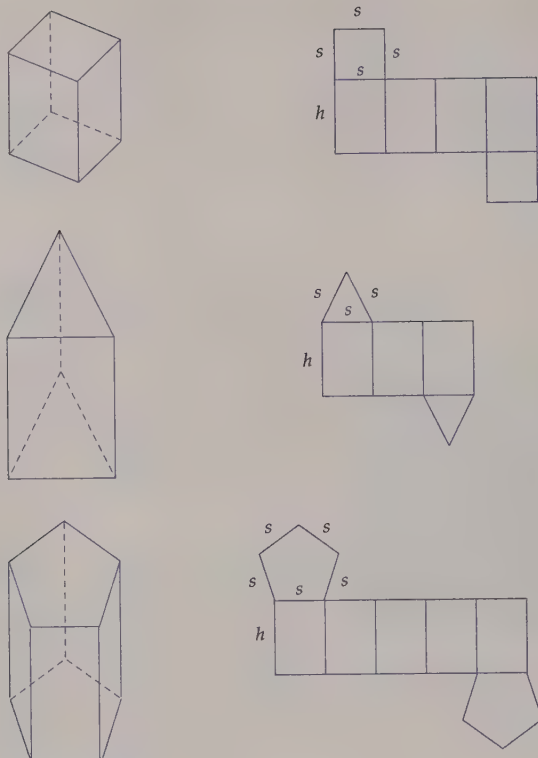


FIGURE 10.23

If you constructed nets in Exploration 10.15, you saw that each prism has a bottom and a top base that are congruent. Therefore, the areas of the top and bottom bases will be equal. Similarly, the lateral surfaces (sides) of all these prisms are congruent; therefore, their areas will be equal. The number of lateral surfaces is equal to the number of sides of the base. If we let this number be n , we can state a generic formula for the surface area of all prisms whose bases are regular polygons:

Surface area = 2 times the area of one base + n times the area of one lateral face

Because the area of the base will not be the same in all cases, we can let B stand for the area of the base. The area of each of the lateral faces is equal to the side s times the height h (Figure 10.23), or $s \cdot h$, so we can now present a general formula:

$$\text{Surface area of an } n\text{-gon prism} = 2B + nsh$$

We can further simplify the formula by noting that ns is equal to the perimeter p of the base. Thus, an alternative representation of this formula (one that will be more useful later) is

$$\text{Surface area of a prism} = 2B + ph$$

Understanding the Surface Area of Cylinders

Just as we saw a relationship between cylinders and prisms when we first examined them in Chapter 8, there is a relationship between the formulas

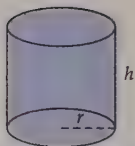


FIGURE 10.24

for the surface areas of cylinders and prisms. In other words, the formulas for the surface areas of cylinders and prisms are connected. How do you think we might determine the surface area of a cylinder (see Figure 10.24)? Even if you can't figure out the whole formula, write down what you do know. Then read on. . . .



DISCUSSION

One of the connections between prisms and cylinders is that both have two congruent bases. Thus, to find the surface area of a cylinder, we have to find the area of one base and multiply that by 2. What about the lateral surface area? How do you think we might find that? If you are not sure, find a cylinder. (Use an empty toilet paper tube or paper towel tube, or wrap a piece of paper around a soup can or soda can.) What is the shape of the lateral face? Do this yourself before reading on. . . .



Yes, it is a rectangle! The height of the rectangle is equal to the height of the cylinder. What about the length of the base of the rectangle? How is it connected to the cylinder's base? Think and then read on. . . .



The length of the base of the rectangle is equal to the circumference of the circle! See Figure 10.25. With this knowledge, see if you can now discover the formula for the surface area of any cylinder in terms of the radius of the cylinder and the height of the cylinder. Then read on. . . .

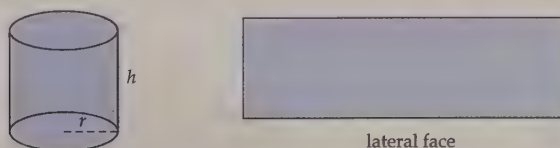


FIGURE 10.25

- The area of the base is πr^2 , and because we have two bases, the area of the two bases is $2\pi r^2$.
- The area of the rectangle = (the circumference of the circle) times (the height of the cylinder); that is, the area of the rectangle = $2\pi rh$.

Thus, the surface area of a cylinder = $2\pi r^2 + 2\pi rh$

Understanding the Surface Area of Pyramids

Whether we have a triangular pyramid, a square pyramid, or whatever, we have a base, and we have as many lateral faces as we have sides of the base. We will develop the formula for pyramids in which the apex is directly above the center of the base so that the lateral faces will be congruent triangles. Can you determine the surface area of the pyramid in Figure 10.26, if s represents the length of each side of the base and l is the slant height of the pyramid—that is, the altitude of each of the lateral faces? Try to do so before reading on. . . .



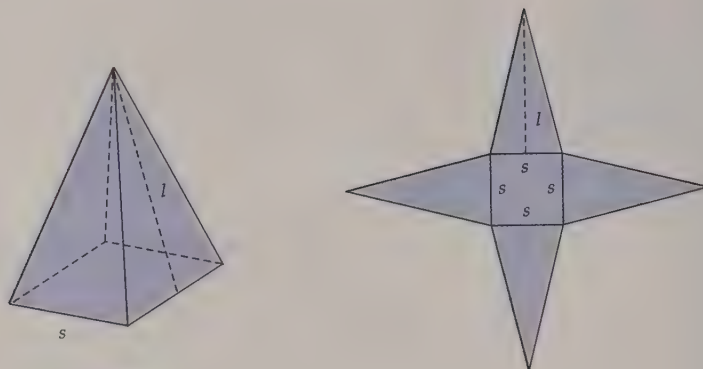


FIGURE 10.26

$$\text{The surface area of the pyramid above} = s^2 + 4\left(\frac{1}{2}sl\right) = s^2 + 2sl$$

In order to determine the general formula, we need to make two substitutions. First, as before, we will let B represent the area of the base of the pyramid. Second, the perimeter of the base of the pyramid in the formula above is equal to $4s$. Because we have a $2s$ in the second part of the formula, we have, in essence, one-half of the perimeter. Thus, we will substitute $\frac{1}{2}p$ for $2s$. It will soon become apparent why we want to make this substitution. Thus, substituting B for s^2 and $\frac{1}{2}p$ for $2s$, we have

$$\text{Surface area of a pyramid} = B + \frac{1}{2}pl$$

Understanding the Surface Area of Cones

Just as we saw connections between the surface areas of prisms and cylinders, there are connections between the surface areas of pyramids and cones. Both objects have bases, and both have slant heights (see Figure 10.27). If we model the formula for the surface area of a cone after that of a pyramid, we have

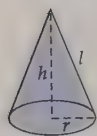


FIGURE 10.27

$$\text{The surface area of a pyramid} = B + \frac{1}{2}(\text{perimeter})(\text{slant height})$$

Thus,

$$\text{The surface area of a cone} = B + \frac{1}{2}(\text{perimeter})(\text{slant height})$$

What do we know about B , the area of the base of the cone? What is the equivalent of $\frac{1}{2}p$? What do you think?

All cones have a circular base, and so the area of the base B is equal to πr^2 . The perimeter (circumference) of a circle is $2\pi r$; thus, the value of $\frac{1}{2}p$ for the cone is πr .

Therefore, we can say that

$$\text{Surface area of any cone} = \pi r^2 + \pi rl$$

Understanding the Surface Area of Spheres

Understanding the derivation of the formula for the surface area of a sphere is somewhat sophisticated, so we will not go into the derivation here.

$$\text{Surface area of a sphere} = 4\pi r^2 \quad \text{where } r \text{ is the radius of the sphere}$$

One interesting connection is that the surface area of the sphere is exactly four times the area of a great circle of the sphere. If you are indeed actively

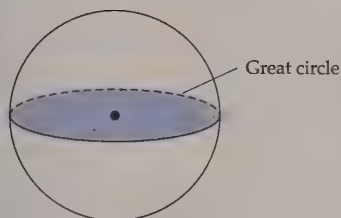


FIGURE 10.28

reading this book, you are now thinking that you need to know what a great circle is to understand this relationship. What do you think a great circle is? Think of the planet Earth as a sphere. What might be a great circle on the Earth's surface? Stop and think before reading on. . . 

Imagine passing a plane through the center of a sphere. The intersection (cross section) of that plane and the sphere is a great circle (see Figure 10.28). Thus, any sphere has an infinite number of great circles. On Earth's surface, the equator and all lines of longitude are great circles.

Volume

In Section 10.2, we connected the terms *perimeter* and *surround*, and we connected the terms *area* and *cover*. With respect to volume, we can say that **volume** deals with how much it will take to *fill* an object. Exploration 10.16 also helps you to understand basic concepts and formulas related to volume.

Understanding the Volumes of Prisms

Let us first develop the formula for the volume of a rectangular prism (more commonly known as a box). Imagine a box whose base is 4 centimeters by 3 centimeters and whose height is 2 centimeters (see Figure 10.29). If we were to fill that box with little cubes, each with dimensions 1 centimeter by 1 centimeter—that is, with a volume of 1 cubic centimeter—how many little cubes would it take?

It would take 24 little cubes—that is, $4 \cdot 3 \cdot 2$. Therefore, it makes sense to say that *we can determine the volume of a rectangular prism by multiplying its length, its width, and its height: $V = l \cdot w \cdot h$.*

Volume of an oblique prism What about an oblique prism? What if the two prisms in Figure 10.30 have the same height and the same base? Will they have the same volume, or will their volumes be different? What do you think?

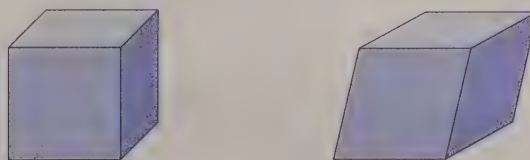


FIGURE 10.30

An analogy that is helpful is to think of a stack of paper. We can arrange the stack of paper to look like each of the diagrams in Figure 10.31. However, we still have the same volume of paper. This principle was first articulated by an Italian mathematician, Bonaventura Cavalieri, in the early seventeenth century. Cavalieri proved that two solids with congruent bases and the same height will have the same volume if the following condition is true: Each cross section that is parallel to the plane of the bases has the same area for each solid.



FIGURE 10.31

Thus, we can say that the volume of any prism can be determined by multiplying the area of its base by its height.

Understanding the Volumes of Cylinders

What about cylinders? Can you connect what we have learned about determining the volume of prisms to determining the volume of cylinders? Think and then read on. . . .

The following discussion does not constitute a proof of the formula for determining the volume of a cylinder. Rather, it is designed to help you make sense of the formula. Recall the sentence in NCTM Curriculum Standard 3: Students should “believe that mathematics makes sense” (*Curriculum Standards*, p. 29).

In Section 10.2, we used our understanding of the area of a square to shape our understanding of the area of a circle. We will do something similar here. Consider the cylinder at the left in Figure 10.32. We can approximate the volume of the cylinder by breaking it down into a number of prisms—in this case, five prisms, as shown in the middle figure. Recall Investigation 10.6 (How Big Is the Footprint?) and Exploration 10.13 (Irregular Areas). We know that the volume of each of the prisms is Bh . Thus, if we denote the area of the bases of the prisms as B_1, B_2, B_3 , and so on, we can say that the volume of the figure in the middle diagram is $B_1h + B_2h + B_3h + B_4h + B_5h = (B_1 + B_2 + B_3 + B_4 + B_5)h$. That is, the volume of this object is equal to the product of the height and the total area of the bases. Thus, it makes sense to conclude that the volume of a cylinder is equal to the product of the height and the total area of the bases. By breaking the cylinder into smaller and smaller prisms, as in the figure at the right, we can come closer and closer to the actual volume, just as we can closely approximate the area of a circle by covering it with a grid with smaller and smaller squares.

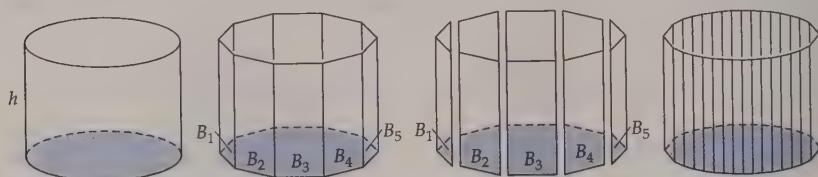


FIGURE 10.32

Finally, just as we applied Cavalieri's principle to state the formula for finding the volume of any prism, not just right prisms, so too we can apply Cavalieri's principle to state the formula for finding the volume of any cylinder, not just right cylinders:

$$V = Bh \quad \text{where } B \text{ represents the area of the base}$$

Understanding the Volumes of Pyramids

Let's look at pyramids. The formula is difficult for most people to develop, but it becomes much more understandable with concrete models.

We will illustrate the formula using a cube. It so happens that three congruent pyramids will fit inside the cube. Most people need to see this to believe it, and Figure 10.33 provides the necessary information to make three congruent pyramids that can be joined together to make a cube.

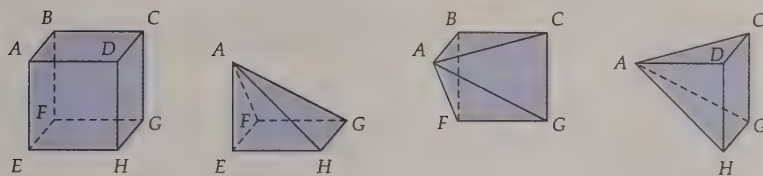


FIGURE 10.33

At this moment, let us examine the implications of this relationship for determining the volume of a pyramid. On the basis of this information, what do you predict will be the formula for the volume of a pyramid? Think before reading on. . .

If three congruent pyramids can be fit into the cube, and the bases of the pyramids are congruent to the base of the cube, this means that the volume of the pyramid is one-third the volume of the cube and that the formula for finding the volume of a pyramid is

$$V = \frac{1}{3} Bh \quad \text{where } B \text{ is the area of the base (any face) of the cube}$$

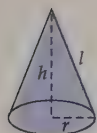


FIGURE 10.34

Understanding the Volumes of Cones

What about cones? Knowing the formula for the surface area of a pyramid and thinking of the similarities between pyramids and cones, can you hypothesize a formula for the volume of a cone with radius r , height h , and slant height l (see Figure 10.34)? Try to do so before reading on. . .

We can deduce that just as the volume of a pyramid is one-third the volume of a prism with the same base and height, the volume of a cone must be one-third the volume of a cylinder with the same base and height:

$$\text{Volume of a cone is } V = \frac{1}{3} Bh$$

Understanding the Volume of Spheres

As was true of the formula for the surface area of a sphere, the derivation of the formula for the volume is rather sophisticated.

$$\text{Volume of a sphere} = \frac{4}{3} \pi r^3 \quad \text{where } r \text{ is the radius of the sphere}$$

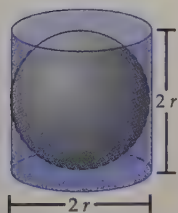


FIGURE 10.35

There is an interesting connection between the volume of a sphere and the volume of a cylinder that you can verify if your instructor has a hollow cylinder and sphere of the same radius. The height of the cylinder must be equal to the diameter of the cylinder. Recall that the volume of a cylinder is Bh ; if the base of the cylinder is a circle of radius r and the height of the cylinder is $2r$, then the volume of that cylinder is $2\pi r^3$.

Imagine first placing the sphere inside the cylinder (Figure 10.35). Next, imagine filling the cylinder with water. Finally, we take out the sphere and put the water back in the cylinder (Figure 10.36). If this experiment is done

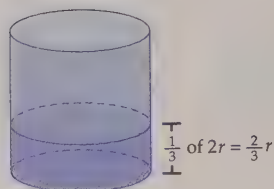


FIGURE 10.36

carefully, the height of the water will be $1/3$ the height of the cylinder. It logically follows that the volume of the sphere is $2/3$ the volume of the cylinder, so we have

$$\text{Volume of sphere} = \frac{2}{3}(2\pi r^3) = \frac{4}{3}\pi r^3$$

Archimedes was the first person to prove the formula for the volume of a sphere.



BEYOND THE CLASSROOM

I hope that in this course, you are coming to realize that mathematics is more than just formulas. With respect to understanding spheres, one of the more interesting mathematical problems is determining how to represent that sphere in two dimensions—for example, making a map of the planet Earth.

If you look at the traditional (Mercator) map of the world (Figure 10.37), you can see that this map misrepresents the relative sizes of the continents. For example, the area of Africa is $1\frac{1}{2}$ times the area of North America. However, North America appears to be much larger than Africa in the Mercator projection. In 1974, the Peters projection (Figure 10.38) was created to represent accurately the

relative sizes of Earth's land masses.

The mathematical problem here is representing a three-dimensional object in two dimensions. Thus, if you choose to make a map of the planet that is rectangular in shape (there are other possibilities), either you can have the shapes accurate (that is, this is what the continents would look like viewed from space) or you can have the relative sizes accurate, but you cannot have both! This is one of but many examples where the mathematical answer to a problem is “There is no perfect solution.” There are many web sites where you can explore this map-making issue in more depth, and there are many web sites with lesson plans for elementary schools.

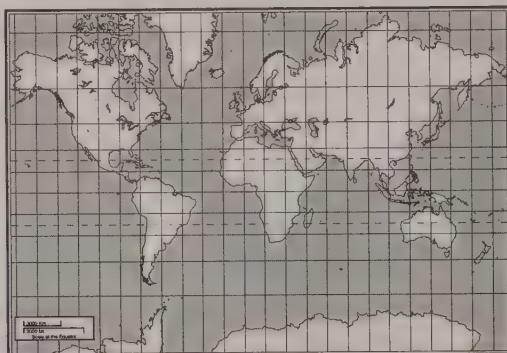


FIGURE 10.37

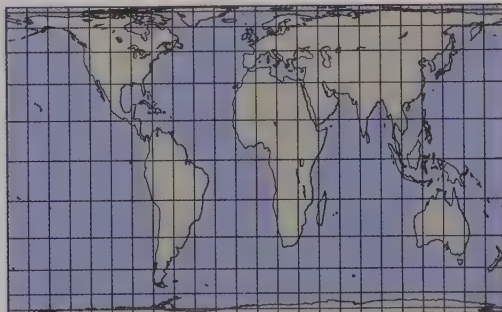


FIGURE 10.38

Source: Mercator projection map and Gall-Peters equal-area projection map, from *Which Map Is Best? Projections for World Maps* (1986). Courtesy of American Congress on Surveying and Mapping. Used with permission.

INVESTIGATION 10.8 Are Their Pictures Misleading?

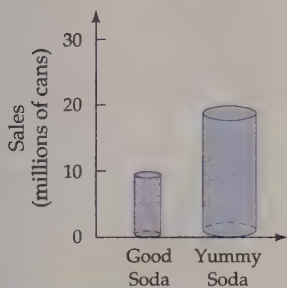


FIGURE 10.39

More and more newspapers, magazines, and brochures that companies put out are using graphs to display data in a way that will catch the reader's eye. The hypothetical example below illustrates the dangers. Let's say that Yummy Soda sold twice as much soda as Good Soda last year. The graph in Figure 10.39 is a pictorial representation of the sales of the two sodas that Yummy included in an ad. Why do you think Good Soda might object to the graph and actually file a lawsuit for unfair advertising? Write down your thoughts before reading on. . . .

DISCUSSION

If the reader looks at this as simply a "cute" bar graph, the height of the Yummy bar is twice the height of the Good bar. However, the diameter of the base of the Yummy can is twice the diameter of the base of the Good can, causing the Yummy can to look more than twice as big as the Good can. What is the ratio of the volumes of the two cans? Determine this before reading on. . . .

One solution is to take the actual dimensions:

- The Good Soda can has a radius of 1 millimeter and a height of 10 millimeters.
- The Yummy Soda can has a radius of 2 millimeters and a height of 20 millimeters.

Using the volume formula for cylinders, $V = \pi r^2 h$, we have

- Volume of Good Soda can = 10π
- Volume of Yummy Soda can = 80π

That is, the Yummy Soda can has 8 times the volume of the Good Soda can.

Note: Do you see why we used π instead of using a decimal approximation? We also could have used the ratios; that is,

$$\text{Volume of Good Soda can} = \pi r^2 h$$

$$\text{Volume of Yummy Soda can} = \pi(2r)^2(2h) = 8\pi r^2 h$$

In either case, we find that the graph gives a visual impression that Yummy Soda is selling much more than twice as much as Good Soda. (It also gives the impression that Yummy cans of soda are bigger!)

Determining the Volumes of Irregularly Shaped Objects

In Section 10.2, we acknowledged that formulas have limited usefulness in real life. Often we need to find the area of an irregularly shaped figure. Similarly, we often need to find the volume of a solid for which the formulas do not apply. How can we do this? Think about this before reading on. . . .

This is actually a very challenging problem that stumped ancient mathematician-scientists for quite some time. One way of addressing this question was developed by one of the greatest mathematicians of all times, Archimedes, who lived in the city of Syracuse, located in modern Italy, over 2000 years ago.

Archimedes was summoned by the king, who suspected that the goldsmith who had made his crown had not made it of solid gold but had mixed some silver in it. He wanted Archimedes to determine whether the crown was pure gold without having to cut into it.

Poor Archimedes. In those days, one did not disappoint kings. In fact, it was not unusual for disappointed kings to banish or even execute advisors who failed to solve their problems.

Archimedes knew that silver was only about half as dense as gold. (We determine the density of an object by dividing its mass by its volume—for example, grams per cubic centimeter.) Archimedes was aware of the formula $\text{volume} \times \text{density} = \text{mass}$ (for example, $\text{cm}^3 \times \text{g}/\text{cm}^3 = \text{g}$). Thus, if he could determine the volume of the crown, he would be able to solve the problem. But how was he to find the volume of the crown accurately?

Some time later, possibly that same day, Archimedes was taking a bath when the solution suddenly came to him: He could determine the volume of the crown by submerging it in water, because two objects with the same volume will displace the same amount of water. Legend has it that he became so excited that he ran outside, still naked, shouting “Eureka,” which means, in Greek, “I have found it!”

How did this discovery enable him to determine whether the crown was counterfeit?

Now that he knew a way to determine the volume with some precision, he could use this number and the density of gold in the formula given above to determine how much the crown would weigh if it were pure gold. (It turns out that the goldsmith had indeed cheated the king!)

INVESTIGATION 10.9 Finding the Volume of a Hollow Box

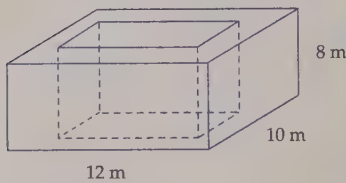


FIGURE 10.40

Imagine a large object that will be used to hold water or other materials (see Figure 10.40). The outer dimensions are 12 meters by 10 meters by 8 meters. The walls are 1 meter thick, and so is the floor. What is the volume of the container in cubic meters? For example, if we were to make this object from concrete, we would need to know how much concrete to order. Work on this until you determine the volume or become stuck. Then read on. . . 

DISCUSSION

STRATEGY 1: Break the problem into parts

Just as a rectangular prism has six surfaces, a prism with no top will have five surfaces. Let us identify them and then determine the dimensions of each of them: bottom, front, right side, left side, back. If you did not think of this strategy, seeing if you can use it to determine the volume will be a useful exercise of your problem-solving tools. If you used a different strategy, do you get the same answer using this strategy? Work on this before reading on. . . 

The volume of the bottom will be

$$12 \text{ m} \cdot 10 \text{ m} \cdot 1 \text{ m} = 120 \text{ m}^3$$

The front and back sides are congruent, and the volume of each will be

$$12 \text{ m} \cdot 7 \text{ m} \cdot 1 \text{ m} = 84 \text{ m}^3$$

(Do you see why it is not 8 m? If not, make a model of the container.)
The right and left sides are congruent, and the volume of each will be

$$8 \text{ m} \times 7 \text{ m} \cdot 1 \text{ m} = 56 \text{ m}^3$$

(Do you understand these dimensions? If not, make a model.)

STRATEGY 2: See the problem from a different perspective

For example, determine what the volume would be if the container were not hollow. Then determine the volume of the hollow region. Then subtract! Do you understand this strategy?

Volume of the whole region:

$$12 \text{ m} \cdot 10 \text{ m} \cdot 8 \text{ m} = 960 \text{ m}^3$$

Volume of the hollow:

$$10 \text{ m} \cdot 8 \text{ m} \cdot 7 \text{ m} = 560 \text{ m}^3$$

What is the difference between the two? Does this answer match the answer from Strategy 1? Exploration 10.17 provides more opportunities to apply your understanding of volume concepts.

Again it is important to note that this discussion illustrated only two of several possible ways to solve this problem.

INVESTIGATION 10.10 Surface Area and Volume

In Section 10.2, we discovered that perimeter and area are not related in a simple way; for example, if you double the perimeter, that doesn't necessarily mean that you double the area. Let us investigate the relationship between surface area and volume. As you can imagine, there are many applications of this relationship in fields outside mathematics!

Consider a set of eight small cubes, each of whose dimensions are 1 centimeter by 1 centimeter by 1 centimeter (see Figure 10.41).



FIGURE 10.41

What arrangement of those cubes will have the smallest surface area?

What arrangement of those cubes will have the largest surface area?

Work on these questions before reading on. . . . 

DISCUSSION

There are two aspects of this question that are not well defined. First, some students, applying this idea to real-life phenomena such as melting ice, consider the "surface area" to be the amount of surface exposed. Thus, they do not count the base. Other students imagine having to paint the whole arrangement, and thus they do count the base. In this investigation, when we say "surface area," we will mean the latter.

The second part of the question that is not well defined is “arrangement.” In this investigation, in order for something to count as an arrangement, at least one entire face of each cube will have to cover an entire face of another cube. Thus, simply putting the six cubes down on the table so that they are all disjointed does not count as an “arrangement” as I am defining this problem. With these two aspects of the problem now well-defined, solve the problem before reading on. . . . 

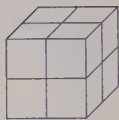


FIGURE 10.42

Arranging the cubes into a large cube will create the smallest surface area (see Figure 10.42). If you didn't find this, make them into a cube (or draw a diagram) and determine the surface area.

Because all six faces of the cube are congruent, we can multiply the area of one face (4 cm^2) by 6 (the number of faces), to get 24 cm^2 .

The largest surface area is obtained by arranging the six cubes in a line (see Figure 10.43). If you didn't find this, please determine the surface area. If you did, you may want to compare your solution path with that of another student. Then read on. . . . 



FIGURE 10.43

STRATEGY 1: See this arrangement as a prism with six sides

The surface area of the front, top, back, and bottom is 32 cm^2 .

The surface area of the two bases (the two sides in this diagram) is 2 cm^2 .

Thus, the total surface area is 34 cm^2 .

STRATEGY 2: See this arrangement as two end cubes and six cubes in the middle

The surface area of the six cubes in the middle is 24 cm^2 .

The surface area of the two cubes on the end is 10 cm^2 .

Thus, the total surface area is 34 cm^2 .

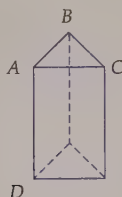
Summary

If this section has been successful, the various formulas for surface area and volume make sense. In order to make sense of these formulas, we used some strategies like the ones we used to make sense of area formulas in Section 10.2 (breaking a figure into parts) and built on previous knowledge. Connections were an important aspect of making sense of these formulas. That is, the formulas for finding the surface area and volume of prisms are fairly straightforward. By seeing how prisms and cylinders are similar, we could then use the prism formulas to understand the cylinder formulas. Similarly, we connected the volume of a pyramid to the volume of a cube in order to understand the formula for the volume of a pyramid. After we developed the formula for finding the surface area of a pyramid, seeing how pyramids and cones are similar enabled us to make sense of the formula for the surface area of a cone. Understanding the formulas and having a good problem-solving toolbox then enabled us to solve nonroutine problems. Finally, we examined the relationship between surface area and volume, realizing as we did so that this relationship, like that of perimeter and area, is not simple.

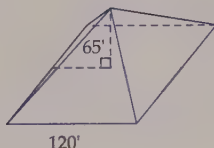
EXERCISES 10.3

1. Determine the surface area and volume of each of the following:

- a. $\triangle ABC$ is a right angle.
 $AB = 10$ inches, $BC = 12$ inches
 $AD = 3$ feet



- b. The base is a square.



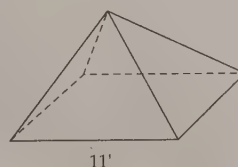
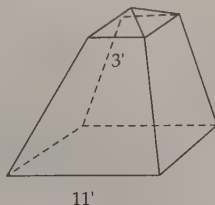
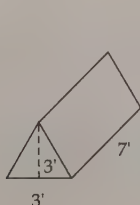
- c. The diameter is 5 feet 6 inches.
 The height is 11 feet 3 inches.



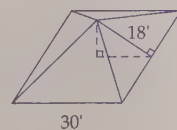
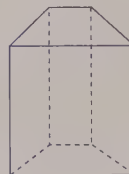
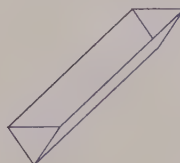
2. a. Determine the surface area and volume of the watering trough below. The length of the trough is 16 feet, the two ends of the trough are isosceles triangles whose base is 2 feet, and the height is 1 foot.

4. Before we had children, my wife and I went camping in a pup tent, shown at the left in the figure that follows. After having two children, we bought a much bigger tent, like the one at the right. Determining the volume of that tent is rather complex, so let us assume that the big tent was actually a square pyramid, like the figure at the lower right, with a height of 8 feet.

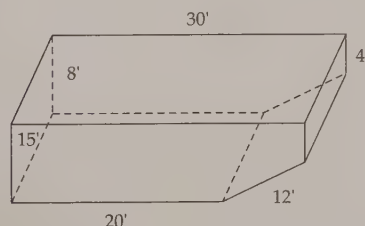
- a. How much bigger is the new tent than the old tent? First, predict the answer and explain the reasoning behind your prediction. Then do the computations.
 b. Compare the amounts of material needed to make these tents. First, predict the answer and explain the reasoning behind your prediction. Then do the computations.
 c. Using what you have learned about two- and three-dimensional figures in the last three chapters, describe your hypotheses for how you might determine the actual volume of the middle tent. The height of the truncated pyramid is 6 feet, and the height of the small pyramid at the top is 2 feet.



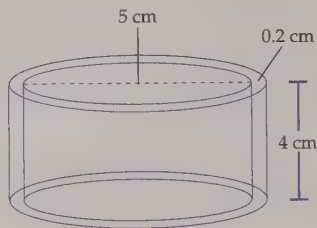
- b. Determine the surface area of the inside of the room below and the volume of the room. The base of the room is in the shape of an isosceles trapezoid. The longer base is 12 feet 4 inches, and the shorter base is 8 feet 9 inches. Both slant sides are 13 feet. The height of the room is 9 feet.
 c. The base of the pyramid is a square. The slant height of the pyramid is 18 feet. Determine the surface area and volume.



3. a. Determine the surface area and volume of the swimming pool below.
 b. Determine how many gallons of water the pool will hold.



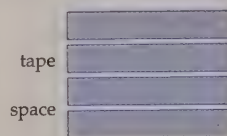
5. The napkin ring pictured is to be resilvered. How many square millimeters of surface area must be covered?



6. This past year the local elementary school has bought 50 boxes of paper, each box containing 10 reams. Each ream contains 500 sheets of paper. There are two shelves in the copy room, each 12 inches wide and 5 feet long. The second shelf is 2 feet above the first shelf and 3 feet below the ceiling. Assuming that the shelves are sturdy, how many sheets of paper can be stored on the shelves? What percent of the yearly purchase is this?
7. The U.S. federal debt, at the time of the writing of this exercise, was \$4.8 trillion. One way to get a sense of the enormity of that number is to imagine how much space 4.8 trillion one-dollar bills would occupy. Determine the space. Then recommend a denomination that would enable a semitrailer to haul that much money.
8. The planned Three Gorges Dam in China will require approximately 950 million cubic feet of concrete. Do something with this number to make it more understandable. For example, it would be equivalent to a block 1 mile long, 1 mile wide, and x feet high.
9. The Great Pyramid was built around 2600 B.C. and is considered to be one of the "Seven Wonders of the World." It was built for Pharaoh Khufu. The pyramid was made from about 2.3 million stone blocks, whose total weight was about 6 million tons! It has been estimated that it took 100,000 workers about 30 years to make it, and we believe they hauled the blocks only during the three months of the year when the Nile was flooded—that is, when the farmers weren't farming. The pyramid has a square base, and each side is 768 feet. The height of the pyramid is 481 feet.
- Compare the volume of the pyramid to the volume of the building in which you are taking this course.
 - Compare the volume of the pyramid to the volume of a building 100 feet high covering a football field (120 yards long and 55 yards wide).
 - Compare the Great Pyramid to the Transamerica Pyramid in San Francisco. The height and base of the Transamerica Pyramid are 870 feet and 117 feet, respectively.
 - When I was researching this book, one of the books I read said that the height of each of the triangular faces of the pyramid was 232 meters. Is that figure compatible with the height of the pyramid given here, 481 feet?

- Many books have been written about many of the "coincidental" measurements of the pyramid. For example, it is said that the area of one of the triangular faces of the pyramid is equal to the square of the height of the pyramid. Are these two dimensions equal?

10. Joe has eight blocks of ice, each measuring 30 centimeters by 30 centimeters by 30 centimeters.
- How should he stack them if he wants them to melt as slowly as possible?
 - How should he stack them if he wants them to melt as quickly as possible?
11. Let's say a company is manufacturing juice boxes that are rectangular prisms with dimensions 4 inches by 3 inches by 4 inches. The company has a warehouse whose dimensions are 40 feet by 30 feet by 20 feet. If the company makes 200,000 juice boxes, will it be able to store all the juice boxes in the warehouse?
12. Sharon has 150 cassettes and has decided to make a wooden shelf to hold them. The space between cassettes need only be 2 millimeters. Design a shelf that will hold at least 200 cassettes.



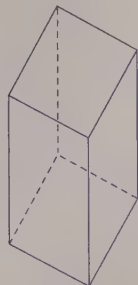
13. Given eight cubes, each with dimensions of 1 centimeter by 1 centimeter by 1 centimeter, can you find arrangements that will have surface areas for every whole number between the minimum and maximum that were found in Investigation 10.10—that is, between 24 square centimeters and 34 square centimeters?
14. One inch is about 2.54 times as long as a centimeter. We could say that it takes just over $2\frac{1}{2}$ centimeters to make an inch.
- Predict how many cubic centimeters it would take to fill a container that was 1 cubic inch in volume. Explain your reasoning.
 - Determine the actual ratio.
 - Assess your prediction, both in terms of percent error and in terms of reasoning.

_____ 1 cm
_____ 1 in.



15. Consider two cubes. The smaller cube measures 4 inches on a side, and the larger cube measures 10 inches on a side. What is the ratio of the surface areas of the two cubes? What is the ratio of the volumes of the two cubes?
16. Consider the two similar rectangular prisms below (not drawn to scale). The dimensions of the smaller prism are

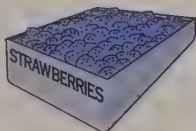
3 inches by 2 inches by 4 inches, and the dimensions of the larger prism are 9 inches by 6 inches by 12 inches. That is, the ratio of their sides is 1:3.



- a. Predict the ratio of their surface areas and explain your reasoning. Then determine the actual ratio of the surface areas. If your prediction was off, explain the error in your prediction.
 - b. Predict the ratio of their volumes and explain your reasoning. Then determine the actual ratio of the volumes. If your prediction was off, explain the error in your prediction.
17. When my children were younger, we bought a set of dominoes. A complete set had every possible combination of numbers from 0 (represented by a blank) to 6. For example:



- a. How many dominoes are in a complete set?
 - b. If each domino is 1 inch $\times$ 2 inches $\times$ $\frac{3}{8}$ inch, design a carrying case that would hold all of them.
18. This problem is taken from a book called *Mathematical Investigations: Book One*.⁸ I have noticed that “flats” of berries and “flats” of young plants are not cubical in shape, but rather are rectangular prisms (see the picture below).



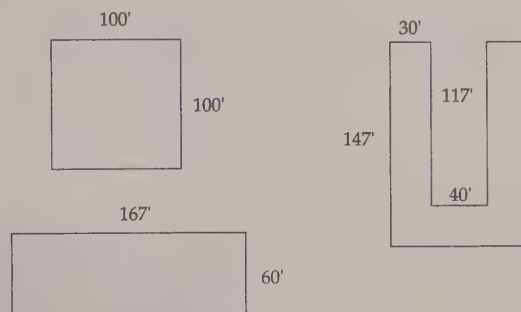
Suppose you wanted a flat that would hold 4000 cubic centimeters of strawberries. What would be the dimensions of the prism that would require the least amount of cardboard to make?

19. a. The cake pictured has been cut into three equal pieces. One more person comes. How can you make one cut in the cake so that each of four persons will get the same amount of cake?

- b. The cake pictured below has been cut into n equal pieces. One more person comes. How can you make one cut in the cake so that each of $n + 1$ persons will get the same amount of cake?



20. Below are three possible floor plans for a new office building. All the plans have almost exactly the same amount of floor space.
- a. Analyze the advantages and disadvantages of each of these designs.
 - b. If each building were to be made of bricks, compare the relative amount of bricks needed for each.
 - c. If the outside walls of each building were 10 feet tall and 1 gallon of paint covers approximately 500 square feet, determine the amount of paint that would be needed to paint each building.



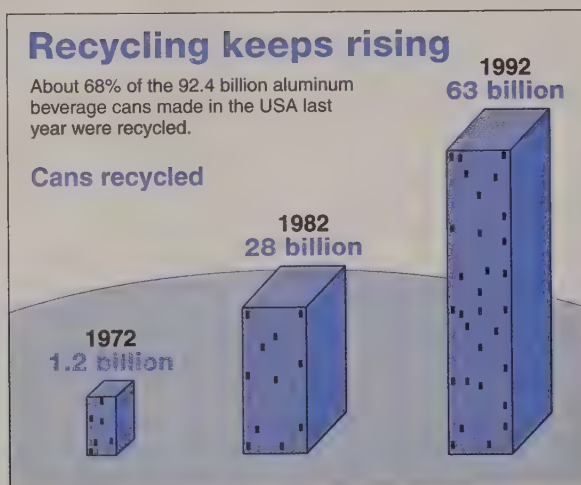
21. Each American produces about 3.5 kilograms of solid waste (garbage) a day. If 1 cubic meter of garbage weighs 90 kilograms, determine the size of the waste produced each day by:
 - a. The population of your college
 - b. The people in your state
 - c. The people in the United States
 In each case, describe the assumptions you made in order to answer the question.
22. Many automobile commercials have stressed that the new model being advertised has more carrying capacity.
 - a. How might you determine the carrying capacity of two similar automobiles?
 - b. Select two comparable automobiles and determine the carrying capacity of each. Look up the carrying capacity given by the manufacturers. Do you agree with their amounts? If not, do you think one of the figures is erroneous, or do you think the term was defined differently?

⁸Randall Souviney, Murray Britt, Salvi Garguilo, and Peter Hughes, *Mathematical Investigations: Book One* (Palo Alto, CA: Dale Seymour Publications, 1988), pp. 10, 14. Reprinted by permission.

23. This problem has been adapted from *Mathematical Investigations: Book One*⁹. When a boat or ship sinks, the Red Cross recommends that people huddle together to stay warm. Let's say four adults huddle together to keep warm. Determine a way to answer the following question: What is the ratio of body surface area exposed to the cold water when floating alone (in a life jacket) to body surface area when huddling?



24. Analyze the accompanying graph for its accuracy. First, predict whether you think it is accurate or not and explain your reasoning. Then do the measuring and calculating and explain your interpretation of the numbers. Then assess your prediction. If the prediction and the reasoning behind it were accurate, great. If either the prediction or the reasoning behind it was not accurate, explain the errors in the prediction.



Source: "Recycling Keeps Rising," from *USA Today*. Copyright 1993, *USA Today*. Reprinted with permission.

25. Make a net for a soccer ball. Describe your process. For example, how did you determine how many faces the soccer ball had and how to put them together? Describe any discoveries, conjectures, and questions that came from this exploration.
26. Take a standard $8\frac{1}{2}$ - by 11-inch sheet of paper and determine how to cut and fold the paper so that you could make a rectangular prism with no top that would contain the greatest volume.
27. Predict how many feet of floor space there are in the building in which you are taking this class. If it is a large building, your instructor may wish to have you determine the floor space in a specific subset of the whole building. As accurately as you can, determine the actual floor space.

⁹Randall Souviney, Murray Britt, Salvi Garguilo, and Peter Hughes, *Mathematical Investigations: Book One* (Palo Alto, CA: Dale Seymour Publications, 1988), pp. 10, 14. Reprinted by permission.

CHAPTER SUMMARY

1. To measure an object means to assign a number to some attribute of the object.
2. Two objects do not have to be congruent to have the same measure. For example, two objects can have different shapes but the same volume.
3. In order to measure, we must decide on a unit.
4. All units are conventions, as opposed to absolutes.
5. Virtually all measurements are approximations.
6. On some occasions, we cannot directly measure an attribute and must rely on indirect measurement.
7. Different attributes of an object—perimeter and area, and surface area and volume—are not related in simple ways.

Basic Concepts

Section 10.1: Systems of Measurement

metric system 551
 U.S. customary system 551
 meter 552
 liter, gram 553
 weight, mass 553

precise 555 accurate 555
 greatest possible error (GPE) 555
 relative error 555

Section 10.2: Perimeter and Area

perimeter 560, circumference 560
 π 560 area, base, height 561
 Pythagorean theorem 565
 Understanding the area formulas for
 parallelograms 563, triangles 564,
 trapezoids 564, and circles 566
 Areas of irregularly shaped figures 568
 Relationships between perimeter and area 569

Section 10.3: Surface Area and Volume

Understanding the surface area formulas for
 prisms 575, cylinders 576, pyramids 577,
 cones 578, and spheres 578
 Understanding the volume formulas for
 prisms 579, cylinders 580, pyramids 580,
 cones 581, and spheres 581.

NCTM Standards*

This appendix includes the NCTM Curriculum Standards for Grades Pre-K through 12 as they appear in *Principles and Standards for School Mathematics*, published in 2000 by the National Council of Teachers of Mathematics. For more information on these standards, visit the NCTM web site at www.nctm.org/.

Standards for School Mathematics

■ Number and Operations

Instructional programs from prekindergarten through grade 12 should enable all students to—

- understand numbers, ways of representing numbers, relationships among numbers, and number systems;
- understand meanings of operations and how they relate to one another;
- compute fluently and make reasonable estimates.

■ Algebra

Instructional programs from prekindergarten through grade 12 should enable all students to—

- understand patterns, relations, and functions;

*Reprinted with permission from *Principles and Standards for School Mathematics*, Copyright © 2000 by the National Council of Teachers of Mathematics. All rights reserved.

- represent and analyze mathematical situations and structures using algebraic symbols;
- use mathematical models to represent and understand quantitative relationships;
- analyze change in various concepts.

Geometry

Instructional programs from prekindergarten through grade 12 should enable all students to—

- analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
- specify locations and describe spatial relationships using coordinate geometry and other representational systems;
- apply transformations and use symmetry to analyze mathematical situations;
- use visualization, spatial reasoning, and geometric modeling to solve problems.

Measurement

Instructional programs from prekindergarten through grade 12 should enable all students to—

- understand measurable attributes of objects and the units, systems, and processes of measurement;
- apply appropriate techniques, tools, and formulas to determine measurements.

Data Analysis and Probability

Instructional programs from prekindergarten through grade 12 should enable all students to—

- formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them;
- select and use appropriate statistical methods to analyze data;
- develop and evaluate inferences and predictions that are based on data;
- understand and apply basic concepts of probability.

Problem Solving

Instructional programs from prekindergarten through grade 12 should enable all students to—

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

Communication

Instructional programs from prekindergarten through grade 12 should enable all students to—

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely.

Connections

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics.

Representation

Instructional programs from prekindergarten through grade 12 should enable all students to—

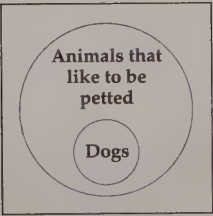
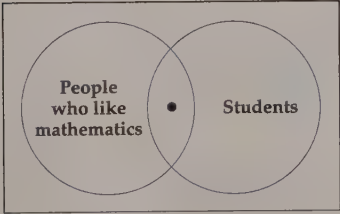
- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

ANSWERS TO SELECTED EXERCISES

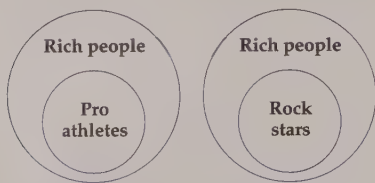
Chapter 1 Exercises, page 27

1. 30 pigs and 139 chickens
3. 500 \$2 tickets and 100 \$5 tickets
5. 8 tricycles and 24 bicycles
11. 675 cases in five days
13. 7 economy models
15. 73,500 apples
19. Between 35,000 and 39,000, depending on assumptions made.
23. $73 + 82$; $91 + 64$
25. The millionth second occurs after 11.57 days, which will be on the 12th day—that is, January 12.
26. The ball bounced 4 times.
27. a. 12 ways
28. a. $\square$ c. 13 e. 25 g. 2
30. a. One wording: There are some 4s at the beginning, followed by some 8s, and then a 9 at the end. We can be more specific if we talk about the 1st, 2nd, 3rd, etc. product. In this case, the n th product will consist of $(n + 1)$ 4s, (n) 8s, and a 9 at the end.
- c. Answers will vary. You might explore what happens if you increase the first multiplicand but keep the second one at 67; i.e., 667×67 , 6667×67 , etc. You might explore what happens if there is one more 6 in one multiplicand; i.e., 667×67 , 6667×6667 . You might explore what happens if you try different numbers; e.g., 57×57 , 557×557 ; in this case, the pattern is not as “elegant,” but all the numbers do end in 249.
35. a. Words: The sum of the first n terms is equal to 1 less than the $(n + 2)$ th term.
Notation: $A_1 + A_2 + A_3 + \cdots + A_n = (A_{n+2}) - 1$
b. Words: The product of the n th and $(n + 2)$ th terms is always 1 more or 1 less than the square of the $(n + 1)$ th term.
Notation: $A_n \times A_{n+2} = (A_{n+1})^2 \pm 1$
 $|(A_n \cdot A_{n+2}) - (A_{n+1})^2| = 1$
40. 42 rectangles
44. a. One solution is $519 + 327 = 846$.

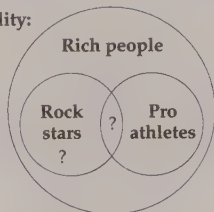
Chapter 1 Exercises, page 52

2. a. If you add two fractions that both have a 2 in the numerator, then the numerator of the product is double the sum of the two denominators, and the denominator of the product is the product of the two denominators.
In notation: $\frac{2}{a} + \frac{2}{b} = \frac{2(a + b)}{ab}$
3. a. 
- c. 
4. a. If it is fruit, then it contains sugar.
- c. If it is a cat, then it does not bark when the mail carrier comes to the door.
- e. If you want to drive a car, then you must pass a written test.
5. a. Inverse: If you do not exercise three times a week, then you will get sick.
Converse: If you do not get sick, then you exercise three times a week.
Contrapositive: If you get sick, then you did not exercise three times a week.
- c. Inverse: If Washington had crossed the Delaware, then we would have won independence from Britain.
Converse: If we had not won independence from Britain, then Washington would not have crossed the Delaware.
Contrapositive: If we won independence from Britain, then Washington crossed the Delaware.
6. a. Valid c. Invalid e. Invalid
7. a. Invalid. Nothing is said about the relationship between professional athletes and rock stars. Although it is possible that some rock stars are professional athletes, it does not follow from the two statements.

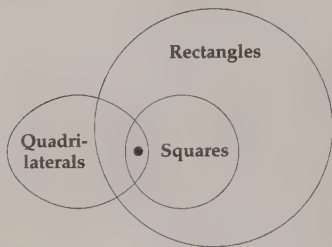
one possibility:



another possibility:



c. Valid



12. 11.25 minutes

13. a. 3 days

14. The deal depends on what your present service is. However, even if your present service costs over 10¢ a minute, it is probably not a good idea. For example, if you make a 1 minute call, you pay 10¢ + \$1.95; thus, that minute costs \$2.05. Even if you make a 30 minute call, you are charged \$3.00 (for the 30 minutes) plus \$1.95, which averages to 16.5 cents per minute.

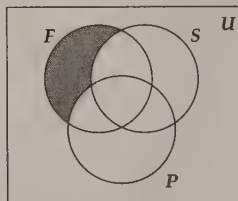
19. 10,440 handshakes

22. a. 17 bottles

Exercises 2.1, page 70

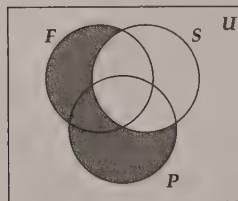
1. a. $0 \notin \emptyset$ or $0 \notin \{\}$ b. $3 \notin B$
2. a. $D \not\subseteq E$ b. $A \subseteq U$
3. a. $\{e, l, m, n, t, a, r, y\}$ and $\{x \mid x \text{ is a letter in the word "elementary"}\}$
- c. $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}; \{x \mid x \text{ is a prime number less than } 100\}$
4. b. is not well defined, because the "countries in Europe" are constantly changing and may be defined in various geopolitical terms.
6. b. $\subset$ $\{3\}$ is a subset of the set
- c. $\in$ $\{1\}$ is an element of this set of sets
- e. $\not\subset$ $\{ab\}$ is neither a subset nor an element
- f. $\subset$ the null set is a subset of every set
7. a. 64
- b. A set with n elements has 2^n subsets.

12. a.



$$F \cap (\overline{S \cup P}) \text{ or } F \cap \overline{S} \cap \overline{P}$$

c.



Nonsmokers who either are female or have health problems

Activity Options

LESSON
3

Catch of the Day

Purpose Children continue their study of living things as they measure with nonstandard units the lengths of paper fish.

Activity 1: Measuring Length

DRAWING TO LEARN

How long should we make our fish?

Management: Individuals (15–20 minutes)

Materials: Giant Math Activity Pad page 20, paper clips, construction paper, markers, crayons, scissors, stickers

1 Revisit Giant Math Activity Pad page 20. Draw children's attention to the water. *What animals and plants might live in water?*

2 Help children to chain together three to eight paper clips. Let them try to make paper fish the same length as their chains. They can decorate the fish with colorful patterns.

3 Have each child hold up his or her fish. *About how many paper clips long is this fish?* Record estimates on the chalkboard. Let the child hold up the paper-clip chain next to the fish and count the clips out loud. *How close were our estimates? Do our estimates improve with practice?*

My fish is about five paper clips long.

**Variation**

Materials: Use interlocking cubes instead of paper clips to measure the fish.

Ongoing Assessment

Can the child tell how long the fish are in paper-clip units?

Follow-up: If a child needs more practice measuring with nonstandard units, use the variation.



Journal Opportunity

Let children draw pictures and use invented spelling to describe what they did to make and measure their fish. They might want to tape more paper fish to their pages.

Students Acquiring English

Give children additional practice measuring with their paper-clip chains. Let pairs find classroom objects to measure and talk about. They can share what they find out with others.

Activity 2: Predicting Symmetry

Can we predict a design?

Management: Pairs (20–30 minutes)

Materials: Construction paper, hole punches, crayons

Invite children to predict the number and location of holes you punch in folded construction paper.



1 Fold a sheet of paper in half and punch several holes through both halves. Open the sheet to display the resulting pattern. *How many holes are there? Where are the holes? Is the design symmetrical? Why?*



2 Fold another sheet of paper in half, open it up, and punch holes in only one half.



3 *Where would the matching holes be if you had punched through both halves?* Let a volunteer mark his or her predictions with a crayon.



4 Refold the paper, and punch through the original holes. *How many holes will there be?* Open up the paper to see how close the predictions were.

5 Now have the children work in pairs. One child punches holes in half of a sheet of paper that has a fold line in the center. The other child marks where the symmetrical holes should be and then folds the paper and punches through the original holes to check.

Ongoing Assessment

Can the child accurately predict the placement of the holes?

Follow-up: The children will have opportunities to use visual-spatial reasoning in later modules.



Portfolio Opportunity

Invite the children to decorate the covers of their portfolios with symmetrical designs. Add a date to the cover.

Extension

Have the children consider this question: How many holes will there be if one side has 2 and we punch the same amount on the other side? (4) What about for 3, 4, and 5? (6, 8, 10)

Alternate Strategy

Punch or cut holes in felt shapes and have the children feel with their fingers where the holes are. Then fold the shapes and have them feel where the holes should be.

Activity Options



Measuring with Jars

Purpose Children use jars, bottles, and other containers to explore volume and to sort objects with overlapping attributes.

Activity 1: Measuring Capacity

Which holds more?

Management: Whole group (25–35 minutes)

Materials: Jars and bottles, and water, sand, or tiny seeds

Vocabulary: empty, full, holds more, holds less

- Assemble some jars and bottles from the classroom collection. *What originally came in this container? In this one? What else could you use these jars for?*
- Ask volunteers to place the jars and bottles in a row from shortest to tallest. Select two: for example, a tall, narrow olive jar and a short, wide jam jar.

Which jar is taller? Which jar do you think holds more? How can we find out?

- As one possible solution, let children try filling one jar with sand, water, or tiny seeds and then pouring the contents into the other.
- Use basic capacity words, such as *empty, full, holds more, holds less, and equals* to describe what happened.
- Repeat with other pairs of jars and bottles and then set the materials out for independent exploration.

Variation

Procedure: Pairs of children choose two different-sized jars and try to guess which holds more. They then check by filling one jar and pouring its contents into the other. Label the jar that holds more and compare the result with their estimate.

Ongoing Assessment

- What words does the child use to describe differences in capacity?
 - What reasoning does she or he use in judging comparative capacity?
- Follow-up:* Children will have further opportunities to explore volume in Activity 2 on page 139.



Journal Opportunity

Invite children to use invented spelling to write stories about collections or other things they keep in jars.

Music Connection

Sing the song "There's a Hole in the Bucket." **What could you carry in a bucket with a hole?**

Cultural Connection

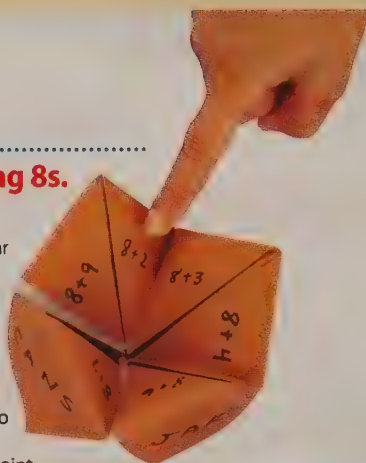
Encourage children to bring in jars and other containers from their own homes. Invite them to share what the original contents of the jar or container were. You might have children guess what they think different jars might have originally held.

Adding

Part B: Let's play Recycling 8s.

Management: Pairs

- To begin, Player A holds up the rectangular folded worksheet in his or her hands so that the two flaps with the words *Paper* and *Glass* read right-side up and the number sentences are hidden in the fold.
- Player A then inserts his or her thumbs under the flaps in front and index fingers under the flaps in back, pushing them into the corners above each word and then bringing the corners together to form a point.
- Player B picks a word from an outside flap. Player A spells out the chosen word, opening the folded worksheet forward and then sideways for each letter in the word. For example, if *Glass*, a five-letter word, is chosen, Player A opens the worksheet five times.
- Player B then picks one of the four number sentences showing, for example $8 + 5$. Player A solves the sentence out loud and opens the worksheet that number of times (13).
- Player A checks the answer by lifting the flap. If the answer is correct, the two players switch roles.



Activity 2: Identifying Attributes

What would a box look like if you laid it flat?

Management: Whole class (20–30 minutes)

Materials: Cardboard boxes, scissors, chart paper, crayons

Vocabulary: net

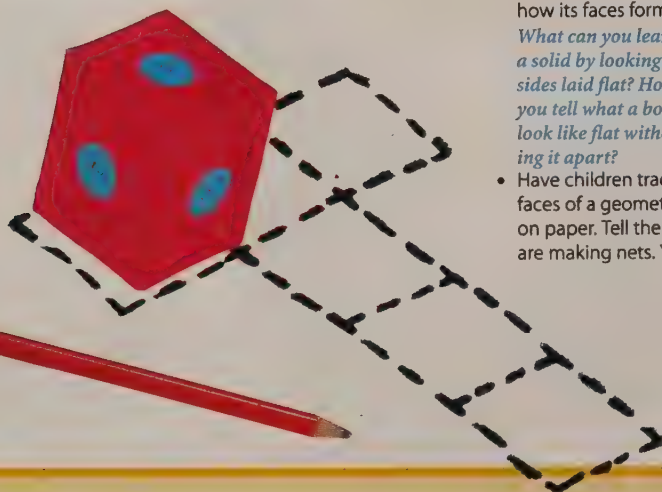
- Have children share their responses to the question above. Then cut down the edges of a box to show how its faces form a solid. *What can you learn about a solid by looking at its sides laid flat? How could you tell what a box would look like flat without taking it apart?*
- Have children trace all the faces of a geometric solid on paper. Tell them they are making nets. You may

need to direct the children to trace one face and then flip the solid and trace another face. *How can you tell if you've traced all the faces?*

- Have some children share their nets. Others can guess which shape was used to make the net. *Are all nets for the same shape alike? Why? Which solids make similar nets? Why? Post the children's nets on a bulletin board.*

Variation

Procedure: Give the children nets printed on construction paper and have them create boxes from them.



Ongoing Investigation

Have the children label objects in the classroom with proper geometric names. Leave this as a permanent display and a resource. Challenge the students to find rarely seen solids such as pyramids.

Extension

Invite children to bring in small, light cardboard boxes, perhaps from cereal or soap. Direct them to use their boxes to make nets. Then have children cut apart their boxes. **Does your net look like your flattened box? Why or why not?**

Alternate Strategy

Children who have difficulty with spatial relationships can first cut apart a box so that all sides lie flat. Then they can trace the flattened box to make a net.

Lesson 9 *continued*

Exercises and Problems

Tiles and Designs

Students look at Italian and Egyptian mosaic and tile patterns and investigate symmetry, congruence, polygons, and line segments.

► **Problem Solving Tip**

Students may find that creating a chart or diagram to help them with these exercises and problems will lead to a swifter, surer solution. Refer them to Problem Solving Strategies, Organize Information, on pages 388–389, in the Tool Kit.

Did You Know?

The earliest use of mosaics dates back as far as 3,000 B.C.; Mosaic flooring was in widespread use by about 300 B.C.

From The World Book Encyclopedia

Exercises and Problems

Tiles and Designs

1. Possible answers: one line: trapezoid; two lines: rectangle; three lines: triangle; four lines: square

Use your Tracing Tool to help you answer the questions.



Italian

Games and sports fields aren't the only places to find symmetry and congruent figures. Look at these old Italian tiles and Arabic patterns.

- 1 Which figures in this tile pattern have one line of symmetry? two? three? four? *See above.*
- 2 Find five line segments in the design. Find three lines of symmetry in the design. Can you find more? *Answers will vary.*
- 3 Find two congruent squares in the pattern. Do you think all the squares are congruent? Why? *Answers will vary; no; they may have sides of different lengths.*

378 Module 8: Puzzles, Shapes, and Games

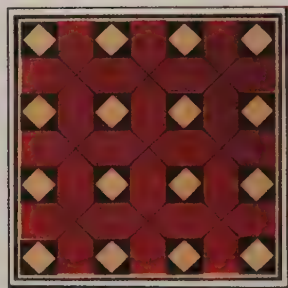
*Teaching Options**Geometry Connection*

Students can connect slides, flips, and turns with the patterns they see. Have students create and sketch a shape. Then have them use that shape to create a pattern using slides, flips, and turns. Students should write a description of their pattern.

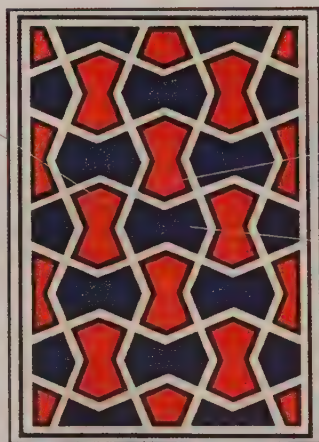
Social Studies Connection

Students might enjoy gathering and sharing tile patterns, mosaics, and designs from other countries. Have them play a show-and-tell in which they point out the lines of symmetry, polygons, and other geometric elements.

4. Possible answer: Triangles, squares, pentagons; 3, 4, 5
5. Triangles, squares, and pentagons; their respective sides and angles are congruent.
6. Yes; the acute angles of the triangles; the obtuse angle of the pentagon, the hexagon, and the octagon



Italian



Arabic

- 4 Find three different kinds of figures in the design above. If you do not know the names of the figures, tell the number of sides.
See above.
- 5 Which figures are congruent? Explain.
See above.
- 6 Find right angles in the pattern. Do you see angles smaller than right angles? larger? Where?
See above.
- 7 List all of the shapes in the pattern above. Triangle, square, pentagon, hexagon, octagon
- 8 Are the black and orange figures above congruent? How do you know?
Yes; their respective sides and angles are congruent.
- 9 Would you use a flip, slide, or turn to place figure A on top of figure B? How would you move figure A on top of figure C? *Slide and turn; slide*
- 10 How many lines of symmetry do the blue and orange figures have? Find one line of symmetry for the entire pattern. Describe it. 2; possible answer: Lengthwise from the midpoint of the top horizontal line to the midpoint of the bottom horizontal line

Section D: Congruence and Symmetry 379

Ongoing Assessment

Are students able to find and identify lines of symmetry? Are students able to draw or paste figures in a symmetrical pattern?

Follow-up Strategy Students who have difficulty might find it helpful to see visuals displayed on an overhead projector. Have students point out and draw lines of symmetry on an overhead transparency showing simple shapes.

Section Assessment

An Observation Checklist (Assessment Master 52) is available in *Assessment Options*.

Extension

Some students might be interested in exploring symmetry with three-dimensional objects. Suggest some objects such as balls, cones, and irregular pyramids. Ask students which ones could be split to produce two identical halves.

Cooperative Learning Tip

Have students discuss and answer the questions for the Exercises and Problems in small cooperative groups. This will enable students to share ideas and suggestions about the patterns and mosaics pictured.

Additional Practice and Homework Options

- Extra Practice, p. 440(b)
- Activity Worksheet 76
- Skill Worksheets 191–192
- Alternate Strategy Worksheet 64
- Math to Go 41

End of Lesson 9

Lesson 6 *continued*

Exercises and Problems

Combinations



These pages give students an opportunity to play a game of chance and keep track of the probabilities as they play.

Drawing to Learn



Marking their predictions on a number line allows students to connect the predictions they make in this game to the predictions they made in other activities in this module. Using a number line also offers them an occasion to apply a mathematical tool to a real-life game.

► Using the Tool Kit

Drawing can help students convey information in an exciting way. Refer students to Drawing to Learn, Drawing to Share, on page 407 of the Tool Kit.

Did You Know?

A game of chance played by the Klamath Indians of Oregon is called *shakla*. Four long sticks are arranged under a blanket. One player guesses the arrangement. If the player guesses correctly, the player wins points.

From *Games of the North American Indians*
by Stewart Culin

Exercises and Problems

Combinations

Women of the Twana nation in the state of Washington play a game with beaver teeth. They toss teeth that have carvings on one side. Scoring depends on how the teeth land. You can play this game with your partner, using coins or "beaver teeth" you make from cardboard.



Predict

- 1 Suppose you want only one out of four pennies to show heads. Is the probability of that happening closer to 0 or 1? Explain.
0, because the probability is $\frac{1}{4}$, which is closer to 0 than to 1
- 2 Will the chance of tossing one tail be the same as tossing one head? How do you know?
See below.
- 3 In how many different combinations could the four coins land? 5
- 4 Sketch the combinations. 4 heads, 3 heads and 1 tail, 2 heads and 2 tails, 1 head and 3 tails, 4 tails

Play

- 5 Toss four coins 20 times. Record the results of each trial. *Display students' data.*
 - a. Write ratios to compare the number of times each combination was tossed to the total number of trials.
 - b. Repeat for 20 more trials. What is the new ratio for each combination in 20 trials?

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Module 8: Lucky Summer

2. Yes, because out of the 16 possible outcomes, 4 are 1 head and 3 tails and 4 are 1 tail and 3 heads.

Teaching Options

Cultural Connection

Native American games of chance are of two kinds. In the first kind of game, players randomly throw marked objects. The objects stand for a number or numbers that count toward a sum. In the second kind of game, players guess the locations of concealed objects, worth varying point values.

Computer Opportunity

Invite students to explore probabilities of combinations with *MathKeys*. They can use the coins, spinners, or Number Machine in *Unlocking Probability Vol. II*. Ask students to explain how the bar graph in *Unlocking Probability Vol. II* represents the probability of each combination occurring.

7. Responses should be appropriate to, and reflective of, students' process.

Think

- 6 Which combination did you toss most often?
See students' data.

- 7 Why did that happen?
See above.

- 8 Draw a number line. Mark 0, $\frac{1}{2}$, and 1. Mark where you think the probability of tossing each combination should be. Use your ratios to help you.

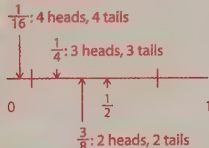
See above right.

- 9 Explain why you placed each combination where you did. Responses should be appropriate to, and reflective of, students' process.

- 10 Which combination of coins should be worth the most points? Why do you think so?

Possible answer: The least likely ones (4 heads or 4 tails)

Possible answer:



Ongoing Assessment

Check students' ratios for Exercise 5. The ratios should be written as the number of times a combination was tossed over the total number of tosses.

Follow-up Strategy Students who have difficulty working with four coins may start with one or two coins.

3. Reflecting and Sharing

Journal Opportunity

Have students explain why some combinations occur more frequently than others and how that frequency affects probability.

Section Assessment

An Observation Checklist (Assessment Master 52) is available in *Assessment Options*.

Calculator Tip

If students want to figure scores for the game, they can assign a point value to each combination. Then, encourage students to use their calculators to total the points they scored with their tosses.

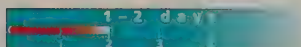
Additional Practice and Homework Options

- Extra Practice, p. 439(b)
- Activity Worksheet 74
- Skill Worksheets 185–186
- Alternate Strategy Worksheet 62
- Math to Go 40
- Gameboard Masters 8A–8B

End of Lesson 6

Investigation

Paper Towel Olympics

**Purpose**

Students will apply knowledge of metric measurements in a problem solving situation and analyze data expressed in metric measures.

Materials

- pan
- three different brands of paper towels
- metric measuring beaker
- centimeter ruler
- gram weights

Calculator access assumed

1. Getting Started

Are all paper towels created equal? Students might discuss what factors would differentiate them, such as thickness or paper quality.

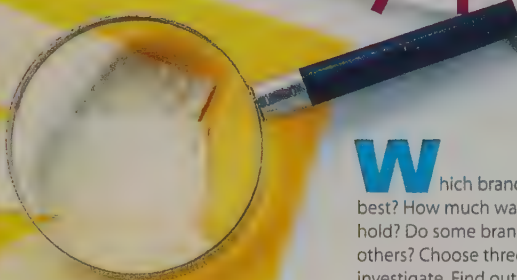
2. Facilitating Investigations

You might suggest that drying the pan and measuring cup between repeats could improve the accuracy of results. Remind students that they need to record their results so they can analyze the data. Discuss the information they need and the best format.

Investigate! Investigate! Investigate! Investigate! Investigate! Investigate! Investigate! Investigate!

Investigation

Paper Towel Olympics



Which brand of paper towels is best? How much water do paper towels hold? Do some brands hold more than others? Choose three different brands to investigate. Find out which deserves the gold medal.



- 1 How much water does each towel hold? Pour a thin layer of water into a pan. How many milliliters of water are in the pan? Place a towel in the pan. Soak it, and remove it. How much water is left in the pan? How much did the towel hold? Repeat this with each brand.

Module 5: Faster, Higher, Stronger

Investigate! Investigate! Investigate! Investigate! Investigate! Investigate! Investigate! Investigate!

Teaching Options

Cultural Connection

According to the 1993 *Information Please Almanac*, paper was invented in China around 100 BC. You might suggest that interested students investigate to find out how the Chinese made their paper.

Cooperative Learning Tip

You may want students to do this investigation in groups. Each group should have a materials handler who collects all the materials the group needs and takes responsibility for putting items away at the end of the activity.

Investigate! Investigate! Investigate! Investigate! Investigate! Investigate! Investigate!

- 2** How strong is each towel? Wet a towel. Place weights on it one at a time. How much will it hold before it breaks? Try this with the other brands. How much did the strongest brand hold?



- 3** How fast does each towel pick up water? Hold the edge of a towel on a water surface for one minute. In millimeters, measure how far the water climbs up the towel. How far does the water climb with the other brands?



Determine the overall winner of the Paper Towel Olympics. For each of the three events, give the first place brand 10 points. Give 5 points for second place and 2 points for third place. Which brand won the most total points?

See investigation rubric in *Assessment Options*.

Ask Yourself

- ☐ What makes a paper towel good?
- ☐ Which towel is the best?
- ☐ Why are milliliters a better unit than liters for this investigation? See above.
- ☐ How can you record the results? Make a table.
- ☐ Did all students get similar results? Explain.

They are smaller and measure small quantities more accurately.

Section B: Sizing It Up 263

Investigate! Investigate! Investigate! Investigate! Investigate! Investigate! Investigate!

Computer Opportunity

Have student groups record the results of their experiments in a graphing program. Suggest they create a bar graph. Ask them to explain how it represents the data. Combine all group data on the computer to create a class graph. Have the groups explain how their graph is similar to the class graph.

Students Acquiring English

Students may need some help recording the result of each experiment. They may also need assistance in translating the Ask Yourself questions. Allow ample time for students to formulate their responses.

Drawing to Learn



You might suggest that students draw a graph to show the comparisons of the three brands of paper towels they have investigated.

3. Reflecting and Sharing

Talk About It

After students complete their investigations, have them discuss the results.

- “How did you measure the water left in the pan in the first experiment?” Possible answer: I used a cup marked in milliliters.
- “What unit of measure did you use to express the results of the second investigation?” Grams
- “How were you able to measure how far the water climbs up the towel?” Possible answer: I measured from the edge of the paper to the highest point the water reached.

Ongoing Assessment

A complete scoring rubric for this investigation is available in *Assessment Options*.

Portfolio Opportunity

Students might want to place the results of this investigation in their portfolios.

LESSON 7

Building Polygons

2-3 days

Purpose

Students use ratios and polygons to make scale drawings.

Vocabulary

- polygons
- quadrilaterals

Materials

- Drawing to Learn: Geometry Tool or protractor Tracing Tool or grid paper
- construction paper
- ruler
- scissors
- crayons or markers
- MathKeys: Unlocking Geometry Vol. II software
- Activity Worksheet 58
- Skill Worksheets 181-184
- Alternate Strategy Worksheet 46

Calculator access assumed

1. Getting Started

Students use triangles to make other polygons. You may want to review various polygons.

2. Facilitating Activities

ACTIVITY 1 Cut and Combine



Encourage students to think about each figure as combinations of triangles.

LESSON 7

Building Polygons

Now use triangles to build **polygons**—closed figures made entirely of line segments.

ACTIVITY 1

Cut and Combine

With Your Group Combine triangles to make **quadrilaterals**, or four-sided polygons. The six shapes at the bottom of this page are quadrilaterals. See students' drawings and other work.



Fold a sheet of paper in half. Cut out a triangle along the folded edge. Then unfold the triangle.



What kind of quadrilateral do you have? What can you say about the two triangles that make it up? Type of quadrilateral varies depending on triangle used; they are congruent.



Cut across the fold line. Arrange the triangles into other quadrilaterals. Record your results.



Repeat steps 1-3 with another triangle. What generalizations can you make about the triangles that make up quadrilaterals? Describe your quadrilaterals. Check student's answers. quadrilateral

Teaching Options

Math Minute

Have students write each decimal as a fraction.

1. 0.66 2. 3.75 3. 0.1 4. 2.8

1. $\frac{2}{3}$; 2. $3\frac{3}{4}$; 3. $\frac{1}{10}$; 4. $2\frac{2}{5}$

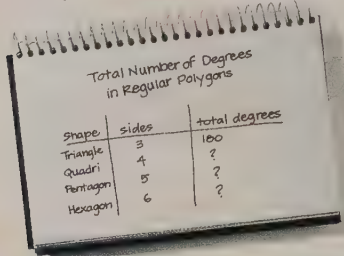
Students Acquiring English

You may want to have students use **tangram** pieces to learn the names of different **polygons**. Say a name, display it on an index card, and have students construct the polygon using the pieces. A trapezoid is made with two triangles and a square, for example.

Reasoning and Problem Solving. Possible answer: By adding, subtracting, multiplying, and/or dividing, an expression can make a pattern; the total number of degrees increases in multiples of 180; $(n - 2) \times 180$

ACTIVITY 2 More Than Four

With Your Group Measure the angles in a quadrilateral. Add them up. Record the number. Do the same with a pentagon and a hexagon. What pattern do you see? See below.



Shape	sides	total degrees
Triangle	3	180
Quadrilateral	4	?
Pentagon	5	?
Hexagon	6	?

Quadrilateral: 360°; pentagon: 540°; hexagon: 720°; for each additional side, add 180°.

What You'll Need

- Geometry Tool or protractor
- ruler



What do you know about expressions that can help you find the pattern? First describe the pattern in words. Then use numbers and a variable. For instance, start with $n \times 180$. See above.

What You'll Need

- Geometry tool or protractor
- ruler

Vocabulary Strategy

Many students confuse rhombuses and trapezoids. You might help students remember that a rhombus is a parallelogram by mentioning that both words contain the letter *m*.

ACTIVITY 2 More Than Four



Students should recall that the number of degrees in the angles of a triangle is 180. Students should discover that each time they add a side to a figure, the total number of degrees in the angles increases by 180°.

Problem Solving Tip

Some students may not see the pattern as the number of sides increases. You may want to have students find the difference between the number of degrees in each figure and record their data in a chart.

ACTIVITY 3 Venning Shapes

With Your Group Look at the Venn diagram below. The outside shape stands for all quadrilaterals. What do the inside shapes stand for?

Parallelograms, rectangles, squares

Why are squares shown inside rectangles and parallelograms? See below.

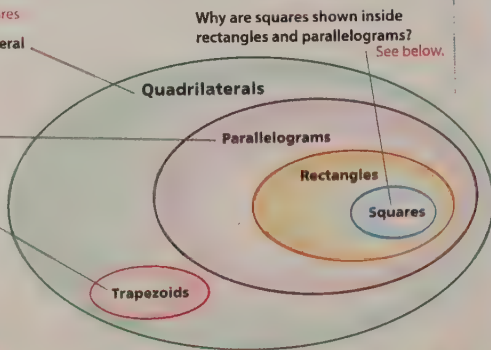
Draw a quadrilateral that is not a parallelogram.

Draw a parallelogram that is not a rectangle.

Why are trapezoids and parallelograms separate? Trapezoids are not parallelograms.

Which part of the diagram stands for figures that do not have two sets of parallel sides?

Quadrilaterals and trapezoids



Section C: Scale Drawing 331

Squares are special types of rectangles and parallelograms.

Computer Opportunity

Have students make quadrilaterals with *MathKeys: Unlocking Geometry Vol. II*. Students can use the electronic grid or pattern blocks to build their own quadrilaterals. Ask students to find out how many quadrilaterals they can form. In the Notepad, have them categorize and describe each shape. You may want to print out students' drawings to publish a geometry resource book for the class.

Literature Connection

One part of Norton Juster's book *The Phantom Tollbooth* features Mr. Dodecahedron, a three-dimensional twelve-sided figure. You may wish to read the story aloud to students.

ACTIVITY 3 Venning Shapes



You may want to ask students questions to help them interpret the Venn diagram.

- "Are all trapezoids quadrilaterals?" Yes.
- "Are all quadrilaterals trapezoids?" No, trapezoids are a subset.
- "How does the Venn diagram tell you this?" The trapezoid section is within the quadrilateral section but the quadrilateral section is not within the trapezoid section.

Lesson 7 continues . . .

Investigation

Circles and Paths

**Purpose**

Students analyze patterns created by using circles and points on the circle. They analyze the relationship between the number of points on the circle and the number of points they use to make their patterns.

Materials

- compass
- *Drawing to Learn:* Fraction Tool or protractor and ruler
- **MathKeys:** Unlocking Geometry Vol. II software

1. Getting Started

Review the number of degrees in a circle. Also review how to mark equidistant points on a circle with a compass.

2. Facilitating Investigations

Students should discover that their patterns involve the common factors of the total number of points and the number of points skipped each time. If these numbers share no common factor, there is only one path. For numbers with a common factor, the number of paths is the GCF of the two numbers.

Investigation

Circles and Paths

There's more than one path you can take to get around a circle. Make some circular designs and see if you notice any interesting mathematical patterns.

- 1 Choose a number—for example, 8. Use your Fraction Tool or compass to mark eight points at equal distances around the circumference of the circle.



- 2 Pick a second number that is less than the first, such as 2. Choose a place to start. Use a straightedge to connect every second point.



- 3 Keep going until you return to the starting point. This is a path. If all points are not connected, move over one point and begin a new path. You might want to do this next path in a different color.



86 Module 2: Getting Around

Investigate! Investigate! Investigate! Investigate! Investigate! Investigate! Investigate!

Teaching Options

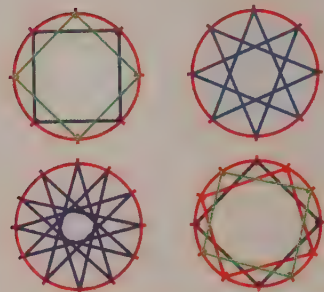
Science Connection

Some spiders use circles and paths for their webs. Once the spider has spun the boundaries of the web, it spins an inner and an outer spiral. Interested students might do research and report to the class on different types of spider webs.

Investigate! Investigate! Investigate! Investigate! Investigate! Investigate! Investigate!

See investigation rubric in *Assessment Options*.

- 4 How many paths are in your circle? Make a chart like the one on this page to show your results. Describe your design. What do you notice about the paths and the number of points you used to create the design? Explain.
- 5 Now try using other numbers to create different designs. Review your explanation. Does it apply to your new designs?



number of points	connect every...	number of paths
8	2	2
8	3	
12		
12		
24		

Ask Yourself

- What mathematical relationship do you see between the number of paths and the numbers you used?
- ☐ How did you find the relationship between the paths and the number of points?
- ☐ What math did you use—addition, multiplication, division?
- How did the circles you created affect your explanation of the patterns?
- How could you explain these circle patterns to a friend?



Investigate! Investigate! Investigate! Investigate! Investigate! Investigate! Investigate!

Geometry Connection

Students may connect the shapes they create with their paths to the number of degrees in a circle. You may wish to suggest the use of a protractor to measure the inscribed angles. Have students describe any relationships.

Extension

Suggest that students start with one point on the circle and connect it to every other point. Ask them to describe the patterns these paths create. Have them also tell how these paths differ from those in the investigation.

Ongoing Assessment

Can students clearly explain how the paths and numbers relate? Although the patterns on the circles show the greatest common factor of two numbers, students may find other relationships as well.

A complete scoring rubric for this investigation is available in *Assessment Options*.

Follow-up Strategy Students who have problems creating circles may trace the ones on these two pages. Ask students if they can predict how many paths will occur in a 64-pointed circle counting every 20 points (4 paths with 16 points in each).

3. Reflecting and Sharing

Talk About It

Ask students to share and discuss their explanations with the class. You may want to ask students to think about how they worked through this activity.

Portfolio Opportunity

Students may want to add their circles and explanations to their portfolios.

2. Facilitating Activities

ACTIVITY 1 Around and About



Students might use pattern blocks to model slides, flips, and turns as they write definitions.

ACTIVITY 2 In Perfect Balance



Students choosing the first option may wish to trace the buildings' outlines on notebook paper, then draw lines of symmetry. For option 2, suggest students label points on their diagrams, then refer to examples of symmetry in their notes.

Questioning Strategy

Help students recall what they know about symmetry.

- "Where would you draw a line to show two symmetrical halves of the Taj Mahal?" From the top of the tower to the base of the Taj Mahal.
- "Is each door and window symmetrical? Explain." Yes; a vertical line through the center of each forms two matching halves.
- "How can you make sure you have found a line of symmetry?" Possible answer: Trace, then fold along the presumed line of symmetry.

Drawing to Learn



Have students make diagrams to show symmetry of each shape in their Dictionary of Shapes.

What You'll Need

- Tracing Tool or ruler



How many lines of symmetry can you find on a square? on a regular octagon or a circle? How can you use what you know about symmetry to add to your definitions of polygons? Possible answer: 4; 8; an infinite number; a regular polygon has as many lines of symmetry as sides.

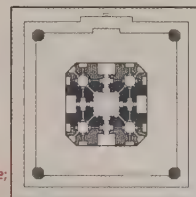
In the early 1600s in what is now northern India, Mumtaz Mahal died. Her husband, the Mogul Emperor Shah Jahan, had the Taj Mahal built as his and her burial place.

ACTIVITY 2 In Perfect Balance

With Your Partner The Taj Mahal uses flips to create symmetry. You know a **line of symmetry** divides a figure into two congruent halves. Choose one of these two activities to complete.

- 1 Make a diagram to show what you find.
- 2 Design a building that uses symmetry in its face and its floor plan. Describe how you used slides and flips to create symmetry and list all the ways in which your building is symmetrical.

See students' drawings for 1–2.

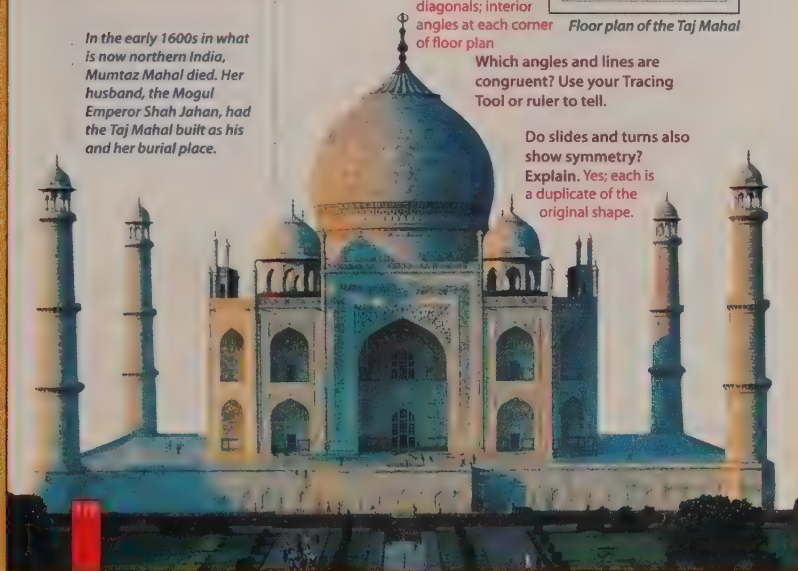


Floor plan of the Taj Mahal

Possible answers:
Length of each side;
diagonals; interior
angles at each corner
of floor plan

Which angles and lines are congruent? Use your Tracing Tool or ruler to tell.

Do slides and turns also show symmetry? Explain. Yes; each is a duplicate of the original shape.



Teaching Options

Computer Opportunity

Challenge students to use the electronic pattern blocks in *MathKeys: Unlocking Geometry Vol. II* software to design a rug or quilt square. They can flip, slide, and rotate the blocks to make designs. Encourage students to combine pattern blocks to make larger shapes. Invite them to use the Notepad to describe the symmetry of their pattern.

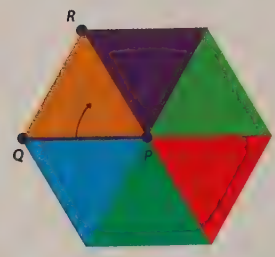
Extension

Ask students to study the symmetry of the structure of their choice. They might select a well-known building or a structure in their neighborhood. Invite students to draw a diagram of the building and list the ways it is symmetrical.

ACTIVITY 3 **Round and Round**

With Your Partner An architect needs to describe the **angle of rotation**, or number of degrees a shape is turned, so someone else can follow the plan.

- 1 Trace an equilateral triangle. Label its vertices. Rotate and trace the triangle. Line up the edges as you go until you meet your original shape. What shape did you make? *Hexagon*
- 2 Estimate how far the triangle was turned each time. Use your protractor to check your answer. *60°*



Red; it moved 60° each time
 $3 \times 60^\circ = 180^\circ$
a. Which color panel is a 180° turn of the yellow panel around point P? Explain how you know.

c. Which kind of triangle could you rotate to make an octagon? How far would you need to rotate it each time?
Isosceles; 45°

b. Which color panel shows a 240° turn of the yellow panel around point P?
Dark green; $4 \times 60^\circ = 240^\circ$

With Your Partner Draw your own tile pattern using turns of 90°, 180°, and 270°.

- What You'll Need**
- Tracing Tool or tracing paper
 - Geometry Tool or protractor and ruler
 - crayons or markers



Architect R. Buckminster Fuller designed geodesic domes like the one above. The domes use polygons for the faces. Fuller designed the one that housed the United States exhibit at Expo 67 in Montreal in 1967.



You know there are 360° in a circle. How could you use this information to find the angle of rotation, or turn, of a regular polygon? Divide 360° by the number of sides.

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ACTIVITY 3 **Round and Round**



Students might make stencils, trace pattern blocks, or use their Geometry Tool to show turns.

► **Questioning Strategy**
Help students relate the activity to polygons.

- “Why is it important to know that there are 360° in a circle when finding the angle of rotation?” Possible answer: Because if you keep rotating the figure, you make a circle; 360° divided by the number of turns gives you the angle of rotation.
- “How can you decide which angle measurements could be rotated to make regular polygons?” Any measure, such as 10, 20, 45, or 60, that is a factor of 360

► **Ongoing Assessment**
Students should be able to figure out the angle of rotation within any polygon.

Follow-up Strategy Have students cut out an isosceles triangle and rotate it around the vertex between the two congruent sides. Then ask them to divide 360° by the number of turns.

- “What is the angle of rotation? Is the repeated angle a factor of 360°?” Possible answers: 90°, yes; 36°, yes

Lesson 7 continues . . .

Combination Classes

Younger students might use manipulatives to help them find the angle of rotation. In addition, isometric dot paper or grid paper might help some students draw repeated shapes more quickly.

Cultural Connection

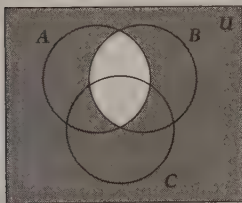
Islamic buildings are often covered in tile patterns. Ask students to do research on tile patterns in different countries. You might suggest they begin by looking up the Great Mosque of Córdoba and the Alhambra in Granada, Spain.

e. $\bar{F} \cap (S \cap P)$

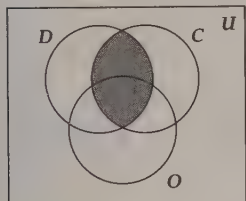
Males who smoke and have a health problem

14. b. All numbers that don't evenly divide 12, 15, or 20; $\overline{A \cup B \cup C}$ or $\bar{A} \cap \bar{B} \cap \bar{C}$

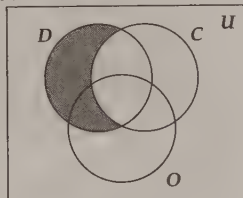
- d. All numbers except 1 and 3



15. a. Students who have at least one cat and at least one dog



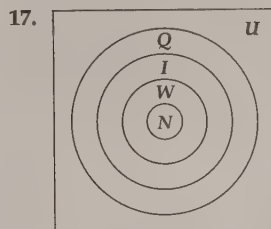
d. $D \cap \bar{C}$



g. $\overline{D \cup C \cup O}$ or $\bar{D} \cap \bar{C} \cap \bar{O}$

Students who have no pets

16. The circles enable us to easily represent visually all the possible subsets. The diagram is not equivalent because there is no region corresponding to elements that are in all three sets.

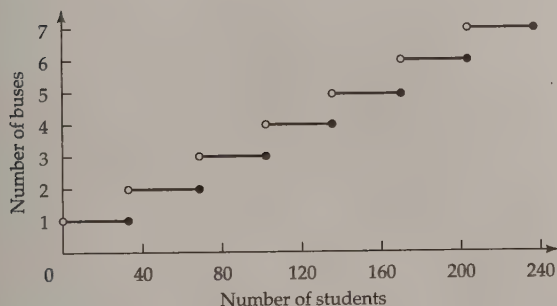


21. b. Technically, this subset represents a lesson in which the students use concrete materials, but not in a lab approach and not in small groups. Pedagogically this doesn't make sense.

Exercises 2.2, page 91

4. a. 7 buses. Each bus will carry a maximum of 34 students and two adults. 7 buses $\times$ 34 students per bus = 238 students, so there will be 6 extra seats.
b.

School Buses for Field Trip



6. 164.9 cm, or 1.65 m
9. No
12. No. Two students could spend the same amount of time and get different grades.
13. a. $T_n^2 - T_{n-1}^2$ b. It is a function.
15. It would be less expensive to rent a car for a week. The minimum cost renting by the day is $(\$45) \times (3 \text{ days}) + (\$0.30 \text{ per mile}) (320 \text{ miles round trip}) = \$231$.
16. Three solution paths are presented here:
Solution path 1: Guess-check-revise
Find when the first plan costs \$5.

Checks	Charge	Calculations	Reflection/Analysis
0	2.00	2.00	The charge for checks will be \$2.
10	3.50	$2.00 + 10(0.15)$	10 checks added \$1.50. Another \$1.50 will make \$5.
20	5.00	$2.00 + 20(0.15)$	Got it!

Solution path 2: Represent the problem algebraically

Find when the charges are equal.

Let x = the number of checks written.

Charges in first plan: $2 + 0.15x$

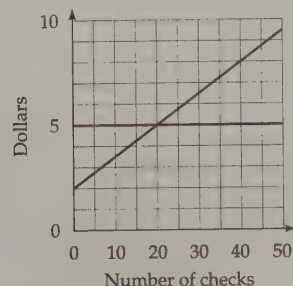
Charges in second plan: 5

$$2 + 0.15x = 5$$

$$0.15x = 3$$

$$x = 20$$

Solution path 3: Represent the problem with a graph
Where the two lines cross is the point where you will be charged the same fee.



18. a. \$3400 b. \$800 c. In
- $6\frac{1}{4}$
- years

23. Two solution paths are shown.

Solution path 1: You can view the parts of the figures as a rectangle plus 1 square sticking out at the top right and 1 square sticking out at the bottom left.

	2nd figure	3rd figure	4th figure	n th figure
Middle rectangle	$3 \cdot 1$	$4 \cdot 2$	$5 \cdot 3$	$(n+1)(n-1)$
Little squares "sticking out"	2	2	2	2

Thus, the number of squares in the n th figure is $(n+1)(n-1) + 2$, which simplifies to $n^2 + 1$.

Solution path 2: You can view the parts of the figure as a top and a bottom row and then a middle square.

	2nd figure	3rd figure	4th figure	n th figure
Top and bottom row	$2 + 2$	$3 + 3$	$4 + 4$	$n + n$
Middle square	1	$2 \cdot 2$	$3 \cdot 3$	$(n-1)(n-1)$

Thus, the number of squares in the n th figure is $n + n + n^2 - 2n + 1$, which simplifies to $n^2 + 1$.

Exercises 2.3, page 107

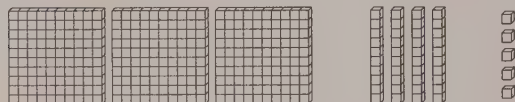
2. a. 3031 c. 1666 f.
- $75,602(21 \times 60^2 + 0 \times 60 + 2)$

3. a. Egyptian
- 𐪓𐪓𐪓𐪓
-
- Roman CCCXII
-
- Babylonian
- $\text{IIIIII} \lll$

- c. Egyptian
- 𐪓𐪓𐪓𐪓𐪓𐪓
-
- Roman MMMMMM
-
- Babylonian
- $\text{I} \lll \lll \text{II}$

$$(1 \times 60^2 + 40 \times 60 + 0)$$

5. a.

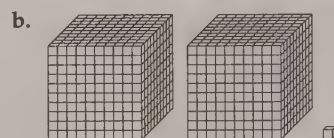


26. Here we will assume that the area of each face is 1 square unit.

- a.
- Solution path 1:*
- Make a table and look for patterns. If the stack has 1 cube, the surface area is 6 square units. If the stack has 2 cubes, the surface area is 10 square units. If the stack has 3 cubes, the surface area is 14 square units. In this case, the increase is 4 each time, indicating that this relationship can be represented by a linear equation whose slope is 4 and that includes the ordered pair (1, 6). Using what we learned in Algebra 1, we deduce that the equation is
- $y = 4x + 2$
- . We can transfer this equation to the question at hand directly: A stack of
- n
- cubes will have a surface area of
- $4n + 2$
- square units.
-
- Solution path 2:*
- Break the problem into parts. Each stack will have four equal "sides" (technically, they are faces) and a top and bottom.

	Height of the tower			
	1 cube	2 cubes	3 cubes	n cubes
Number of cubes on the sides	4	$4 \cdot 2$	$4 \cdot 3$	$4 \cdot n$
Top + bottom	2	2	2	2

Thus, the surface area of a tower of n cubes is $4n + 2$.



- b.
6. a. $300 + 40 + 5$ 7. a. 4859
8. a. $500 + 10 + 10 + 5 + 1 + 1 = 527$
c. $\text{HHH} \triangle \triangle \triangle \text{IIIIII}$
9. a. 32,570 c. $\text{III} \equiv \text{T}$
10. c. $(15 \times 360) + (0 \times 20) + (7 \times 1) = 4707$
d. III
11. a. 40_5 g. 130_5 12. e. 1001_2 g. 113_4
13. b. $(5 \times 6) + (5 \times 1) = 35$
d. $(2 \times 125) + (1 \times 25) + (4 \times 1) = 279$
14. a. 134_5 c. 1011100_2 g. 112_6
16. $x = 9$

Exercises 3.1, page 134

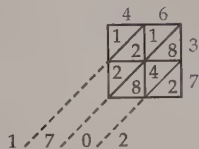
6. 5 hundreds + 10 tens + 13 ones
-
- $= 5 \text{ hundreds} + 11 \text{ tens} + 3 \text{ ones}$
-
- $= 6 \text{ hundreds} + 1 \text{ ten} + 3 \text{ ones}$
-
- $= 613$

7. a. After adding
- $6 + 8 = 14$
- , the student wrote the 1 in the ones place, carried the 4, and added it to
- $3 + 2$
- .

8. a. The greatest sum, 1593, is obtained when the 7 and 8 are in the hundreds places, the 3 and 6 are in the tens places, and the 1 and 2 are in the ones places.
10. $N = 9, P = 2$
11. 372
168 There are many solutions.
459
999
13. a. $a - b \neq b - a$, because $b - a = -(a - b)$.
17. a. The student subtracted $8 - 6$. Some students automatically subtract the smaller number from the larger number, even if it changes the order of the numbers in the problem.
- e. The student changed the zero in the ones place to 10 but didn't rename the 7 in the tens place as 6.
18. a. $876 - 123 = 753$
19. There are many possibilities. Two are shown here:
 $968 - 734, 946 - 712$.
20. a. $7 - 5 = 2$
c. $3 + 4 = 7$
e. $3 \cdot 2 = 6$ (repeated addition)
23. Adam: 55 pounds, Ben: 50 pounds, third boy: 65 pounds

Exercises 3.2, page 161

3. c. 20
5. $\{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
12. a. $\begin{array}{r} 25 \\ 50 \\ 100 \\ 200 \\ 400 \end{array}$ 17
8
4
2
1
 $400 + 25 = 425$
13. a. $45^2 - 9^2 = 2025 - 81 = 1944$
15. a. The 1 being carried represents 10. It is carried so that it can be added to the other values in the tens column.
17. a. Set up the numbers as shown below.
Find the four partial products and place them in the appropriate places: the ones digit in the lower spot and the tens digit in the higher spot of each cell. *Note:* If the product of the two numbers is less than 10, you can either leave the top cell blank or place a 0 in that spot.
To find the product, find the sum of the numbers on each of the diagonal rows. If the sum of any diagonal row is above 10, carry the digit in the tens place to the next place.



19. a. 8×6 : The student wrote down the tens digit and carried the ones digit.
c. 2×46 : The student wrote the product in the tens and ones places, even though the 2 represents 20.
20. a. $530 \cdot 71 = 37,630$
22. a. 55,999,999,944 Below is one solution path.
 $999,999,999 \cdot 56 = (1,000,000,000 - 1)56$
 $= 56,000,000,000 - 56$
27. a. $3 \cdot 2 = 6$ (repeated addition)
30. a. The student didn't use zero as a place holder in the quotient to indicate that 2 (in the tens place) divided by 7 is less than 1 (ten).
31. a. $2\overline{)974} = 487$
33. $x = 7$ and $y = 13$, or vice versa
34. a. 14 buses
35. 60 cans of juice, assuming 8 oz/glass; \$53.40
37. 25¢ 38. 11 cartons
40. 160 tapes 41. 700 rolls
42. 2,190,000 gizmos
48. a. 4950 cubes 49. a. 36 squares
- 51.
- | | | | | |
|----|------|-----|---------|-------------|
| a. | a | b | $a + b$ | $a \cdot b$ |
| | 65 | 66 | 131 | 4290 |
| | 234 | 766 | 1000 | 179,244 |
| | 1208 | 72 | 1280 | 86,976 |
- | | | | | |
|----|-----|-----|---------|------------|
| b. | a | b | $a + b$ | $a \div b$ |
| | 19 | 1 | 20 | 19 |
- | | | | | |
|----|-----|-----|-------------|------------|
| d. | a | b | $a \cdot b$ | $a \div b$ |
| | 10 | 2 | 20 | 5 |
52. a. $74 \cdot 86 = 6364$
e. $1530 = 31 \cdot 48 + 42$
55. Alike: The sum (product) of any row, any column, and both diagonals are identical.
Difference: In a multiplication magic square, it seems that the numbers in the square don't all need to be different.

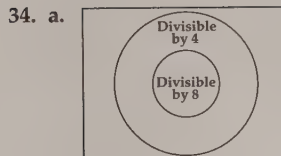
Exercises 3.3, page 181

1. Answers will vary. One possibility for each is given below.
 - a. $47 + 53 = 100$ (compatible numbers). Now add 2 more to get 102.
 - c. $150 + 15 = 165$
 - e. $575 + 125 = 700$. Now add 3 to get 703.
 - g. $340 + 16 = 356$
 - i. $387 + 24 = 400 + 11 = 411$. Then $411 + 53 = 464$
2. Answers will vary. One possibility for each is given below.
 - a. Leading digit: 1173
 $(100 + 160 = 1160; 1160 + 13 = 73)$
 - c. Leading digit and rounding: 146,000
 Add leading digits: $8 + 3 + 2 = 13$ (representing 130,000). Then round each number to nearest thousand. And add second leading digit: $9 + 3 + 4 = 16$ (representing 16,000). $130,000 + 16,000 = 146,000$.
 - e. Leading digit, compatible numbers and rounding: 2800
 $4 + 3 + 1 + 5 + 9 + 3 = 25$, representing 2500.
 $73 + 34 \approx 100; 45 + 55 = 100; 65 + 43 \approx 100$.
 $2500 + 100 + 100 + 100 = 2800$.
 - g. Leading digit and rounding: 1,400,000,000
3.
 - a. $\begin{array}{r} 37 \\ 24 \\ 185 \end{array}$
 - c. $\begin{array}{r} 56 \\ 47 \\ 153 \end{array}$
 - e. $\begin{array}{r} 74 \\ 97 \\ 274 \end{array}$
4. Answers will vary. One possibility for each is given below.
 - a. Add 1 to each number: $88 - 30 = 58$.
 - c. $34 + 50 = 84$; the answer is 48.
 - e. $206 + 300 = 506$; the answer is 296.
 - g. $475 + 25 = 500$; add 125 more; the answer is 150.
 - i. Add one to each number $507 - 30$; count backwards by tens; 507, 497, 487, 477.
5. Answers will vary. One possibility for each is given below.
 - a. Leading digit: 2100
 - c. 28 to 30 is 2, 30 to 73 is 43; an estimate is 45,000.
 - e. 285 to 300 is 15, to 413 is 113 more. $113 + 15 = 128$; estimate is 128,000.
 - g. Rounding: about 1,200,000
6. a. 475 764 c. 382 723
7. Answers will vary. One possibility for each is given below.
 - a. $16 \times 16 = 32 \times 8 = 64 \times 4 = 256$
 - c. $66 \times 10 = 660$, so $66 \times 5 = 330$
 - e. $6 \times 3 = 18$, so $60 \times 30 = 1800$
 - g. $35 \times 10 = 350, 35 \times 2 = 70$; answer is $350 + 70 = 420$.
 - i. Double 736 twice: 1472, then 2944
8. Answers will vary. One possibility for each is given below.
 - a. $40 \times 70 = 2800$
 - c. $\begin{array}{r} 60 + 2 \\ 80 + 3 \\ \hline 4800 + 200 + 200 = 5200 \end{array}$
 - e. $30,000 \times 50 = 1,500,000$
9. Part (b) is wrong.
10. $34 \times 65 = 2210$
13. b. A reasonable estimate is 4200.
15. a. $587 \times 40 = 23,480$
 $587 \times 305 = 179,035$
24. Rough: About 500
 Refined: About 670
26. Rough: Just under \$50,000
 Refined: About \$45,000
29. Rough: About \$90
 Refined: About \$103
32. Rough: About 8000 days
 Refined: About 7300 days
34. About 8 years
37. About 3600 miles. Answers on cost will vary depending on price of gas and miles per gallon that the car gets.
39. About \$240
47. Answers may vary. 3 million crimes per year would mean about 17,000 crimes per school day (dividing by 180 school days per year). If we divide 17,000 by 50 states, that's about 340 crimes per day per state. That still seems high, so the term *crime* must be broad. It must include such things as vandalism, petty theft, and minor assault. Does it mean "crimes reported to police"?

Exercises 4.1, page 202

3. a. 1, 3, 7, 9 or 1, 3, 5, 11
6. a. $\begin{array}{r} 7, 7, 5, 1, 1 \\ 7, 7, 3, 3, 1 \end{array}$ $\begin{array}{r} 7, 5, 5, 3, 1 \\ 7, 5, 3, 3, 3 \end{array}$ $\begin{array}{r} 5, 5, 5, 5, 1 \\ 5, 5, 5, 3, 3 \end{array}$
8. a. Hint: 
9. b. $8 \cdot 10 = 92 - 12$
10. b. $123,456,789 \cdot 8 + 9 = 987,654,321$
11. a. 30 and 15 b. 36 and 12
13. a. $37 \times 3 \times 1 = 111; 37 \times 3 \times 2 = 222;$
 $37 \times 3 \times 3 = 333$. The repeated digit in the product is the factor multiplied by 37×3 in the problem.
19. 467,712 21. 301 cards
26. c. The last two digits must be 00, 25, 50, 75.
 d. There are several possibilities. One is that the sum of the digits must be an even number. The most elegant is that the number is divisible by 2 iff there are an even number of even digits in the number.
27. a. True
 c. False; eg., $6/4 \cdot 9$, but 6×4 and 6×9
 e. True $6 \mid 4$

28. a. True c. True
 29. a. The sum of three consecutive numbers will always be divisible by 3 and by the middle number in the sequence. $x + (x + 1) + (x + 2) = 3x + 3$
 31. a. Yes 32. a. 1



Exercises 4.2, page 212

- Any prime number has exactly two factors.
- The first 20 prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71.
- $2 \times 2 \times 2 \times 2 \times 2 \times 3$
 - $5 \times 5 \times 3$
 - $2 \times 2 \times 23$
 - $2 \times 2 \times 2 \times 2 \times 3 \times 3$
 - $2 \times 2 \times 7 \times 7$
 - $2 \times 2 \times 2 \times 3 \times 3 \times 7$
- 61 is prime
 - Composite: 5×13
 - 71 is prime.
 - 89 is prime.
 - Prime f. Prime
 - Composite: 19×23
- 1951; 1973; 1979; 1987; 1993; 1997; 1999
- 24
- Answers may vary: 313, 317, 373
- 2520
 - 1369; factors: 1, 37, 1369
 - 2, 4, 5, 10, 20, 25, 50
 - 3, 7, and 1

Exercises 4.3, page 225

- 15
 - 9
 - 1
 - 27
 - 3
- a. 3
- 60
 - 150
 - 132
- a. 300
- 5; 200
 - 36; 216
 - 84; 1,260
- 143
- $224 \text{ mm} \times 224 \text{ mm}$; 28 dominoes
- a is a factor of b , and b is a multiple of a .
 - a
- True
 - False; $\text{GCF}(12, 18) = 6$
 - True
- Answers vary. One possibility is $120 \text{ cm} \times 120 \text{ cm} \times 120 \text{ cm}$.
- A 2 by 2 square
 - A 3 by 3 square
 - The GCF of x and y .
- A train that is 12 units long; for example, a train made by joining an orange rod and a red rod
 - A train that is 24 units long
 - A train whose length is the LCM of x and y
- The purple rod—that is, a rod 4 units long
 - The yellow rod—that is, a rod 5 units long
 - A train whose length is the GCF of x and y .
- The LCF of any two numbers is 1, by definition. Thus, the LCF is not useful. The GCM of any two numbers is infinitely large, and therefore not useful.

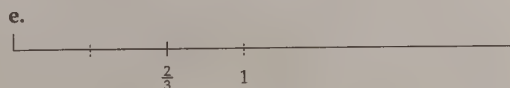
Exercises 5.1, page 237

- 218
 - 19
 - 14
 - 7
 - 221
- 238
 - 417
 - 213
- 35
 - 5
 - 5° Celsius
 - 5° F
- 20 mph wind speed
 - 180 mph maximum speed with no wind
- 19 hours 14. 3764 B.C.
- All numbers are estimates from the graph.
 - The lowest trade balance was during 1986. It was approximately -150.
 - 80
 - The biggest decline was in 1983; approximately -50.
- Always positive.
 - Always positive.
- Always positive.
 - Always positive.
- 176 pounds

Exercises 5.2, page 252

3. a. 

c. 



g. 7/10 is one possibility.

5. Note: Answers will vary. The figure shows one valid representation for each part of 3/5.



● ● ● ○ ○

7. a. 5/8 c. 9/10 e. 7/9 g. 29/30
 11. a. $a/b > a/c$ c. $b/c > a/c$
 13. 1/10 of your blood
 15. a. 1/3 b. 1/9
 17. Approximately 3/5
 19. Approximately 2/3
 21. Religion Fraction
 Christianity 1/3

Islam	1/6
Hinduism	1/8
Buddhism	1/16
Judaism	1/300

23. a. $5/16 \approx 1/3$ c. $199/307 \approx 2/3$
 25. The new fraction is larger than the original fraction.
 27. 1/2 of the plants survived.
 30. a. 7/12
 c. Yes. Copper and zinc have different prices.
 31. a. The 1/125 shutter speed means that the shutter is open 4 times as long.

Exercises 5.3, page 274

1. a. $80\frac{17}{34}$ b. $-2\frac{23}{120}$ c. $23\frac{13}{20}$ d. $191\frac{11}{16}$
 e. $-3\frac{7}{12}$ f. $66\frac{7}{24}$ g. 99 h. 135
 i. $13\frac{1}{2}$
 2. a. 13 3/8 d. 19 2/5 g. 44
 3. a. Less than 10
 c. Greater than 2
 e. Greater than 20
 5. Estimate: 4 ounces
 Exact answer: $4\frac{23}{64}$ ounces
 7. Estimate: 72 inches or 75 inches
 Exact answer: $73\frac{1}{2}$ inches
 9. Estimate: 90 million people
 Exact answer: $2/3 \times 136,800,000 = 91,200,000$
 12. Estimate: Maybe yes, maybe no. It will be close.
 Exact answer: Yes, exactly.
 14. 120 freshman women.
 16. a. $\frac{1}{2}$ or 1/1024 of the germs remain.
 b. 19,531 germs (theoretically). $1/1024 \times 20,000,000 = 2 \times 10^7 \div 1024$. A better answer might be to say approximately 1/1000 of 20 million $\approx 20,000$ germs.
 18. 3/7 are single.
 20. The person who eats 1 piece eats 1/8 of the total.
 The person who eats 2 pieces eats $2/8 = 1/4$ of the total. The other two each eat 5/16 of the total.
 22. 415,220; 1,245,660
 24. a. 4 pressings; 6 pressings

$$27. \frac{bc^2 - bd}{ac}$$

28. a. One way: $26 \times 11 \div 12$

$$33. \frac{9\frac{1}{4}}{3\frac{3}{4}} = \frac{9 + \frac{1}{4}}{3 + \frac{3}{4}} \neq \frac{9}{3} + \frac{\frac{1}{4}}{\frac{3}{4}}. \text{ Rather, } \frac{9\frac{1}{4}}{3\frac{3}{4}} = \frac{9}{3\frac{3}{4}} + \frac{\frac{1}{4}}{3\frac{3}{4}}.$$

35. a. They added the denominators.
 c. They cross-multiplied, one numerator by the other denominator.

$$38. \frac{7}{8} + \frac{1}{9}$$

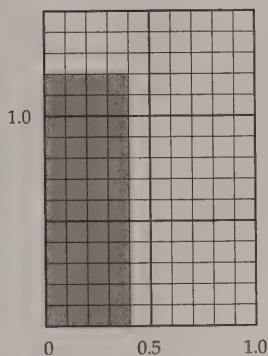
42. $\frac{7}{8} \div \frac{1}{9}$ if you assumed proper fractions; $\frac{8}{2} \div \frac{1}{9}$ if you did not.

Exercises 5.4, page 299

1. a. 0.075 c. 0.03 e. 1/200
 2. a. $4 + \frac{6}{10}$, $4\frac{6}{10}$ c. $1 + \frac{2}{10} + \frac{3}{100} + \frac{4}{1000}$, $1\frac{234}{1000}$
 3. a. Thirty-two and four hundredths
 4. a. 2.4 c. 2.40 e. 0.988
 5. a. 0.4 c. 0.05
 6. a. 0.0084, 0.058, 0.56, 0.6
 8. a.



9. b.



10. \$24,440,000 or \$24.44 million

14. a. 160 c. 0.8 e. 0.425 g. 0.24

15. a. 1.23456789×10^8 c. 5.6×10^{-10} 16. a. 4.182×10^{18} c. 1.53×10^{-17}

17. 0.00075 cm

19. a. $1.84467441 \times 10^{19}$ 21. a. The right answer is \$108. There are several ways to get \$1080. One is 24×45 .23. a. Answers and justifications will vary. The range of temperatures in Fahrenheit is 97.16° (for 36.2°C)

to 99.5° (for 37.5°C). The mean (average) of these temperatures is 98.33°F .

24. 102 is how many vials can be completely filled, and .4 is the fraction of another vial that can be filled.

28. a. 32.7 miles per gallon

b. \$18.22

c. Her average speed was $405.6 \text{ miles} / 11 \text{ hours} = 36.9 \text{ miles per hour}$. Also acceptable is $405.6 / 10.5 = 38.6 \text{ miles per hour}$. This is her average speed while the car was moving.

30. 845 items; \$1.21

32. 21,735.3 dominoes per minute, which is 362.3 dominoes per second

34. 27.7¢/day

36. a. Option (b) b. 30 checks

38. a. 2550 dots b. 8.415×10^6 dots

41. At his current job he makes approximately \$19,000. Not taking into consideration benefits, the work environment, and other factors, the annual salary of \$24,000 seems better.

43. a. 72°F c. 38°C

Exercises 6.1, page 320

1. 143.75 calories. Answer is an approximation.

3. 165 dentists. Answer is an approximation.

5. a. \$750; this is an exact answer.

c. \$977.08; this answer is rounded to two decimal places.

7. 860,000 people. Answer is an approximation (rounded to two significant figures).

9. Approximately 51 feet, 5 inches.

11. Yes; at this rate she will row 2,228.6 meters in 10 minutes, which exceeds her goal.

13. $11 \frac{3}{4}$ inches. This is an approximate answer.

15. a. 45 miles. This is an approximate answer.

18. a. 104 km/h. 19. \$500

21. 40 miles. 22. 10 questions

24. The man with five loaves should get 7 cents, and the man with three should get 1 cent.

25. 3.5 feet

26. a. 23 miles per hour

27. 37 handshakes per minute, which is one handshake in less than 2 seconds

29. 2520 students

30. $1 \frac{1}{2}$ hours more

33. More prison inmates. For a U.S. population of 270,000,000, there are approximately 492,000 physicians and 543,000 prison inmates.

36. $199/325 = 61.2\%$ and $5/8 = 62.5\%$. These percents are close enough to say that the advertisement is accurate.

39. The person who gets paid twice a month will have a larger paycheck.

40. a. In order to get 40 pledges in 3 hours, they would need 20 pledges by 7:30.

b. Two lines of reasoning are appropriate: One is that the announcer may have felt that it would turn people off to say that they were behind; the other is that the announcer might realize that they will get more callers from 7:30 to 9 than from 6 to 7:30, and so they would catch up then.

Exercises 6.2, page 338

1. a. 18 c. 1,211.8 e. 0.8625 g. 80%

i. 8.52% k. 18.75% m. 46.35 o. 7.192

q. 32.4 s. 1,185.6

3. 35% 5. 700 patients

7. a. 131% c. 43%

9. Approximately 6%; 9.6 lb 11. 5%

13. Approximately 2,160,000 (2.16 million) Americans are homeless.

15. Approximately \$90,300 17. \$4,544.40

19. \$440 21. 18%

24. Having the public pick apples will provide more income.

27. \$41.06

29. a. The average elementary principal makes an average of 63% more than the average teacher, the average junior high principal makes 74%

more, and the average senior high principal makes 87% more.

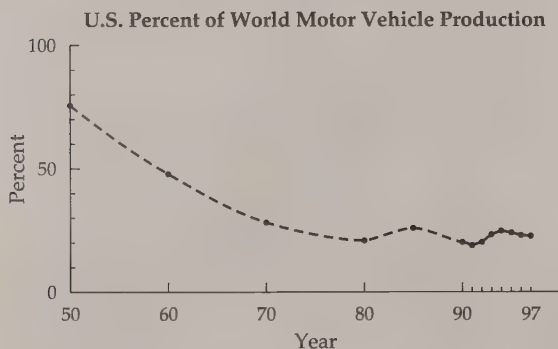
- b. The average junior high principal makes 7% less than the average senior high principal, the average elementary principal makes 13% less, and the average teacher makes 47% less.
31. a. 5 out of 1,000, or 1 out of 200, children will have a severe reaction.
33. In the United States, inflation caused prices to rise 5%, so a \$10 item would cost \$10.50. In a country with a 400% inflation rate, a \$10 item would cost \$50 a year later. The value of the item hasn't changed, but the value of the currency used to purchase the item has diminished.
35. a. \$17,405
37. a.

Country	Change 1993–2000	Estimated % change	Actual % change
United States	1.8	$1/5 = 20\%$	-21.4%
Cuba	1.3	$1/9 \approx 11\%$	-12.4%
Egypt	12.6	$1/6 \approx 17\%$	-16.1%
Haiti	7.3	$1/15 \approx 7\%$	-6.7%
Japan	0.2	$1/20 = 5\%$	-4.7%
Mexico	8.1	$1/4 = 25\%$	-28.1%
Somalia	75.6	$1/2 = 50\%$	-46.5%

39. Answers will vary. It is a 20% drop, but only 0.02 change in blood alcohol content.
41. This interpretation represents an interesting misconception. If we selected 100 students at random, we might or might not find that 8 of them were working full-time. The 8% means that if we take the fraction of students that are working full-time and convert that fraction to a fraction whose denominator is 100, that fraction will be closer to $8/100$ than to $7/100$ or $9/100$.
43. a. 600 square feet of window space
b. It depends on the size of the windows. If the windows are 5 feet $\times$ 4 feet, it would be 30 windows.
45. \$121,885.98
49. $2\frac{1}{2}$ hours
53. a. 80 seconds

Exercises 7.1, page 362

2. a.

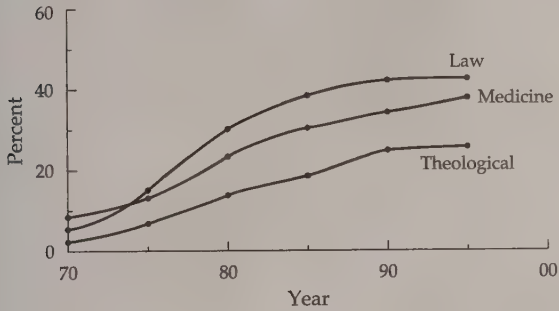


Technically, we should either connect the first six points with dotted lines or not connect them at all, because doing so would imply that the decline or increase during these periods was steady. As we can see from the 1990–1997 period, there has been moderate fluctuation. The key error many students make here is to use the same interval between all of the years reported, whereas in fact, the space between 1950 and 1960 should not be the same as the space between 1996 and 1997.

- c. Although the United States produced about 50% more motor vehicles in 1997 than in 1950, its percentage of the world total production has fallen from 76% to 23%. That is, 3 out of every 4 motor vehicles made in 1950 were made in the United States, whereas less than 1 in 4 made in 1997 were made in the United States.
- d. World production is about 5 times as much in 1997 as it was in 1950.

3. a.

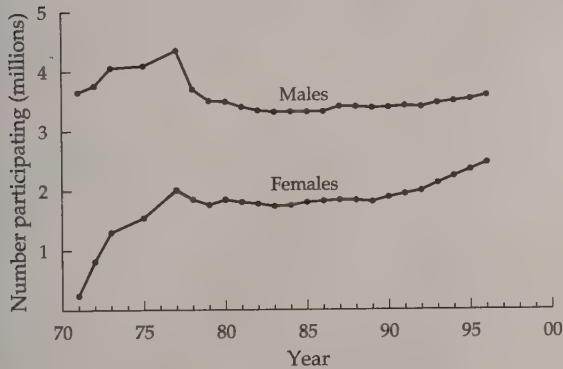
Percent of First Professional Degrees Earned by Women



- c. The percentages of women receiving degrees in these fields have risen dramatically in the past 25 years. By 1995 about 4 out of every 10 degrees in medicine and law went to women, and about 1 in 4 theological degrees went to women.

5. a.

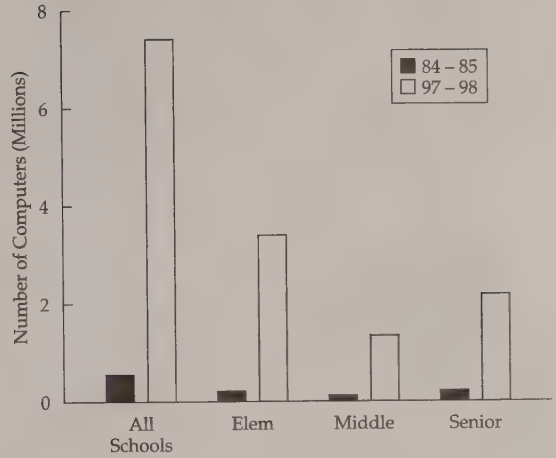
Participation in High School Athletic Programs



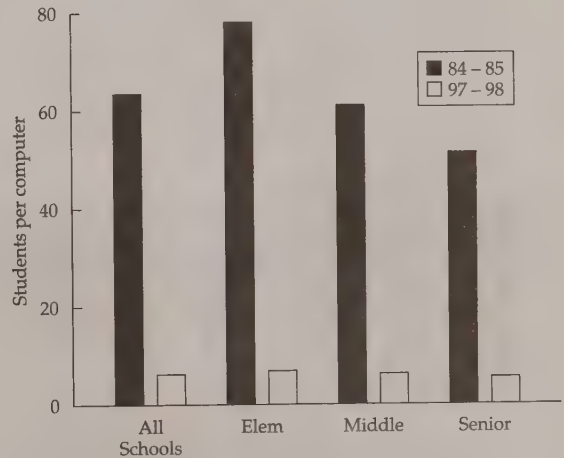
- c. The number of females participating in high school athletic programs increased sharply for several years, declined for two years, remained relatively constant during the 1980s, and rose gradually in the 1990s. Since 1990, the number of females participating has increased by 33%. On the other hand, the number of males participating in high school athletic programs increased between 1971 and 1978 and then declined for several years. The numbers in the 1980s were fairly stable. Since 1990, the number of males participating has increased by 9%.

8. a.

Computer Use in Schools



Computer Use in Schools



- b. Bar graphs or stacked bar graphs are the only reasonable choices for comparing the two years. Technically, you could make side-by-side circle graphs for the "Number of computers" data, but we are interested not in the changes in percentages of computers at each level but in the relative availability of computers to students.
- d. There are more than 13 times as many computers in schools in 1998 as there were in 1984. From another perspective, the number of students per computer has dropped from about 1 computer for every 63 students to 1 computer for every 6 students.
- e. I would use the "Students per computer" numbers because this is a rate that "equalizes" the numbers. There are many more elementary students than middle or high school students, so

- the raw numbers don't tell us as much. The rates tell us that, proportionately, there are significant differences in ratios in 1984, whereas in 1998 the ratios were all in the range of 1 student to every 5 or 6 computers.
9. You could make a bar graph showing the 1990 and 2025 populations of each region next to each other. You could also make two circle graphs side by side so that you could compare the sizes of the slices for the two years.
11. a. A bar graph would work well to compare the victims' reasons for not reporting crimes according to the categories listed.
b. Answers will vary. The reason most commonly given for not reporting a crime was that the crime was not successfully completed. Other common reasons given were lack of proof, that it was considered a private matter, that the stolen property did not have ID numbers, and that the victims thought that the police did not want to be bothered.
c. The data probably come from a survey.
d. Without more information, its accuracy is questionable. The reported reasons may not be people's real reasons.
e. Possibilities include correlating the type of crime with the reason for not reporting it and describing the sampling techniques used in the survey.
- 13.

	Valid/ invalid	Advantages/ strengths	Disadvantages/ limitations
Line	Invalid		We are not comparing data over time.
Bar	Valid	Can show the numbers of people with each type.	
Circle	Invalid		Although you could add the 10 numbers to get a total, this number is meaningless.

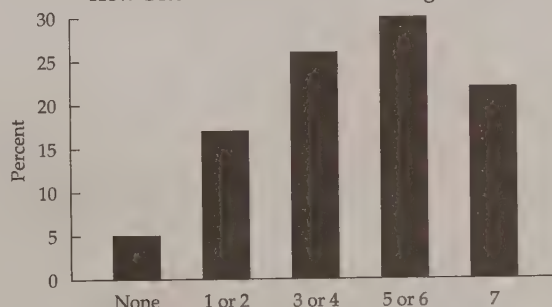
15. a. The number of hours the average worker in the United States needs to work to earn a 12-item food basket dropped from almost 10 hours in 1920 to about $1\frac{1}{2}$ hours in 1997.
b. What items were in the food basket? My guess would be milk, eggs, bread, meat, fruit, and vegetables. I would hope the food would
- represent a balanced diet. Where did the numbers come from? Did the researchers take an average worker? Who would be an average worker?
- c. A line graph is a good choice—we are comparing changes over time.
d. The x and y axes make sense and are labeled well, but we really need a better sense of how the numbers were determined.
e. See the answers to parts b and d.
17. a. The unemployment rates decline steadily as the amount of education increases. (To speak more mathematically, the unemployment rates are inversely proportional to the amount of education.) During the 1992–1996 period, the unemployment rates of all groups declined each year.
b. No concerns except how unemployment is determined—for example, totally unemployed vs. underemployed, and unemployed actively looking for work vs. unemployed and not actively looking for work.
c. A line graph is a good choice—we are comparing changes over time.
d. Don't see the need for the arrows. Would be nice to have all the text describing each line at the right of the graph.
21. a. The more education one has, the more one makes, the average earnings of people with advance degrees being more than triple the average earnings of people with less than a high school diploma. The average earnings of all groups increased between 1975 and 1995.
c. The choice of graph is appropriate.
d. I would like to have had a horizontal line for each of the numbers on the y axis. For example, the 1995 number for Bachelor's degree is somewhat higher than 35,000, but that's about all we can see. The three dimensional bars make it harder to approximate the numbers than if traditional bars had been used.
22. a. When asked to describe what they wanted students to learn, from the lessons that were videotaped, over $1/2$ of the German and U.S. teachers' responses fell into a "Skills" category, compared to $1/4$ of the Japanese teachers. Conversely, roughly $1/4$ of the German and U.S. teachers' responses fell into a "Thinking" category, compared to almost $3/4$ of the Japanese teachers.
c. Side-by-side bar graphs make it easier to compare data.
d. It would be nice if the bars weren't so close together.

Exercises 7.2, page 401

2. b. Mean: 65.15; standard deviation: 2.43 c. 68%
4. a. Grouped frequency bar graph, histogram, or circle graph, box-and-whisker, and line plot are appropriate.
b. Because the distribution is skewed, the mean, median, and mode are not convergent. The median is 18. The mode is 16. The mean is 24.2.

- c. The range is from 12 to 75; the data are skewed to the right; 51 and 75 are outliers.
7. Over $1/3$ of our sample had no alcohol in the past week and almost $1/3$ had more than 5 drinks, so it is hard to say what the average person looks like in this case. Furthermore, the three "averages" are all quite different: The mode is 0, the median is 2, and the mean is about $4\frac{1}{2}$ drinks if we count the "10–15" as 12.5.
8. b. The graph here almost has to be a bar graph. A circle graph is technically valid, but 12 categories make for many slices. Also, because the numbers are small, a change of one person would make the slice much bigger or smaller.
- c. Is it "your favorite sport" for participation or as a spectator? We cannot be sure everyone was answering the same question.
- d. The mode is basketball. There is no median or mean.
10. The lengths of the rivers range from about 2300 to about 4200 miles. One-fourth of the rivers are longer than 3600 miles and one-fourth are between 2300 and 2600 miles in length. The middle 9 rivers are between 2600 and 3600 miles long.
11. 93
13. 40 miles per hour
16. 2
22. The mode will have the smallest value, the median will have the middle value, and the mean will have the highest value.
24. a. The estimated mean becomes a reference point from which all the scores are measured. The mean of the differences tells you how far your estimated mean is from the actual mean.
- b. Answers will vary.
28. 61, 71; 1% or 15 boys
30. The mean will increase by that amount. The standard deviation will not be affected.
32. Without z-scores, one can only approximate the area under the curve. Approximately 35%, or 700 tires, will wear out before 45,000 miles.
34. Math
37. Julie is right.
39. Rounding scores generally has an insignificant effect on the final grade.
43. a. Did you get a representative sample, including, for example, both rich and poor, both urban and rural, and different parts of the country? For example, if most of your sample is from cities, then my guess is that the numbers with two or more TVs and two or more VCRs are higher than the actual data would be.
- b. Do you have a television in your house? (If the answer is no, do not count this person.) If the answer is yes, how many TVs? How many VCRs?
- d. One could make two bar graphs to show the data, but a circle graph quickly shows the percentages of each question.
- e. I don't see any advantage to the graphs. I could already see that $3/4$ of the households that had a TV had more than one TV and that about $1/4$ of the households that had a TV had two or more VCRs. *Note:* There is one problem with these data; they imply that every household that has a television has at least one VCR, because the VCR percentages add to 100%. I don't believe it!
46. a. Whom did you survey? I would not say that over half of the families I know eat dinner together 5 or more days a week. Does it count if only part of the family is there? Were these data gathered from two-parent families or from one- and two-parent families?
- b. How often does your family eat dinner together in an average week during the school year? (I would give them the categories in the table below or I would ask for a specific number. For example, a response of "2 or 3" would create problems in comparing to the data given.)
- c.

How Often Families Eat Dinner Together



- d. A circle graph would have been OK here, but there is a numerical progression (from none to 1 or 2 to 3 or 4 to 5 or 6 to 7), and it is easier to follow this progression with a bar graph. One main advantage of a circle graph is that it gives you the part of the whole; in this case, the data are in percentages, so you get that from the bar graph also.
- e. The bar graph does help you see that the percentages increase as the number of days per week eating together increases, up to eating dinner together every day.

Exercises 7.3, page 425

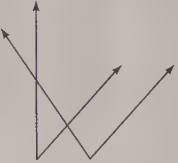
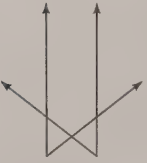
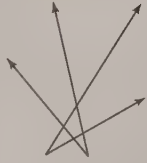
1. a. The probability of picking a black is the same in both cases.
- b. Pick from 6 blacks and 2 whites.
3. a. $11/1105 \approx 0.01$

- b. 27/2197
 c. $3/13 + 3/13 + 3/13$ would be the probability of drawing a face card the first time *or* the second time *or* the third time.
5. No. The class is not a representative sample of the people who died last year.
 7. No, it is not a fair game.
 9. Possible results: 0, 1, 2, 3, 4, 5. Most likely difference: 1. Probability that the difference is 1: $10/36 = 5/18$.
 11. 1 in 256; $1/256 = 0.0039$, or thirty-nine ten-thousandths.
 14. $1/216$
16. Answers will vary. Here is one possibility: One die must have the same number on all its faces; the other die could have 1, 1, 3, 3, 5, 5 or three of one odd number and three of another.
 18. Yes, $P(WWR) = P(WRW) = P(RWW) = 1/5$
 20. a. $81/256$ b. $3/32$
 24. Many solutions are possible. If the shape of the spinner remains the same, one possibility is $\$6(1/8) + \$4(1/8) + \$2(1/8) + \$2(1/8) + \$1(1/4) + \$0(1/4) = \$2$.
 26. No. The player can only win half of the cost of playing the game.

Exercises 7.4, page 436

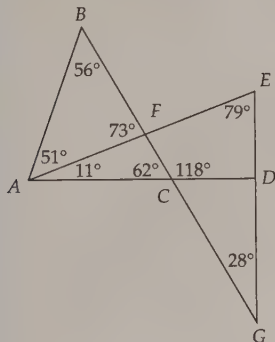
1. 720 3. 5040 5. $1/20$ 7. 220
9. If the flavors were scooped in any order, there would be 84 possibilities. If you specified the order of the flavors, there would be 504 possibilities.
11. $\frac{26}{27}$ 13. a. 24 b. 96
 15. $1/17$ 17. $1/5040$ 19. ${}_nP_n = n!$
 22. a. $11!$ or 39,916,800 possible words
 b. Answers will vary.

Exercises 8.1, page 455

1. $\overrightarrow{AC}; \overrightarrow{BC}; \overrightarrow{CA}; \overrightarrow{BA}$
 3. 
4. a. 
 b. 
 c. 
5. a. False; skew lines do not intersect and are not parallel
 c. True e. True
 g. False; think of two lines on the plane determined by this sheet of paper and a third line perpendicular to this plane.
 i. False; the planes could intersect in a line
 k. True l. True
7. Estimates will vary. a. 45° c. 125°
 11. Several answers are possible. Examples are given.
- a. $\overline{AB}$ and $\overline{DC}$
 c. $\overline{AB}$ and $\overline{AD}$
 e. $\overline{AB}$ and $\overline{EH}$
 g. All intersecting line segments are perpendicular.
13. a. $119^\circ 29'$
 14. a. 24 times c. 150° (and 210°)

Exercises 8.2, page 474

1.



4. a. Concave polygon
c. Not a polygon
e. Convex polygon
5. a. Square, rectangle, right isosceles triangle (two different sizes), 9-gon
8. Answers will vary. These are some examples.

a.



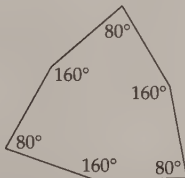
c.



e.



g.

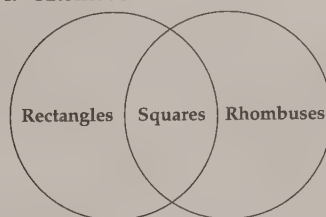


12. 15 cm is too long for the third side. No matter what angle is formed between the 6 cm and 8 cm sides,

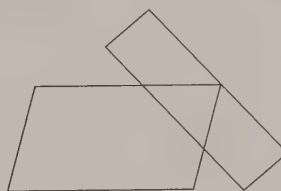
the distance between the endpoints of these sides, which is the length of the third side, will be less than 14 cm.

14. a. Parallelogram, rhombus, rectangle, square
c. Square, rhombus
e. Rectangle, square, isosceles trapezoid
g. Parallelogram, rhombus, rectangle, square
i. Rhombus

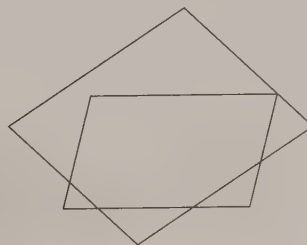
16.



20. a.



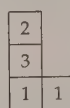
c.



Exercises 8.3, page 485

1. Answers will vary. Several correct answers.
a. $\triangle ABC$ b. A c. $\overline{AB}$
2. a. Octagonal prism c. Rectangular prism
3. The polyhedron on the right is convex.
6. Yes. If all the edges of the base are of different lengths, then the triangular faces will not be congruent.
8. $n(n-1)$ diagonals
10. $n+1$ vertices; $n+1$ faces; $2n$ edges
13. The sides are isosceles triangles.
15. (i) The figure will have 7 cubes.
(ii) The left-hand view will be a reflection of the right-hand view.

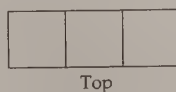
(iii)



(iv)



17. The first one
20.

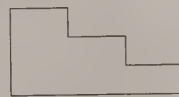


Top

18. The first one



Right side

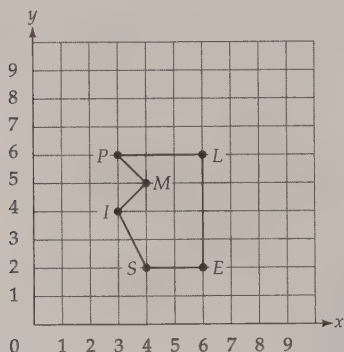


Front

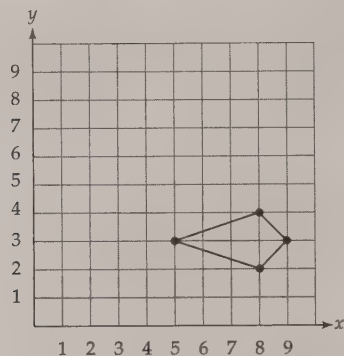
Exercises 9.1, page 506

1. Each point of the figure has been moved 2 units to the right and 3 units down.

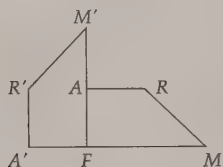
Also, the figure has been moved approximately 3.6 units along a vector that makes a 55-degree angle with the x -axis. See diagram.



3. See diagram.

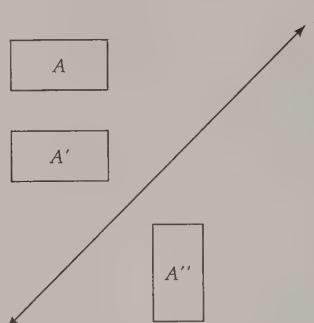


6. a. The first and third cases are reflections of each other. The second case is not a reflection because a point and its image are not equidistant from the line of reflection.
8. See diagram.

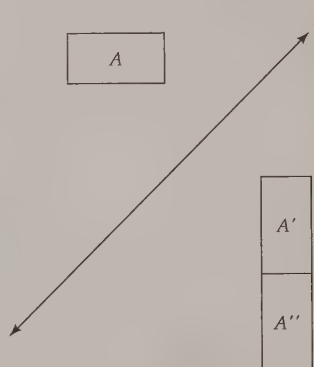


10. Point A' Point B' Point C' Point D'
- a. (4, 7) (6, 7) (6, 6) (4, 4)
- c. (4, 3) (6, 3) (6, 2) (4, 0)

12. a. Answers will vary.
- b. See diagram.

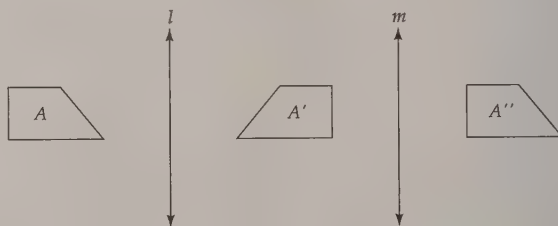


- c. See diagram.

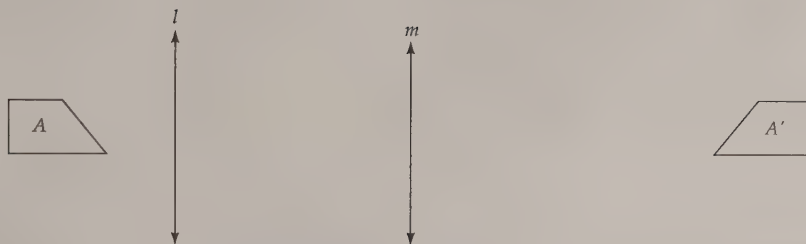


The image from translating and then reflecting does not coincide with the image from reflecting and then translating.

14. a. See diagram.

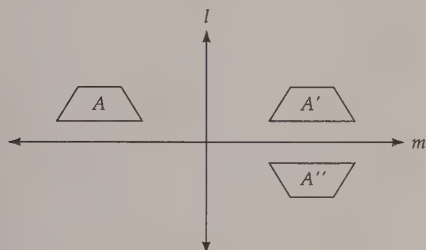


- b. Answers will vary.
c. See diagram.



The trapezoid A'' in part a is the image produced by reflecting across line l and then across line m . Trapezoid A' above is the image produced by reflecting across line m . The image produced by reflecting across line m and then line l is off the page (to the left).

16. a. See diagram.



- b. Answers will vary.
c. The double reflection produces the same image, no matter what the order.
18. a. Translate $\overrightarrow{ST}$ along the vector $\overrightarrow{SU}$.

- b. Rotate the figure 180 degrees about point M .
c. No; in order for the image of $\overline{UM}$ to be $\overline{MT}$, $\overline{UM}$ would need to be perpendicular to $\overline{SR}$, which would be the case only if $STRU$ were a rhombus.
20. a. Rotate $\triangle SRO$ 180 degrees around point Y .
24. Answers will vary.
a. Baby Blocks: If we see the figure as composed of rhombuses, each rhombus can be mapped onto a neighboring rhombus by a 60-degree rotation. If we see the figures as hexagons (comprising three different-colored rhombuses), each hexagon can be translated onto any other hexagon.
28. a. 9 o'clock

Exercises 9.2, page 531

1. a. 60-, 120-, 180-, 240-, and 300-degree rotation symmetry; no reflection symmetry
c. Horizontal and vertical line symmetry; point symmetry
3. a. Vertical line symmetry
c. No symmetry because of the line segments
4. a. Point symmetry
b. Vertical line symmetry
5. a. Horizontal, vertical, and diagonal line symmetry; 90-, 180-, and 270-degree rotation symmetry
b. 90-, 180-, and 270-degree rotation symmetry
6. a. Each of the right isosceles triangles has one line of symmetry.
The parallelogram has point symmetry.
The square has vertical and horizontal symmetry and two diagonal lines of symmetry; it also has 90-, 180-, and 270-degree rotation symmetry.
8. Answers may vary, according to how the digits are written.
1—no symmetry
2—no symmetry
3—horizontal line symmetry
4—no symmetry
5—no symmetry
6—no symmetry
7—no symmetry
8—horizontal and vertical line symmetry; point symmetry
9—no symmetry
0—horizontal and vertical line symmetry; point symmetry
10. a. False b. True c. False
20. b. The diamond-shaped unit is translated horizontally; horizontal and vertical line symmetry; point symmetry.
16. a. Note: All strip patterns, by definition, have translation symmetry. ll; no symmetry.
c. lg; glide reflection symmetry
e. l2; point symmetry
33. a. The pattern has several translation symmetries. It has vertical and horizontal reflection symmetry.
It has 180-degree rotation symmetry.
It has glide reflection symmetry.
37. a. A and F. Descriptions will vary.

- c. A and C (and D and E) are 180-degree rotations of each other about their points of intersection. B , C , and D (and E , F , and G) are 120-degree rotations of each other about the points where

the three chevrons in each cluster intersect.

38. a. Tessellates b. Tessellates c. Tessellates
d. Tessellates k. Does not tessellate

Exercises 9.3, page 543

1. a. 13.5 cm
3. Yes; the angles are the same, and the sides of the figure to the right are twice as long as the sides of the smaller triangle.
4. a. No; different isosceles triangles can have different angles.
c. Yes; all squares have four 90-degree angles and proportionate sides.

- e. Yes; corresponding sides and angles of congruent polygons are congruent.
6. Yes, in general, but it may be distorted by the angles of the mirrors.
8. c. ≈ 666.87 ft
10. The length and width of the document are enlarged by 21 percent. The copy is 121 percent the size of the original.

Exercises 10.1, page 556

1. Answers may vary. Following are examples for the given objects.
 - a. surface area, volume, amount of pollutants, temperature of the water (at various levels), depth
 - c. height, surface area of sides (for painting), surface area of roof (for shingles), surface area of windows, surface area of floors, ratio of area of windows to area of floors (to determine adequacy of ventilation)
2. Answers may vary.
 - a. centimeters or meters
 - c. millimeters or centimeters
 - e. milligrams
3. a. 0.5 c. 0.670 e. 0.450 g. 0.024
6. About 34 meters. If we use their best times, Alan would be 7 seconds behind when Bill crosses the finish line. At Alan's pace (400 meters/82 seconds), he will run about 34 meters in 7 seconds.
8. The difference in conversions is less than an inch.
12. a. The makers of the table essentially set the American/London foot as the unit, with 1000 parts; the other numbers in this column enable us to see the length of other "foots" in relation to

the American/London foot. By setting the unit foot as 1000 parts, they were able to avoid decimals.

14. b. $48.3 \text{ liters} \times 34 \text{ oz/liter} \times 1 \text{ gallon}/128 \text{ oz} = 12.83 \text{ gallons}$; or, by approximation, $48.3 \text{ liters} \times 1 \text{ quart}/1 \text{ liter} \times 1 \text{ liter}/4 \text{ quarts} = 12.08 \text{ gallons}$. The difference would be about $3/4$ gallon.
17. 5° Celsius, because each degree Celsius is equivalent to 1.8 degrees Fahrenheit.
19. a. $35 \text{ ft}/9 \text{ sec} \times 1 \text{ mi}/5280 \text{ ft} \times 60 \text{ sec}/1 \text{ min} \times 60 \text{ min}/1 \text{ hr} \approx 2.65 \text{ miles per hour}$
21. $13\frac{1}{3}\%$. 20% of 12 oz = 2.4 oz. $2.4 \text{ oz milk}/18 \text{ oz coffee} \times 100 = 13\frac{1}{3}\%$.
27. About 100 meters! The height depends on the thickness of the paper. Multiply the thickness by the number of sheets of paper; $2^{20} = 1,048,567$ sheets of paper. I measured the height of 2000 sheets of paper (4 reams) and got 190 millimeters. This ratio is equivalent to 0.095 millimeter per sheet.
30. a. 1 kask is about 11 kilometers. $1 \text{ kask} \times 30 \text{ ush/kask} \times 60 \text{ gar/ush} \times 60 \text{ palm/gar} \times 4 \text{ in/palm} \times 2.54 \text{ cm/in} \times 1 \text{ m}/100 \text{ cm} \times 1 \text{ km}/1000 \text{ m} = 10.97 \text{ km}$

Exercises 10.2, page 570

1. a. $P = 69 \text{ mm}$, $A = 195 \text{ mm}^2$
b. $P = 80 \text{ mm}$, $A = 148.5 \text{ mm}^2$
c. $P = 6 \text{ cm}$, $A = \sqrt{3} \approx 1.7 \text{ cm}^2$
d. $P \approx 37.68 \text{ cm}$, $A \approx 113.04 \text{ cm}^2$
e. $P = 40 \text{ cm}$, $A = 72 \text{ cm}^2$
f. $P = 35.8 \text{ cm}$, $A \approx 51.9 \text{ cm}^2$
3. a. $A = 15$ square units
4. a. Approximately 8.81 square centimeters
5. Approximately 11.2 feet
6. ≈ 7.4 feet

7. Solution paths will vary.

$$\begin{aligned} \text{a. } A &= (5 \times 10) + (5 \times 5) + (10 \times 10) = 50 + 25 + 100 = 175 \text{ square centimeters} \\ A &= (20 \times 10) - (5 \times 5) = 200 - 25 = 175 \text{ square centimeters} \end{aligned}$$

8. 14.14 inches on a side. The length of the sides of a square with an area of 200 sq. in. is $\sqrt{200} = 10\sqrt{2} \approx 10 \times 1.414 \approx 14.14$.
9. $\$39.90$. $9600 \text{ sq. ft.}/1050 \text{ sq. ft. per bag} = 9.14 \text{ bags}$. She needs to buy 10 bags.

14. a. ≈ 11.1 square centimeters = 100 sq. cm./9 small triangles.
 16. b. πr
 18. Approximately 344 square centimeters
 20. a. 1250π square feet
 22. a. 3 meters
 b. About 2.24 meters
 c. About 1.73 meters
 d. The longest distance is along the edges of the cube; the shortest distance is directly through the cube.
38. a. 10π
 39. a. 84 square feet
 40. a. 25 square inches b. Not possible
 41. a. Need to assume that the lengths of the shorter sides that are not marked are all 2 meters and that the angles that look like right angles are right angles. Area is 60 square meters.

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1. a. S.A. ≈ 1474 square inches
 $V = 2160$ cubic inches
 2. a. S.A. ≈ 47 square feet
 $V = 16$ cubic feet
 c. S.A. = 1980 square feet
 $V = 2985$ cubic feet
 4. a. Predictions and explanations will vary.
 Volume of new tent ≈ 322.7 cubic feet
 Volume of old tent = 31.5 cubic feet
 The new tent is 291 cubic feet larger than the old tent or has about 10 times the volume!
 6. Assuming 1 ream is 2 inches in height, you can get 210 reams, which equals 105,000 sheets, and 42% of the yearly purchase.
8. 34 feet
 10. a. Stack them into a cube each side being 60 cm long.
 14. a. Prediction: 15 cubic centimeters
 b. ≈ 16.387 cubic centimeters
 16. a. S.A. of small prism : S.A. of large prism = 1:9
 This is the square of the ratio of their sides.
 b. Volume of small prism : volume of large prism = 1:27
 This is the cube of the ratio of their sides.
 17. a. 28 dominoes
 19. a. The cut should be one-fourth of the way from the end.

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TRIANGLES

The dome structure shown was first conceptualized by the architect Buckminster Fuller in 1927. The structure displays many triangles that are the same shape and size. Such triangles, known as congruent triangles, are also seen in the Ferris wheel displayed in Exercise 37 of Section 3.3.

As we shall see, the properties of triangles found in this chapter are also applied in the development of the characteristics of quadrilaterals that we study in Chapter 4.

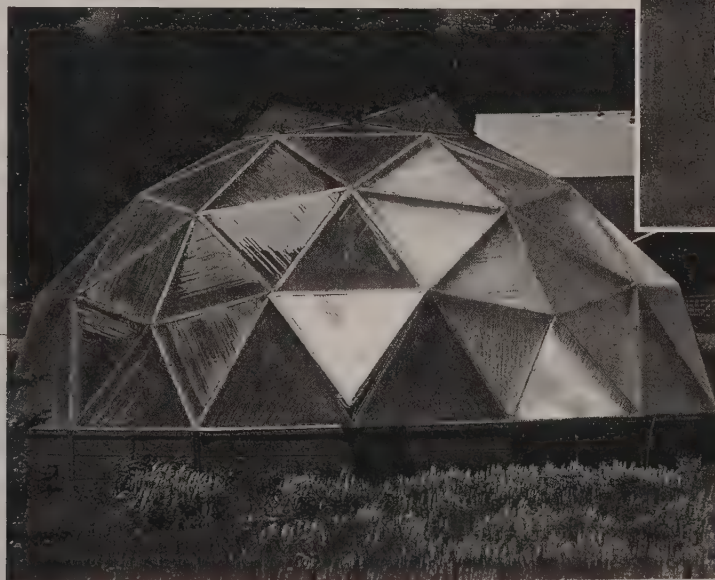
CHAPTER OUTLINE

- 3.1 Congruent Triangles
- 3.2 Corresponding Parts of Congruent Triangles
- 3.3 Isosceles Triangles
- 3.4 Basic Constructions Justified
- 3.5 Inequalities in a Triangle

A Look Beyond:
Historical Sketch of Archimedes

Summary

For online student resources, visit this textbook's web site at math.college.hmco.com/students.

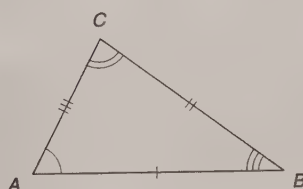




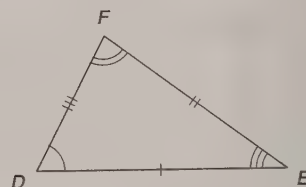
CONGRUENT TRIANGLES

Two triangles are **congruent** if one coincides with (fits perfectly over) the other. In Figure 3.1, we say that $\triangle ABC \cong \triangle DEF$ if these congruences hold:

$$\begin{array}{ll} \angle A \cong \angle D & \overline{AB} \cong \overline{DE} \\ \angle B \cong \angle E & \overline{BC} \cong \overline{EF} \\ \angle C \cong \angle F & \overline{AC} \cong \overline{DF} \end{array}$$



(a)



(b)

FIGURE 3.1

From the indicated congruences, we also say that vertex A corresponds to vertex D , as does B to E and C to F . In symbols, this correspondence is represented by

$$A \leftrightarrow D \quad B \leftrightarrow E \quad C \leftrightarrow F$$

The claim $\triangle MNQ \cong \triangle RST$ orders corresponding vertices of the triangles (not shown), so we can conclude from this statement that

$$M \leftrightarrow R, \quad N \leftrightarrow S, \quad \text{and} \quad Q \leftrightarrow T$$

This correspondence of vertices implies the congruence of corresponding parts such as $\angle M \cong \angle R$ and $\overline{NQ} \cong \overline{ST}$.

Conversely, if the correspondence of vertices of two congruent triangles is $M \leftrightarrow R$, $N \leftrightarrow S$, and $Q \leftrightarrow T$, we can write $\triangle MNQ \cong \triangle RST$, $\triangle NQM \cong \triangle STR$, and so on.

DISCOVER!

Holding two sheets of construction paper together, use scissors to cut out two triangles. How do the triangles compare?

ANSWERS
The triangles are congruent.

DEFINITION: Two triangles are **congruent** when the six parts of the first triangle are congruent to the six corresponding parts of the second triangle.

As always, any definition is reversible! If two triangles are known to be congruent, we may conclude that the corresponding parts are congruent. Moreover, if the six pairs of parts are known to be congruent, then so are the triangles! From the congruent parts indicated in Figure 3.2, we can conclude that $\triangle MNQ \cong \triangle RST$.

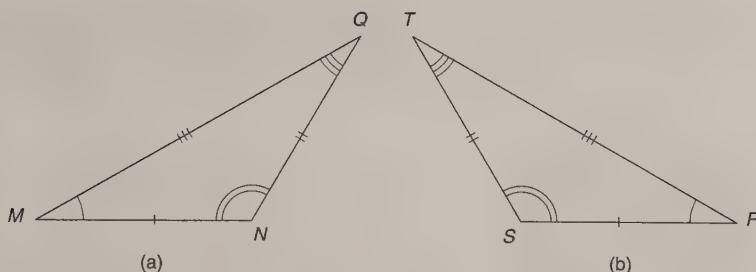


FIGURE 3.2

Here are some of the properties of congruent triangles that are useful in later proofs and explanations:

1. $\triangle ABC \cong \triangle ABC$ (Reflexive Property of Congruence)
2. If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$. (Symmetric Property of Congruence)
3. If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHI$, then $\triangle ABC \cong \triangle GHI$. (Transitive Property of Congruence)

On the basis of the properties above, we see that the “congruence of triangles” is an equivalence relation.

It would be difficult to establish that triangles were congruent if six pairs of congruent parts had to be verified first. Fortunately, it is possible to prove triangles congruent with fewer than six pairs of congruences. To suggest a first method, consider the construction in Example 1.

EXAMPLE 1

Construct a triangle whose sides have the lengths of the segments provided in Figure 3.3(a).

Solution

Choose $\overline{AB}$ as the first side of the triangle and mark its length as shown in Figure 3.3(b).

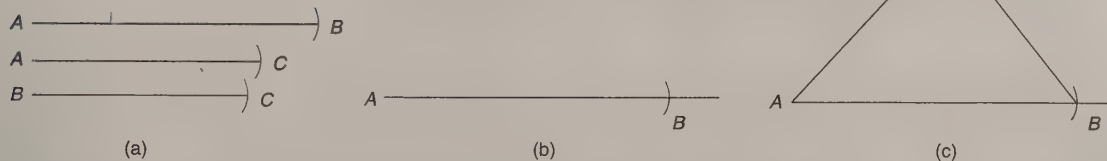
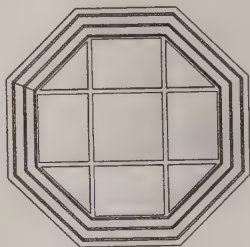


FIGURE 3.3

GEOMETRY

IN THE REAL WORLD



The four triangular panes in the octagonal window are congruent triangles.

Using the left endpoint A , mark off an arc of length equal to that of $\overline{AC}$. Now mark off an arc the length of $\overline{BC}$ from the right endpoint B so that these arcs intersect at C , the third vertex of the triangle. Joining point C to A and then to B completes the desired triangle. [See Figure 3.3(c).]

Consider Example 1 once more. If a “different” triangle were constructed by choosing $\overline{AC}$ to be the first side, it would be congruent to the one shown. It might be necessary to flip or rotate it to have corresponding vertices match, but that is perfectly acceptable! The point of Example 1 is that it does provide a method for establishing the congruence of triangles, using only three pairs of parts. If corresponding angles are measured in the previous triangle or in any other triangle constructed with the same lengths for sides, these pairs of angles will also be congruent!

SSS (METHOD FOR PROVING TRIANGLES CONGRUENT)

POSTULATE 12:

If the three sides of one triangle are congruent to the three sides of a second triangle, then the triangles are congruent (SSS).

The designation SSS will be cited as a reason in the proof that follows. The three S letters refer to the three pairs of congruent sides.

EXAMPLE 2

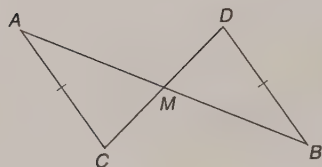


FIGURE 3.4

Given: $\overline{AB}$ and $\overline{CD}$ bisect each other at M
 $\overline{AC} \cong \overline{DB}$

(See Figure 3.4.)

Prove: $\triangle AMC \cong \triangle BMD$

PROOF

Statements	Reasons
1. $\overline{AB}$ and $\overline{CD}$ bisect each other at M	1. Given
2. $\overline{AM} \cong \overline{MB}$ $\overline{CM} \cong \overline{MD}$	2. If a segment is bisected, the segments formed are $\cong$
3. $\overline{AC} \cong \overline{DB}$	3. Given
4. $\triangle AMC \cong \triangle BMD$	4. SSS

NOTE: In steps 2 and 3, the three pairs of sides were shown to be congruent; thus SSS is cited to justify that $\triangle AMC \cong \triangle BMD$.

The two sides that form an angle of a triangle are said to **include that angle** of the triangle. In $\triangle TUV$ in Figure 3.5(a), sides $\overline{TU}$ and $\overline{TV}$ form $\angle T$; therefore, $\overline{TU}$ and $\overline{TV}$ include $\angle T$. In turn, $\angle T$ is said to be the included angle for $\overline{TU}$ and $\overline{TV}$.

Similarly, any two angles of a triangle must have a common side, and these two angles are said to **include that side**. In $\triangle TUV$, $\angle U$ and $\angle T$ share the common side $\overline{UT}$; therefore, $\angle U$ and $\angle T$ include the side $\overline{UT}$. $\overline{UT}$ is the side included by $\angle U$ and $\angle T$.

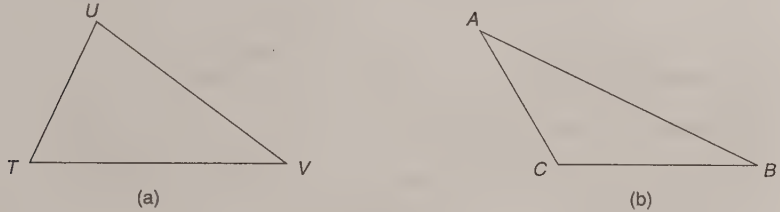


FIGURE 3.5

EXAMPLE 3

In $\triangle ABC$ of Figure 3.5(b):

- Which angle is included by $\overline{AC}$ and $\overline{CB}$?
- Which sides include $\angle B$?
- What is the included side for $\angle A$ and $\angle B$?
- Which angles include $\overline{CB}$?

Solution

- $\angle C$ (because it is formed by $\overline{AC}$ and $\overline{CB}$)
- $\overline{AB}$ and $\overline{BC}$ (because these form $\angle B$)
- $\overline{AB}$ (because it is the common side for $\angle A$ and $\angle B$)
- $\angle C$ and $\angle B$ (because $\overline{CB}$ is a side of each angle)

SAS (METHOD FOR PROVING TRIANGLES CONGRUENT)

A second way of establishing that two triangles are congruent involves showing that two sides and the included angle of one triangle are congruent to two sides and the included angle of a second triangle. If two people each draw a triangle so that two of the sides measure 2 cm and 3 cm and the included angle measures 54° , then those triangles are congruent. (See Figure 3.6.)

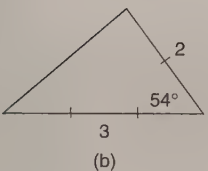
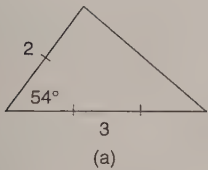


FIGURE 3.6

POSTULATE 13:

If two sides and the included angle of one triangle are congruent to two sides and the included angle of a second triangle, then the triangles are congruent (SAS).

The order of the letters SAS in Postulate 13 helps us remember that the two sides that are named have the angle “between” them. That is, in each triangle, the two sides form the angle.

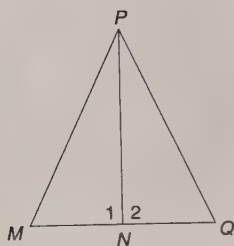
In Example 4, which follows, the two triangles to be proved congruent share a common side; the statement $\overline{PN} \cong \overline{PN}$ is justified by the Reflexive Property of Congruence, which is sometimes referred to as Identity. However stated, this

justification applies when triangles (or perhaps other polygons) have a part in common. In Example 4, note the use of SAS as the final reason.

EXAMPLE 4

Given: $\overline{PN} \perp \overline{MQ}$
 $\overline{MN} \cong \overline{NQ}$
 (See Figure 3.7.)

Prove: $\triangle PNM \cong \triangle PNQ$

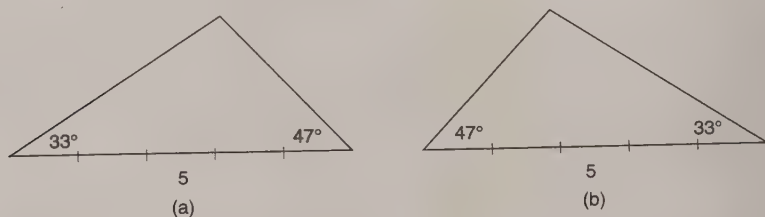
**FIGURE 3.7****P R O O F**

Statements	Reasons
1. $\overline{PN} \perp \overline{MQ}$	1. Given
2. $\angle 1 \cong \angle 2$	2. If two lines are $\perp$, they meet to form $\cong$ adjacent $\angle$ s
3. $\overline{MN} \cong \overline{NQ}$	3. Given
4. $\overline{PN} \cong \overline{PN}$	4. Identity (or Reflexive)
5. $\triangle PNM \cong \triangle PNQ$	5. SAS

NOTE: In $\triangle PNM$, $\overline{MN}$ (step 3) and $\overline{PN}$ (step 4) include $\angle 1$; thus SAS is used to verify that $\triangle PNM \cong \triangle PNQ$.

ASA (METHOD FOR PROVING TRIANGLES CONGRUENT)

The next method for proving triangles congruent requires a combination of two angles and the included side. If two people each draw a triangle for which two of the angles measure 33° and 47° and the included side measures 5 centimeters, then those triangles are congruent. See Figure 3.8.

**FIGURE 3.8****POSTULATE 14:**

If two angles and the included side of one triangle are congruent to two angles and the included side of a second triangle, then the triangles are congruent (ASA).

Although this method is written compactly as ASA, you must be careful as you write these abbreviations! For example, ASA refers to two angles and the included

FIGURE 3.9

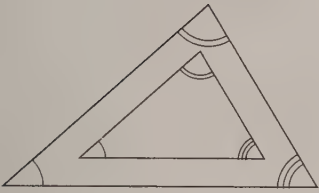
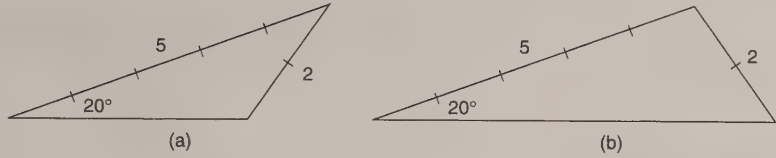


FIGURE 3.10

EXAMPLE 5

Given: $\overline{AC} \cong \overline{DC}$
 $\angle 1 \cong \angle 2$
(See Figure 3.11.)
Prove: $\triangle ACE \cong \triangle DCB$

P R O O F	
Statements	Reasons
1. $\overline{AC} \cong \overline{DC}$ (See Figure 3.12.)	1. Given
2. $\angle 1 \cong \angle 2$	2. Given
3. $\angle C \cong \angle C$	3. Identity
4. $\triangle ACE \cong \triangle DCB$	4. ASA

Next we consider a theorem (proved by the ASA postulate) that is convenient as a reason in many proofs.

AAS (METHOD FOR PROVING TRIANGLES CONGRUENT)

THEOREM 3.1.1: If two angles and a nonincluded side of one triangle are congruent to two angles and a nonincluded side of a second triangle, then the triangles are congruent (AAS).

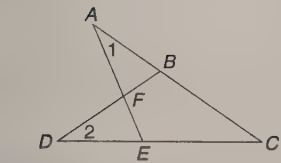


FIGURE 3.11

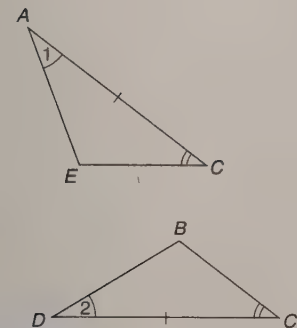
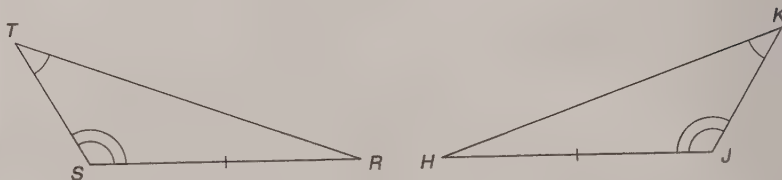


FIGURE 3.12

Given: $\angle T \cong \angle K$, $\angle S \cong \angle J$, and $\overline{SR} \cong \overline{HJ}$ (See Figure 3.13.)

Prove: $\triangle TSR \cong \triangle KJH$

FIGURE 3.13



PROOF

Statements	Reasons
1. $\angle T \cong \angle K$	1. Given
$\angle S \cong \angle J$	
2. $\angle R \cong \angle H$	2. If two $\angle$ s of one $\triangle$ are $\cong$ to two $\angle$ s of another $\triangle$, then the third $\angle$ s are also congruent
3. $\overline{SR} \cong \overline{HJ}$	3. Given
4. $\triangle TSR \cong \triangle KJH$	4. ASA

In summary, you may use SSS, SAS, ASA, or AAS to prove that triangles are congruent. Identity or Reflexive may be used to state a self-congruence when a side or angle is common to two triangles.

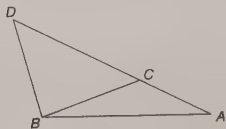
WARNING

Do not use AAA or SSA, because they are simply not valid for proving triangles congruent; with AAA the triangles have the same shape but are not necessarily congruent.

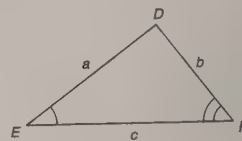
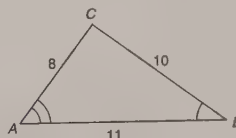
3.1 EXERCISES

In Exercises 1 to 8, use drawings as needed to answer each question.

1. Name a common angle and a common side for $\triangle ABC$ and $\triangle ABD$. If $\overline{BC} \cong \overline{BD}$, can you conclude that $\triangle ABC$ and $\triangle ABD$ are congruent? Can SSA be used as a reason for proving triangles congruent?



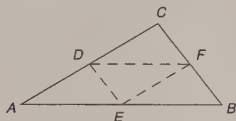
2. With corresponding angles indicated, the triangles are congruent. Find values for a , b , and c .



Exercises 2–4, 6

3. With corresponding angles indicated, find $m\angle A$ if $m\angle F = 72^\circ$.
4. With corresponding angles indicated, find $m\angle E$ if $m\angle A = 57^\circ$ and $m\angle C = 85^\circ$.

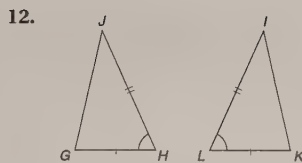
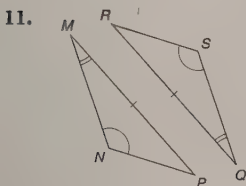
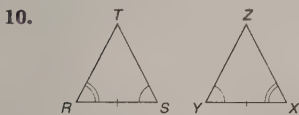
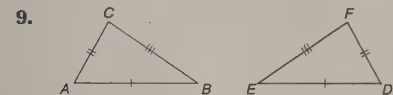
5. In a right triangle, the sides that form the right angle are the **legs**; the longest side (opposite the right angle) is the **hypotenuse**. Some textbooks say that when two right triangles have congruent pairs of legs, the right triangles are congruent by the reason LL. In our work, LL is just a special case of one of the postulates in this section. Which postulate is that?
6. In the figure for Exercise 2, write a statement that the triangles are congruent, paying due attention to the order of corresponding vertices.
7. In $\triangle ABC$, the midpoints of the sides are joined. What does intuition tell you about the relationship between $\triangle AED$ and $\triangle FDE$? (We will prove this relationship later.)



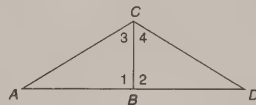
8. Suppose that you wish to prove that $\triangle RST \cong \triangle SRV$. Using the reason Identity, name one pair of corresponding parts that are congruent.



In Exercises 9 to 12, congruent parts are indicated by like dashes (sides) or arcs (angles). State which method (SSS, SAS, ASA, or AAS) would be used to prove the two triangles congruent.



In Exercises 13 to 18, use only the given information to state the reason why $\triangle ABC \cong \triangle DBC$. Use marks like those used in Exercises 9 to 12.



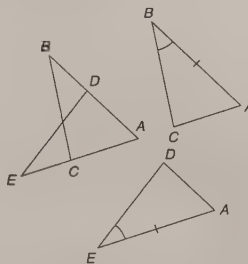
Exercises 13–18

13. $\angle A \cong \angle D$, $\overline{AB} \cong \overline{BD}$, and $\angle 1 \cong \angle 2$
14. $\angle A \cong \angle D$, $\overline{AC} \cong \overline{CD}$, and B is the midpoint of $\overline{AD}$
15. $\angle A \cong \angle D$, $\overline{AC} \cong \overline{CD}$, and $\overline{CB}$ bisects $\angle ACD$
16. $\angle A \cong \angle D$, $\overline{AC} \cong \overline{CD}$, and $\overline{AB} \cong \overline{BD}$
17. $\overline{AC} \cong \overline{CD}$, $\overline{AB} \cong \overline{BD}$, and $\overline{CB} \cong \overline{CB}$ (by Identity)
18. $\angle 1$ and $\angle 2$ are right $\angle$ s, $\overline{AB} \cong \overline{BD}$, and $\angle A \cong \angle D$

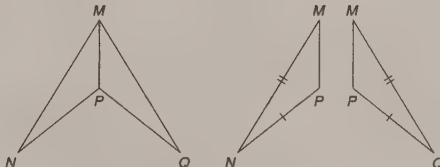
In Exercises 19 and 20, the triangles to be proved congruent have been redrawn separately. Congruent parts are marked.

- a) Name an additional pair of parts that are congruent by Identity.
- b) Considering the congruent parts, state the reason why the triangles must be congruent.

19. $\triangle ABC \cong \triangle AED$

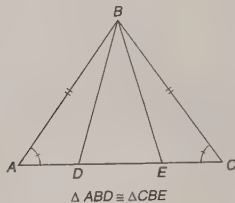


20. $\triangle MNP \cong \triangle MQP$

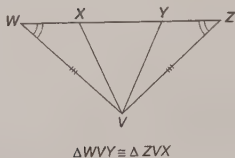


In Exercises 21 to 24, prove that the triangles named are congruent. Considering the congruent pairs marked, name the additional pair of parts that must be congruent for us to use the method named.

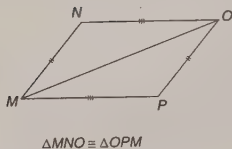
21. SAS



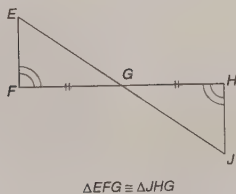
22. ASA



23. SSS



24. AAS

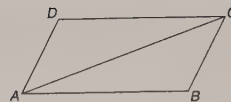


In Exercises 25 and 26, complete each proof.

25. Given: $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$
Prove: $\triangle ABC \cong \triangle CDA$

P R O O F

Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$	1. ?
2. ?	2. Identity
3. $\triangle ABC \cong \triangle CDA$	3. ?



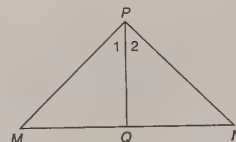
Exercises 25, 26

26. Given: $\overline{DC} \parallel \overline{AB}$ and $\overline{AD} \parallel \overline{BC}$
Prove: $\triangle ABC \cong \triangle CDA$

P R O O F

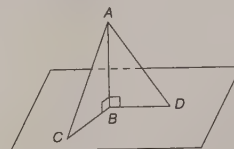
Statements	Reasons
1. $\overline{DC} \parallel \overline{AB}$	1. ?
2. $\angle DCA \cong \angle BAC$	2. ?
3. ?	3. Given
4. ?	4. If two $\parallel$ lines are cut by a transversal, alt. int. $\angle$ s are $\cong$
5. $\overline{AC} \cong \overline{AC}$	5. ?
6. ?	6. ASA

In Exercises 27 to 32, use SSS, SAS, ASA, or AAS to prove that the triangles are congruent.

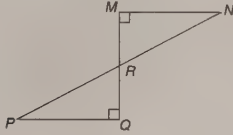


Exercises 27, 28

27. Given: $\overrightarrow{PQ}$ bisects $\angle MPN$
 $\overline{MP} \cong \overline{NP}$
Prove: $\triangle MPQ \cong \triangle NPQ$
28. Given: $\overline{PQ} \perp \overline{MN}$ and $\angle 1 \cong \angle 2$
Prove: $\triangle MPQ \cong \triangle NPQ$
29. Given: $\overline{AB} \perp \overline{BC}$ and $\overline{AB} \perp \overline{BD}$
 $\overline{BC} \cong \overline{BD}$
Prove: $\triangle ABC \cong \triangle ABD$



30. *Given:* $\overline{PN}$ bisects $\overline{MQ}$
 $\angle M$ and $\angle Q$ are right angles
Prove: $\triangle PQR \cong \triangle NMR$



31. *Given:* $\angle VRS \cong \angle TSR$ and $\overline{RV} \cong \overline{TS}$
Prove: $\triangle RST \cong \triangle SRV$

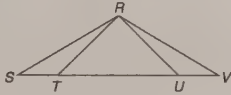


Exercises 31, 32

32. *Given:* $\overline{VS} \cong \overline{TR}$ and $\angle TRS \cong \angle VSR$
Prove: $\triangle RST \cong \triangle SRV$

In Exercises 33 to 36, the methods to be used are SSS, SAS, ASA, and AAS.

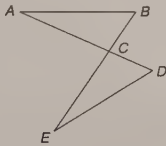
33. Given that $\triangle RST \cong \triangle RVU$, does it follow that $\triangle RSU$ is also congruent to $\triangle RVT$? Name the method, if any, used in arriving at this conclusion.



Exercises 33, 34

34. Given that $\angle S \cong \angle V$ and $\overline{ST} \cong \overline{UV}$, does it follow that $\triangle RST \cong \triangle RVU$? Which method, if any, did you use?

35. Given that $\angle A \cong \angle E$ and $\angle B \cong \angle D$, does it follow that $\triangle ABC \cong \triangle DEC$? If so, cite the method used in arriving at this conclusion.



Exercises 35, 36

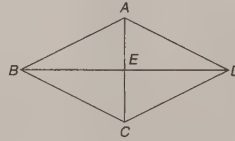
36. Given that $\angle A \cong \angle E$ and $\overline{BC} \cong \overline{DC}$, does it follow that $\triangle ABC \cong \triangle DEC$? Cite the method, if any, used in reaching this conclusion.

37. In quadrilateral $ABCD$, $\overline{AC}$ and $\overline{BD}$ are perpendicular bisectors of each other. Name *all* triangles that are congruent to:

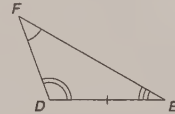
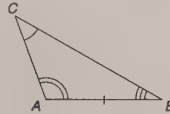
a) $\triangle ABE$

b) $\triangle ABC$

c) $\triangle ABD$

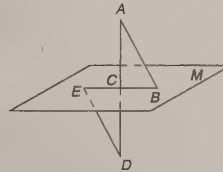


38. In $\triangle ABC$ and $\triangle DEF$, you know that $\angle A \cong \angle D$, $\angle C \cong \angle F$, and $\overline{AB} \cong \overline{DE}$. Before concluding that the triangles are congruent by ASA, you need to show that $\angle B \cong \angle E$. State the postulate or theorem that allows you to confirm this statement ($\angle B \cong \angle E$).

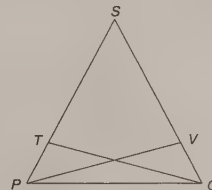


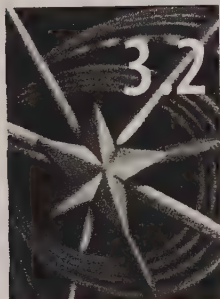
In Exercises 39 and 40, complete each proof.

39. *Given:* Plane M
 C is the midpoint of $\overline{EB}$
 $\overline{AD} \perp \overline{BE}$ and $\overline{AB} \parallel \overline{ED}$
Prove: $\triangle ABC \cong \triangle DEC$



40. *Given:* $\overline{SP} \cong \overline{SQ}$ and $\overline{ST} \cong \overline{SV}$
Prove: $\triangle SPV \cong \triangle SQT$ and $\triangle TPQ \cong \triangle VQP$





CORRESPONDING PARTS OF CONGRUENT TRIANGLES

Recall that the definition of congruent triangles states that *all* six parts (three sides and three angles) of one triangle are congruent respectively to the six corresponding parts of the second triangle. If we have proved that $\triangle ABC \cong \triangle DEF$ by SAS (the congruent parts are marked in Figure 3.14), then we can draw conclusions such as $\angle C \cong \angle F$ and $\overline{AC} \cong \overline{DF}$. The following reason is often cited for drawing such conclusions and is based on the definition of congruent triangles.

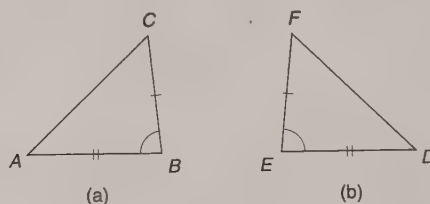


FIGURE 3.14

CPCTC: Corresponding parts of congruent triangles are congruent.

For triangles that have been proved congruent, CPCTC may be used to establish that either two line segments (corresponding sides) or two angles (corresponding angles) are congruent.

EXAMPLE 1

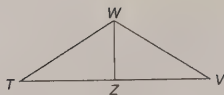


FIGURE 3.15

Given: $\overrightarrow{WZ}$ bisects $\angle TWV$
 $\overline{WT} \cong \overline{WV}$

(See Figure 3.15.)

Prove: $\overline{TZ} \cong \overline{VZ}$

PROOF

Statements	Reasons
1. $\overrightarrow{WZ}$ bisects $\angle TWV$	1. Given
2. $\angle TWZ \cong \angle VWZ$	2. The bisector of an angle separates it into two $\cong$ $\angle$ s
3. $\overline{WT} \cong \overline{WV}$	3. Given
4. $\overline{WZ} \cong \overline{WZ}$	4. Identity
5. $\triangle TWZ \cong \triangle VWZ$	5. SAS
6. $\overline{TZ} \cong \overline{VZ}$	6. CPCTC

REMINDER

CPCTC means “corresponding parts of congruent triangles are congruent.”

In Example 1, we could just as easily have used CPCTC to prove that two angles are congruent. If we had been asked to prove that $\angle T \cong \angle V$, then the final statement would have read

6. $\angle T \cong \angle V$

6. CPCTC

We can take the proof a step further by proving triangles congruent and then using CPCTC to reach another conclusion, such as parallel or perpendicular lines. In Example 1, suppose we had been asked to prove that $\overline{WZ}$ bisects $\overline{TV}$. Then steps 1–6 would have remained as is, and a seventh step would have read

7. $\overline{WZ}$ bisects $\overline{TV}$

7. If a line segment is divided into two $\cong$ parts, then it has been bisected

In our study of triangles, we will establish three types of conclusions:

1. *Proving triangles congruent*, like $\triangle TWZ \cong \triangle VWZ$
2. *Proving corresponding parts of congruent triangles congruent*, like $\overline{TZ} \cong \overline{VZ}$ (Note that two $\triangle$ s have to be proved $\cong$ before CPCTC can be used.)
3. *Establishing a further relationship*, like $\overline{WZ}$ bisects $\overline{TV}$ (Note that we must establish that two $\triangle$ s are $\cong$ and also apply CPCTC before this goal can be reached.)

Little is said in this book about a “plan for proof,” but every geometry student and teacher must have a plan before a proof can be completed.

EXAMPLE 2

Given: $\overline{ZW} \cong \overline{YX}$
 $\overline{ZY} \cong \overline{WX}$
 (See Figure 3.16.)

Prove: $\overline{ZY} \parallel \overline{WX}$

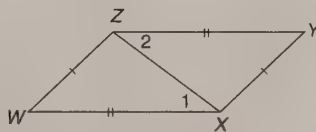


FIGURE 3.16

Plan for Proof: By showing that $\triangle ZWX \cong \triangle XYZ$, we can show that $\angle 1 \cong \angle 2$ by CPCTC. Then $\angle$ s 1 and 2 are congruent alternate interior angles for $\overline{ZY}$ and $\overline{WX}$, which must be parallel.

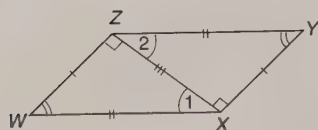


FIGURE 3.17

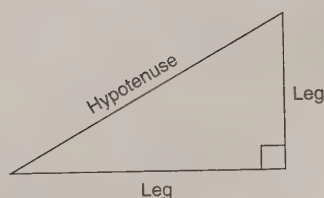


FIGURE 3.18

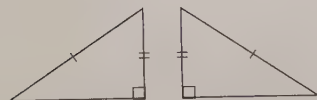


FIGURE 3.19

P R O O F

Statements	Reasons
1. $\overline{ZW} \cong \overline{YX}$; $\overline{ZY} \cong \overline{WX}$	1. Given
2. $\overline{ZX} \cong \overline{ZX}$	2. Identity
3. $\triangle ZWX \cong \triangle XYZ$	3. SSS
4. $\angle 1 \cong \angle 2$	4. CPCTC
5. $\overline{ZY} \parallel \overline{WX}$	5. If two lines are cut by a transversal so that the alt. int. $\angle$ s are $\cong$, these lines are $\parallel$

SUGGESTIONS FOR PROVING TRIANGLES CONGRUENT

Because each proof depends on establishing congruent triangles, we offer the following suggestions.

Suggestions for a proof that involves congruent triangles:

1. Mark the figures systematically, using:
 - a) a *square* in the opening of a right angle;
 - b) the same number of *dashes* on congruent sides; and
 - c) the same number of *arcs* on congruent angles.
 2. Trace the triangles to be proved congruent in different colors.
 3. If the triangles overlap, draw them separately.
- NOTE: See Figure 3.17 for reference.

RIGHT TRIANGLES

In a right triangle, the side opposite the right angle is the **hypotenuse** of the triangle, and the sides of the right angle are the **legs** of the triangle. These parts of a right triangle are illustrated in Figure 3.18.

In addition to the methods discussed earlier for proving triangles congruent, we also have the HL method, which applies exclusively to right triangles. In HL, H refers to hypotenuse and L refers to leg. The proof of this method will be delayed until Section 5.3.

HL (METHOD FOR PROVING TRIANGLES CONGRUENT)

THEOREM 3.2.1: If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of a second right triangle, then the triangles are congruent (HL).

The relationship described in Theorem 3.2.1 is illustrated in Figure 3.19. In Example 3, the construction leads to a unique right triangle.

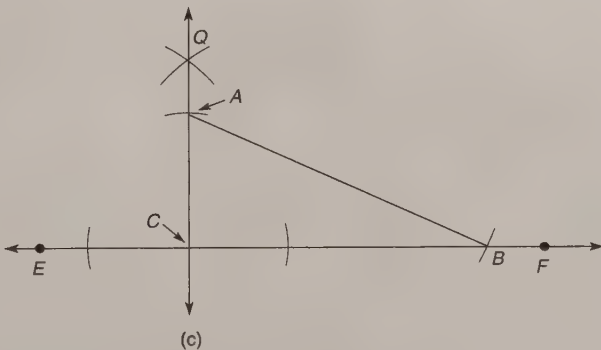
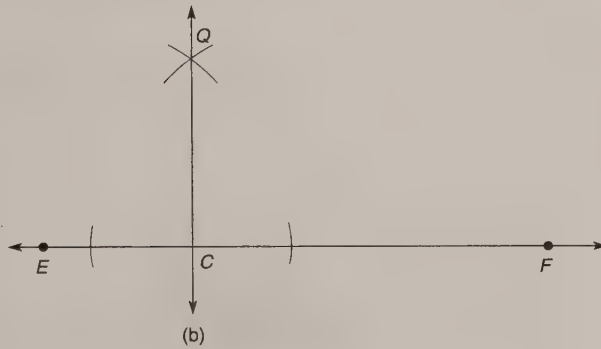
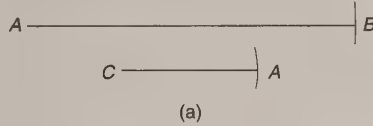
EXAMPLE 3

Given: $\overline{AB}$ and $\overline{CA}$ in Figure 3.20(a)

Construct: The right triangle with hypotenuse of length equal to AB and one leg of length equal to CA

Solution

First, we construct $\overleftrightarrow{CQ}$ perpendicular to $\overleftrightarrow{EF}$ at point C . [See Figure 3.20(b).]
Second, mark off the length of $\overline{CA}$ on $\overleftrightarrow{CQ}$, as shown in Figure 3.20(c).



Finally, with point A as center, mark off a length equal to that of $\overline{AB}$, as shown in Figure 3.20(c). $\triangle ABC$ is the desired right $\triangle$.

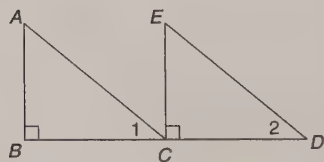
FIGURE 3.20

GEOMETRY

IN THE REAL WORLD



In the manufacturing process, the parts of many machines must be congruent. The two sides of the hinge shown are congruent.

EXAMPLE 4**FIGURE 3.21**

Cite the reason why the right triangles $\triangle ABC$ and $\triangle ECD$ in Figure 3.21 are congruent if:

- a) $\overline{AB} \cong \overline{EC}$ and $\overline{AC} \cong \overline{ED}$
- b) $\angle A \cong \angle E$ and C is the midpoint of $\overline{BD}$
- c) $\overline{BC} \cong \overline{CD}$ and $\angle 1 \cong \angle 2$
- d) $\overline{AB} \cong \overline{EC}$ and $\overline{EC}$ bisects $\overline{BD}$

Solution

- a) HL b) AAS c) ASA d) SAS

The following theorem can be applied only when a triangle is a right triangle (contains a right angle). Proof of the theorem is delayed until Section 5.3.

PYTHAGOREAN THEOREM: The square of the length (c) of the hypotenuse of a right triangle equals the sum of squares of the lengths (a and b) of the legs of the right triangle; that is, $c^2 = a^2 + b^2$.

In applications of the Pythagorean Theorem, we often arrive at statements such as $c^2 = 25$. Using the following property, we see that $c = \sqrt{25}$ or $c = 5$.

SQUARE ROOTS PROPERTY: Let x represent the length of a line segment, and let p represent a positive number. If $x^2 = p$, then $x = \sqrt{p}$.

The *square root* of p , symbolized $\sqrt{p}$, represents the number that when multiplied times itself equals p . As we indicated earlier, $\sqrt{25} = 5$ because $5 \times 5 = 25$. When a square root is not exact, a calculator can be used to find its approximate value; for instance, $\sqrt{22} \approx 4.69$ because $4.69 \times 4.69 = 21.9961 \approx 22$.

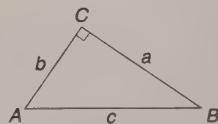
EXAMPLE 5

Find the length of the third side of the right triangle. (See the figure below.)

- a) Find c if $a = 6$ and $b = 8$.
- b) Find b if $a = 7$ and $c = 10$.

Solution

- a) $c^2 = a^2 + b^2$, so $c^2 = 6^2 + 8^2$
or $c^2 = 36 + 64 = 100$.
Then $c = \sqrt{100} = 10$.
- b) $c^2 = a^2 + b^2$, so $10^2 = 7^2 + b^2$ or
 $100 = 49 + b^2$. Subtracting yields
 $b^2 = 51$, so $b = \sqrt{51} \approx 7.14$.

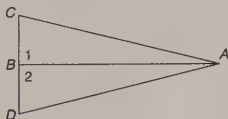


3.2 EXERCISES

In Exercises 1 to 8, plan and write the two-column proof for each problem.

1. Given: $\angle 1$ and $\angle 2$ are right $\angle$ s
 $\overline{CA} \cong \overline{DA}$

Prove: $\triangle ABC \cong \triangle ABD$

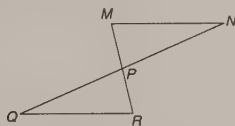


Exercises 1, 2

2. Given: $\angle 1$ and $\angle 2$ are right $\angle$ s
 $\overline{AB}$ bisects $\angle CAD$

Prove: $\triangle ABC \cong \triangle ABD$

3. Given: P is the midpoint of both $\overline{MR}$ and $\overline{NQ}$
 Prove: $\triangle MNP \cong \triangle RQP$

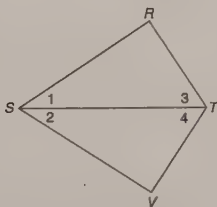


Exercises 3, 4

4. Given: $\overline{MN} \parallel \overline{QR}$ and $\overline{MN} \cong \overline{QR}$
 Prove: $\triangle MNP \cong \triangle RQP$

5. Given: $\angle R$ and $\angle V$ are right $\angle$ s
 $\angle 1 \cong \angle 2$

Prove: $\triangle RST \cong \triangle VST$



Exercises 5–8

6. Given: $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$
 Prove: $\triangle RST \cong \triangle VST$

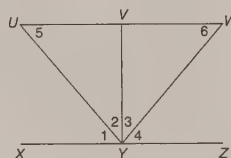
7. Given: $\overline{SR} \cong \overline{SV}$ and $\overline{RT} \cong \overline{VT}$
 Prove: $\triangle RST \cong \triangle VST$

8. Given: $\angle R$ and $\angle V$ are right $\angle$ s
 $\overline{RT} \cong \overline{VT}$

Prove: $\triangle RST \cong \triangle VST$

9. Given: $\overline{UW} \parallel \overline{XZ}$, $\overline{VY} \perp \overline{UW}$, and $\overline{VY} \perp \overline{XZ}$
 $m\angle 1 = m\angle 4 = 42^\circ$

Find: $m\angle 2$, $m\angle 3$, $m\angle 5$, and $m\angle 6$



Exercises 9, 10

10. Given: $\overline{UW} \parallel \overline{XZ}$, $\overline{VY} \perp \overline{UW}$, and $\overline{VY} \perp \overline{XZ}$
 $m\angle 1 = m\angle 4 = 4x + 3$

$$m\angle 2 = 6x - 3$$

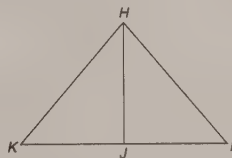
Find: $m\angle 1$, $m\angle 2$, $m\angle 3$, $m\angle 4$, $m\angle 5$, and $m\angle 6$

In Exercises 11 and 12, complete each proof.

11. Given: $\overline{HJ} \perp \overline{KL}$ and $\overline{HK} \cong \overline{HL}$
 Prove: $\overline{KJ} \cong \overline{JL}$

PROOF

Statements	Reasons
1. $\overline{HJ} \perp \overline{KL}$ and $\overline{HK} \cong \overline{HL}$	1. ?
2. $\angle$ s HJK and HJL are rt. $\angle$ s	2. ?
3. $\overline{HJ} \cong \overline{HJ}$	3. ?
4. ?	4. HL
5. ?	5. CPCTC



Exercises 11, 12

12. Given: $\overrightarrow{HJ}$ bisects $\angle KHL$
 $\overrightarrow{HJ} \perp \overrightarrow{KL}$

Prove: $\angle K \cong \angle L$ (See Figure on previous page.)

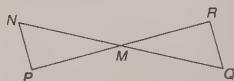
PROOF

Statements	Reasons
1. ?	1. Given
2. $\angle JHK \cong \angle JHL$	2. ?
3. $\overrightarrow{HJ} \perp \overrightarrow{KL}$	3. ?
4. $\angle HJK \cong \angle HJL$	4. ?
5. ?	5. Identity
6. ?	6. ASA
7. $\angle K \cong \angle L$	7. ?

In Exercises 13 to 16, first prove that triangles are congruent, then use CPCTC.

13. Given: $\angle P$ and $\angle R$ are right $\angle$ s
 M is the midpoint of $\overline{PR}$

Prove: $\angle N \cong \angle Q$



Exercises 13, 14

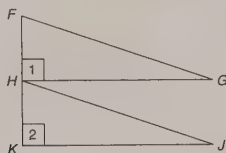
14. Given: M is the midpoint of $\overline{NQ}$
 $\overline{NP} \parallel \overline{RQ}$ with transversals
 $\overline{PR}$ and $\overline{NQ}$

Prove: $\overline{NP} \cong \overline{QR}$

15. Given: $\angle 1$ and $\angle 2$ are right $\angle$ s
 H is the midpoint of $\overline{FK}$

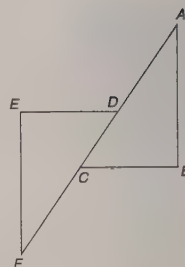
$\overline{FG} \parallel \overline{HJ}$

Prove: $\overline{FG} \cong \overline{HJ}$



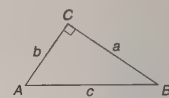
16. Given: $\overline{DE} \perp \overline{EF}$ and $\overline{CB} \perp \overline{AB}$
 $\overline{AB} \parallel \overline{FE}$
 $\overline{AC} \cong \overline{FD}$

Prove: $\overline{EF} \cong \overline{BA}$



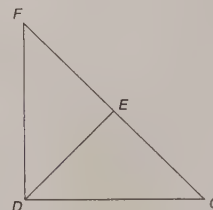
In Exercises 17 to 22, $\triangle ABC$ is a right triangle. Use the given information to find the length of the third side of the triangle.

17. $a = 4$ and $b = 3$
18. $a = 12$ and $b = 5$
19. $a = 15$ and $c = 17$
20. $b = 6$ and $c = 10$
21. $a = 5$ and $b = 4$
22. $a = 7$ and $c = 8$



Exercises 17-22

In Exercises 23 to 25, prove the indicated relationship.



Exercises 23-25

23. Given: $\overline{DF} \cong \overline{DG}$ and $\overline{FE} \cong \overline{EG}$
Prove: $\overline{DE}$ bisects $\angle FDG$

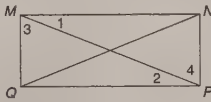
24. Given: $\overline{DE}$ bisects $\angle FDG$
 $\angle F \cong \angle G$

Prove: E is the midpoint of $\overline{FG}$

25. Given: E is the midpoint of $\overline{FG}$
 $\overline{DF} \cong \overline{DG}$

Prove: $\overline{DE} \perp \overline{FG}$

In Exercises 26 to 28, draw the triangles to be shown congruent separately.



Exercises 26–28

26. *Given:* $\angle MQP$ and $\angle NPQ$ are rt. $\angle$ s
 $\overline{MQ} \cong \overline{NP}$

Prove: $\overline{MP} \cong \overline{NQ}$

(**HINT:** Show $\triangle MQP \cong \triangle NPQ$.)

27. *Given:* $\angle 1 \cong \angle 2$ and $\overline{MN} \cong \overline{QP}$

Prove: $\overline{MQ} \parallel \overline{NP}$

(**HINT:** Show $\triangle NMP \cong \triangle QPM$.)

28. *Given:* $\overline{MN} \parallel \overline{QP}$ and $\overline{MQ} \parallel \overline{NP}$

Prove: $\overline{MQ} \cong \overline{NP}$

(**HINT:** Show $\triangle MQP \cong \triangle PNM$.)

29. *Given:* $\overrightarrow{RW}$ bisects $\angle SRU$

$\overline{RS} \cong \overline{RU}$

Prove: $\triangle TRU \cong \triangle VRS$

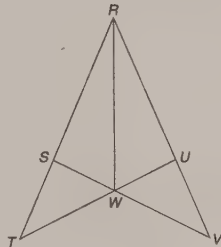
(**HINT:** First show that $\triangle RSW \cong \triangle RUW$.)

30. *Given:* $\overline{DB} \perp \overline{BC}$ and $\overline{CE} \perp \overline{DE}$

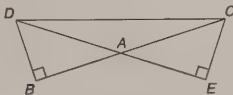
$\overline{AB} \cong \overline{AE}$

Prove: $\triangle BDC \cong \triangle ECD$

(**HINT:** First show that $\triangle ACE \cong \triangle ADB$.)



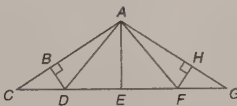
Exercise 29



Exercise 30

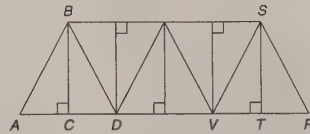
31. In the roof truss shown, $AB = 8$ and $m\angle HAF = 37^\circ$.
 Find:

- a) AH b) $m\angle BAD$ c) $m\angle ADB$

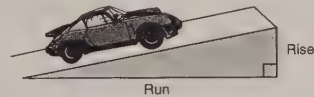


32. In the support system of the bridge shown, $AC = 6$ ft and $m\angle ABC = 28^\circ$. Find:

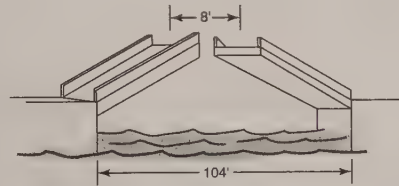
- a) $m\angle RST$ b) $m\angle ABD$ c) BS



33. As a car moves along the roadway in a mountain pass, it passes through a horizontal run of 750 feet and through a vertical rise of 45 feet. To the nearest foot, how far does the car move along the roadway?



34. A drawbridge that is 104 feet in length is raised at its midpoint so that the uppermost points are 8 feet apart. How far has each of the mid-sections been raised?

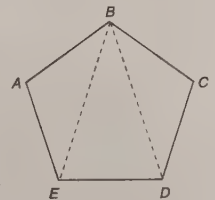


35. *Given:* Regular pentagon $ABCDE$ with diagonals $\overline{BE}$ and $\overline{BD}$

Prove: $\overline{BE} \cong \overline{BD}$

(**HINT:** First prove

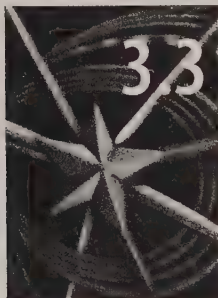
$\triangle ABE \cong \triangle CBD$.)



36. In the figure with regular pentagon $ABCDE$, do $\overrightarrow{BE}$ and $\overrightarrow{BD}$ trisect $\angle ABC$?

(**HINT:** Determine $m\angle ABE$, $m\angle EBD$, and $m\angle DBC$.)

Exercises 35, 36



ISOSCELES TRIANGLES

In an isosceles triangle, the two sides of equal length are **legs** and the third side is the **base**. The point at which the two legs meet is the **vertex** of the triangle, so the angle formed by the legs (and opposite the base) is the **vertex angle**. The two remaining angles are **base angles**. (See Figure 3.22.) If $\overline{AC} \cong \overline{BC}$ in Figure 3.23, then $\triangle ABC$ is isosceles with legs $\overline{AC}$ and $\overline{BC}$, base $\overline{AB}$, vertex C , vertex angle C , and base angles at A and B . By considering Figure 3.23, we see that the base of an isosceles triangle is not necessarily the “bottom” side.

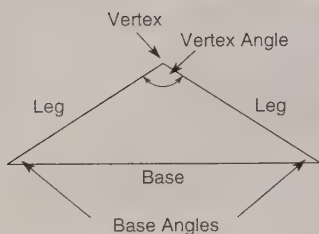


FIGURE 3.22

In any triangle, a number of segments, rays, or lines are related to the triangle. (See Figure 3.23.) Each angle of a triangle has a unique **angle bisector**, and this may be indicated by a ray or segment from the vertex of the bisected angle. Just as an angle bisector begins at the vertex of an angle, so does the **median**, which joins a vertex to the midpoint of the opposite side. Generally, the median from a vertex of a triangle is not the same as the angle bisector from that vertex. An **altitude** is a line segment drawn from a vertex to the opposite side such that it is perpendicular to the opposite side. Finally, the **perpendicular bisector** of a side of a triangle is shown as a line in Figure 3.23. A segment or ray could also perpendicularly bisect a side of the triangle. In Figure 3.23, $\overline{AD}$ is the angle bisector of $\angle BAC$; $\overline{AE}$ is the altitude from A to $\overline{BC}$; M is the midpoint of $\overline{BC}$; $\overline{AM}$ is the median from A to $\overline{BC}$; and $\overline{FM}$ is the perpendicular bisector of $\overline{BC}$.

An altitude can actually lie in the exterior of a triangle. In Figure 3.24, which shows the obtuse triangle $\triangle RST$, the altitude from R must be drawn to an extension of side $\overline{ST}$. Later we will use the length of the altitude $\overline{RH}$ in place of h in the following standard formula for the area of a triangle:

$$A = \frac{1}{2}bh$$

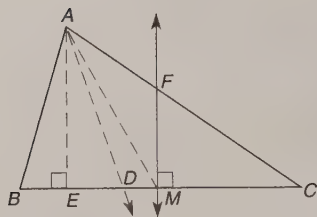


FIGURE 3.23

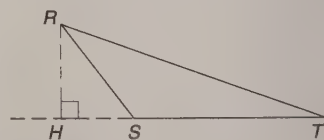


FIGURE 3.24

The angle bisector and the median necessarily lie in the interior of the triangle.

Each triangle has three altitudes—one from each vertex. As these are shown for $\triangle ABC$ in Figure 3.25, do the three altitudes seem to meet at a common point?

We now consider a statement that involves the altitudes of congruent triangles.

THEOREM 3.3.1: Corresponding altitudes of congruent triangles are congruent.

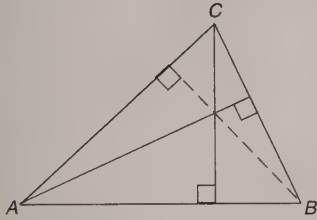
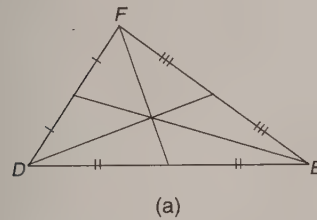
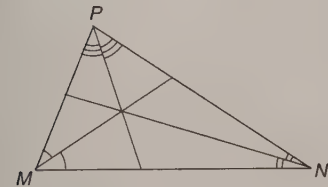


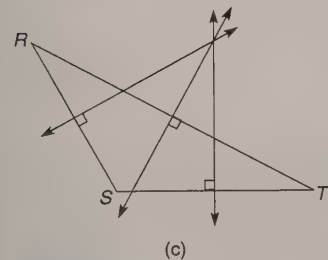
FIGURE 3.25



(a)



(b)



(c)

FIGURE 3.27

Given: $\triangle ABC \cong \triangle RST$
altitudes $\overline{CD}$ to $\overline{AB}$ and $\overline{TV}$ to $\overline{RS}$
(See Figure 3.26.)

Prove: $\overline{CD} \cong \overline{TV}$

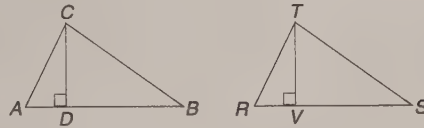


FIGURE 3.26

P R O O F

Statements	Reasons
1. $\triangle ABC \cong \triangle RST$ altitudes $\overline{CD}$ to $\overline{AB}$ and $\overline{TV}$ to $\overline{RS}$	1. Given
2. $\overline{CD} \perp \overline{AB}$ and $\overline{TV} \perp \overline{RS}$	2. An altitude of a $\triangle$ is the line segment from one vertex drawn $\perp$ to the opposite side
3. $\angle CDA$ and $\angle TVR$ are right $\angle$ s	3. If two lines are $\perp$, they form right $\angle$ s
4. $\angle CDA \cong \angle TVR$	4. All right angles are $\cong$
5. $\overline{AC} \cong \overline{RT}$ and $\angle A \cong \angle R$	5. CPCTC (from $\triangle ABC \cong \triangle RST$)
6. $\triangle CDA \cong \triangle TVR$	6. AAS
7. $\overline{CD} \cong \overline{TV}$	7. CPCTC

Each triangle has three medians—one from each vertex to the midpoint of the opposite side. As the medians are drawn for $\triangle DEF$ in Figure 3.27(a), does it appear that the three medians intersect at a point?

Each triangle has three angle bisectors—one for each of the three angles. As these are shown for $\triangle MNP$ in Figure 3.27(b), does it appear that the three angle bisectors have a point in common?

Each triangle has three perpendicular bisectors for its sides; these are shown for $\triangle RST$ in Figure 3.27(c). Like the altitudes, medians, and angle bisectors, the perpendicular bisectors of the sides also meet at a single point.

The angle bisectors and the medians of a triangle always meet in the interior of the triangle. However, the altitudes and perpendicular bisectors of the sides [see Figure 3.27(c)] can meet in the exterior of the triangle. These points of intersection will be given more attention in Chapter 6.

With the following activity, we open the doors to several further discoveries.

DISCOVER!

Using a sheet of construction paper, cut out an isosceles triangle. Now use your compass to bisect the vertex angle. Fold along the angle bisector to form two smaller triangles. How are the smaller triangles related?

ANSWER
They are congruent.

EXAMPLE 1

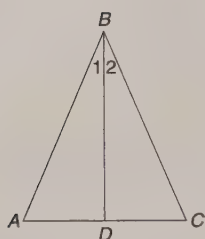
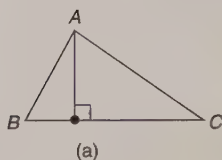
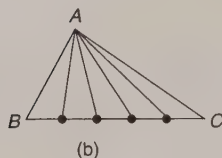


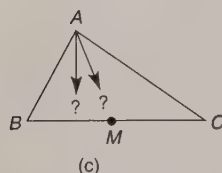
FIGURE 3.28



(a)



(b)



(c)

FIGURE 3.29

Give a formal proof of Theorem 3.3.2.

THEOREM 3.3.2: The bisector of the vertex angle of an isosceles triangle separates the triangle into two congruent triangles.

Given: Isosceles $\triangle ABC$, with $\overline{AB} \cong \overline{BC}$
 $\overline{BD}$ bisects $\angle ABC$
 (See Figure 3.28.)

Prove: $\triangle ABD \cong \triangle CBD$

PROOF

Statements	Reasons
1. Isosceles $\triangle ABC$ with $\overline{AB} \cong \overline{BC}$	1. Given
2. $\overline{BD}$ bisects $\angle ABC$	2. Given
3. $\angle 1 \cong \angle 2$	3. The bisector of an $\angle$ separates it into two $\cong \angle$ s
4. $\overline{BD} \cong \overline{BD}$	4. Identity
5. $\triangle ABD \cong \triangle CBD$	5. SAS

AUXILIARY LINES

In geometry, **auxiliary** (helping) lines or segments are often used to solve a problem or to construct a proof. You must account for the uniqueness of the line, segment, or ray as it is introduced into the existing drawing. That is, each auxiliary figure must be **determined**, but it must not be **underdetermined** or **overdetermined**. A figure is underdetermined when there is more than one possible figure described. On the other extreme, a figure is overdetermined when it is impossible for *all* conditions described to be satisfied.

Consider Figure 3.29 and the following three descriptions, which are coded **D** for determined, **U** for underdetermined, and **O** for overdetermined:

D: Draw a line segment from A perpendicular to $\overline{BC}$ so that the terminal point is on $\overline{BC}$. [*Determined* because the line from A perpendicular to $\overline{BC}$ is unique; see Figure 3.29(a)].

U: Draw a line segment from A to $\overline{BC}$ so that the terminal point is on $\overline{BC}$. [*Underdetermined* because many line segments are possible; see Figure 3.29(b)].

O: Draw a line segment from A perpendicular to $\overline{BC}$ so that it bisects $\overline{BC}$. [*Overdetermined* because the line segment from A drawn perpendicular to $\overline{BC}$ will not contain the midpoint M of $\overline{BC}$; see Figure 3.29(c)].

In Example 2, an auxiliary segment is needed. As you study the proof, note the uniqueness of the segment and its justification in the proof.

EXAMPLE 2

Give a formal proof of Theorem 3.3.3.

THEOREM 3.3.3: If two sides of a triangle are congruent, then the angles opposite these sides are also congruent.

Given: Isosceles $\triangle MNP$
with $\overline{MP} \cong \overline{NP}$
(See Figure 3.30.)

Prove: $\angle M \cong \angle N$

NOTE: Figure 3.30(b) shows the auxiliary segment.

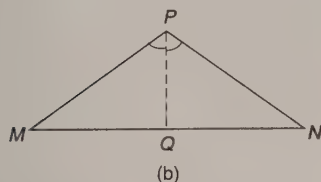
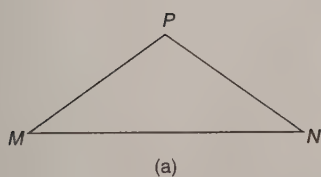


FIGURE 3.30

PROOF

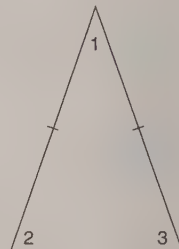
Statements	Reasons
1. Isosceles $\triangle MNP$ with $\overline{MP} \cong \overline{NP}$	1. Given
2. Draw $\angle$ bisector $\overline{PQ}$ from P to $\overline{MN}$	2. Every angle has one and only one bisector
3. $\triangle MPQ \cong \triangle NPQ$	3. The bisector of the vertex angle of an isosceles $\triangle$ separates it into two $\cong \triangle$ s
4. $\angle M \cong \angle N$	4. CPCTC

Theorem 3.3.3 is sometimes stated, "The base angles of an isosceles triangle are congruent." We apply this theorem in Example 3.

EXAMPLE 3

Find the size of each angle of the isosceles triangle shown in Figure 3.31 if:

- a) $m\angle 1 = 36^\circ$
 b) the measure of each base angle is 5° less than twice the measure of the vertex angle

**FIGURE 3.31****Solution**

- a) $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$. Since $m\angle 1 = 36^\circ$ and $\angle 2$ and $\angle 3$ are $\cong$, we have

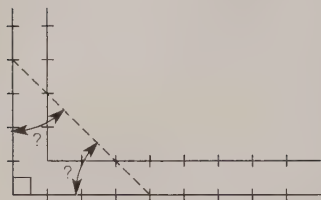
$$\begin{aligned} 36 + 2(m\angle 2) &= 180 \\ 2(m\angle 2) &= 144 \\ m\angle 2 &= 72 \end{aligned}$$

Now $m\angle 1 = 36^\circ$, and $m\angle 2 = m\angle 3 = 72^\circ$.

- b) Let the vertex angle measure be given by x . Then the size of each base angle is $2x - 5$. Because the sum of the measures is 180° ,

$$\begin{aligned} x + (2x - 5) + (2x - 5) &= 180 \\ 5x - 10 &= 180 \\ 5x &= 190 \\ x &= 38 \\ 2x - 5 &= 2(38) - 5 = 76 - 5 = 71 \end{aligned}$$

Therefore, $m\angle 1 = 38^\circ$ and $m\angle 2 = m\angle 3 = 71^\circ$.

**FIGURE 3.32****WARNING**

The converse of an "If, then" statement is not always true.

In some instances, a carpenter may want to get a quick, accurate measurement without having to go get his or her tools. Suppose that the carpenter's square shown in Figure 3.32 is handy but that a miter box is not nearby. If two marks are made at lengths of 4 inches from the corner of the square and these are then joined, what size angle is determined? You should see that each angle measures 45° .

Example 4 shows us that the converse of the theorem "The base angles of an isosceles $\triangle$ are congruent" is also true. However, see the accompanying "Warning."

EXAMPLE 4

Give a formal proof of Theorem 3.3.4.

THEOREM 3.3.4: If two angles of a triangle are congruent, then the sides opposite these angles are also congruent.

Given: $\triangle TUV$ with $\angle T \cong \angle U$ in Figure 3.33(a)

Prove: $\overline{VU} \cong \overline{VT}$

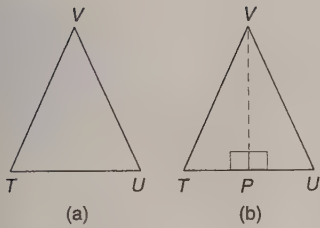


FIGURE 3.33

PROOF

Statements	Reasons
1. $\triangle TUV$ with $\angle T \cong \angle U$	1. Given
2. Draw $\overline{VP}$, the segment from $V \perp \overline{TU}$, as in Figure 3.33(b)	2. There is exactly one perpendicular from a point to a line
3. $\angle VPT \cong \angle VPU$	3. $\perp$ lines meet to form $\cong$ adjacent $\angle$ s
4. $\overline{VP} \cong \overline{VP}$	4. Identity
5. $\triangle TPV \cong \triangle UPV$	5. AAS
6. $\overline{VT} \cong \overline{VU}$	6. CPCTC

When all three sides of a triangle are congruent, the triangle is **equilateral**. If all three angles are congruent, then the triangle is **equiangular**. Theorems 3.3.3 and 3.3.4 can be used to prove the following corollaries.

COROLLARY 3.3.5: An equilateral triangle is also equiangular.

COROLLARY 3.3.6: An equiangular triangle is also equilateral.

DEFINITION: The **perimeter** of a triangle is the sum of the lengths of its sides. Thus if a , b , and c are the lengths of the three sides, then the perimeter P is given by $P = a + b + c$. (See Figure 3.34.)

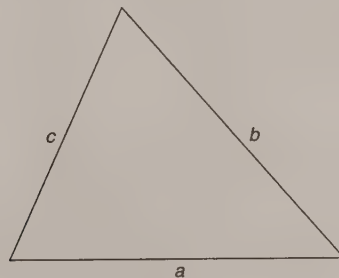


FIGURE 3.34

EXAMPLE 5

Given: $\angle B \cong \angle C$
 $AB = 5.3$ and $BC = 3.6$
 (See Figure 3.35.)

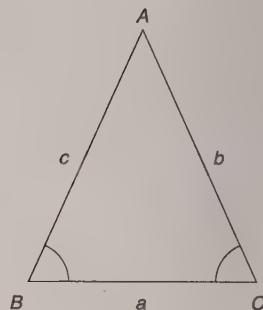
Find: The perimeter of $\triangle ABC$

Solution If $\angle B \cong \angle C$, then $AC = AB = 5.3$. Therefore,

$$P = a + b + c$$

$$P = 3.6 + 5.3 + 5.3$$

$$P = 14.2$$

**FIGURE 3.35**

Many of the properties of triangles that were investigated in earlier sections are summarized in Table 3.1.

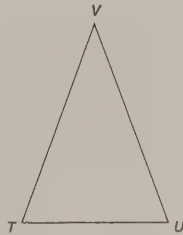
TABLE 3.1*Selected Properties of Triangles*

	Scalene	Isosceles	Equilateral (equiangular)	Acute	Right	Obtuse
						
<i>Sides</i>	no two are $\cong$	exactly two are $\cong$	all three are $\cong$	possibly two or three $\cong$ sides	possibly two $\cong$ sides; $c^2 = a^2 + b^2$	possibly two $\cong$ sides
<i>Angles</i>	sum of $\angle$ s is 180°	sum of $\angle$ s is 180° ; two $\angle$ s $\cong$	sum of $\angle$ s is 180° ; three $\cong 60^\circ$ $\angle$ s	all $\angle$ s acute; sum of $\angle$ s is 180° ; possibly two or three $\cong \angle$ s	one right $\angle$; sum of $\angle$ s is 180° ; possibly two $\cong 45^\circ$ $\angle$ s; acute $\angle$ s are complementary	one obtuse $\angle$; sum of $\angle$ s is 180° ; possibly two $\cong$ acute $\angle$ s

3.3 EXERCISES

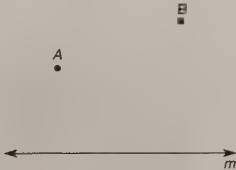
For Exercises 1 to 8, use the accompanying drawing.

1. If $\overline{VU} \cong \overline{VT}$, what type of triangle is $\triangle VTU$?
2. If $\overline{VU} \cong \overline{VT}$, which angles of $\triangle VTU$ are congruent?
3. If $\angle T \cong \angle U$, which sides of $\triangle VTU$ are congruent?
4. If $\overline{VU} \cong \overline{VT}$, $VU = 10$, and $TU = 8$, what is the perimeter of $\triangle VTU$?
5. If $\overline{VU} \cong \overline{VT}$ and $m\angle T = 69^\circ$, find $m\angle U$.
6. If $\overline{VU} \cong \overline{VT}$ and $m\angle T = 69^\circ$, find $m\angle V$.
7. If $\overline{VU} \cong \overline{VT}$ and $m\angle T = 72^\circ$, find $m\angle V$.
8. If $\overline{VU} \cong \overline{VT}$ and $m\angle V = 40^\circ$, find $m\angle T$.



Exercises 1–8

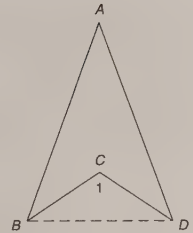
In Exercises 9 to 14, describe the segment as determined, underdetermined, or overdetermined. Use the accompanying drawing for reference.



Exercises 9–14

9. Draw a segment through point A.
10. Draw a segment with endpoints A and B.
11. Draw a segment $\overline{AB}$ parallel to line m .
12. Draw a segment $\overline{AB}$ perpendicular to m .
13. Draw a segment from A perpendicular to m .
14. Draw $\overline{AB}$ so that line m bisects $\overline{AB}$.
15. A surveyor knows that a lot has the shape of an isosceles triangle. If the vertex angle measures 70° and each equal side is 160 feet long, what measure does each of the base angles have?

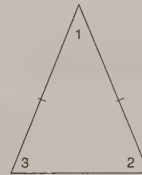
16. In concave quadrilateral $ABCD$, the angle at A measures 40° . $\triangle ABD$ is isosceles, $\overline{BC}$ bisects $\angle ABD$, and $\overline{DC}$ bisects $\angle ADB$. What are the measures of $\angle ABC$, $\angle ADC$, and $\angle 1$?



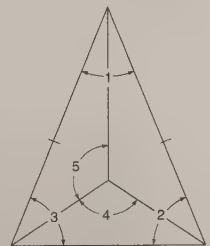
Exercise 16

In Exercises 17 to 22, use arithmetic or algebra as needed to find the measures indicated. Note the use of dashes on equal sides of the given isosceles triangles.

17. Find $m\angle 1$ and $m\angle 2$ if $m\angle 3 = 68^\circ$.



18. If $m\angle 3 = 68^\circ$, find $m\angle 4$, the angle formed by the bisectors of $\angle 3$ and $\angle 2$.
19. Find the measure of $\angle 5$, which is formed by the bisectors of $\angle 1$ and $\angle 3$. Again let $m\angle 3 = 68^\circ$.



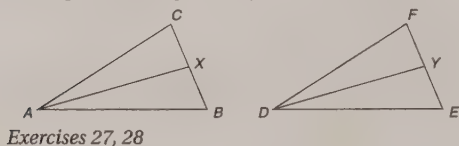
Exercises 18–20

20. Find an expression for the measure of $\angle 5$ if $m\angle 3 = 2x$ and the segments shown bisect the angles of the isosceles triangle.
21. In isosceles $\triangle ABC$ with vertex A (not shown), each base angle is 12° larger than the vertex angle. Find the measure of each angle.

22. In isosceles $\triangle ABC$ (not shown), vertex angle A is 5° more than one-half of base angle B . Find the size of each angle.

In Exercises 23 to 26, suppose that $\overline{BC}$ is the base of isosceles $\triangle ABC$ (not shown).

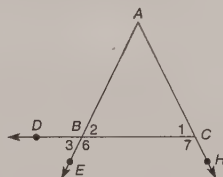
23. Find the perimeter of $\triangle ABC$ if $AB = 8$ and $BC = 10$.
 24. Find AB if the perimeter of $\triangle ABC$ is 36.4 and $BC = 14.6$.
 25. Find x if the perimeter of $\triangle ABC$ is 40, $AB = x$, and $BC = x + 4$.
 26. Find x if the perimeter of $\triangle ABC$ is 68, $AB = x$, and $BC = 1.4x$.
 27. Suppose that $\triangle ABC \cong \triangle DEF$. Also, $\overline{AX}$ bisects $\angle CAB$ and $\overline{DY}$ bisects $\angle FDE$. Are the corresponding angle bisectors of congruent triangles congruent?



28. Suppose that $\triangle ABC \cong \triangle DEF$, $\overline{AX}$ is the median from A to $\overline{BC}$, and $\overline{DY}$ is the median from D to $\overline{EF}$. Are the corresponding medians of congruent triangles congruent?

In Exercises 29 and 30, complete each proof using the drawing below.

29. Given: $\angle 3 \cong \angle 1$
 Prove: $\overline{AB} \cong \overline{AC}$



Exercises 29, 30

PROOF

Statements	Reasons
1. $\angle 3 \cong \angle 1$	1. ?
2. ?	2. If two lines intersect, the vertical $\angle$ s formed are $\cong$
3. ?	3. Transitive Property of Congruence
4. $\overline{AB} \cong \overline{AC}$	4. ?

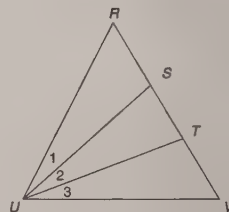
30. Given: $\overline{AB} \cong \overline{AC}$
 Prove: $\angle 6 \cong \angle 7$

PROOF

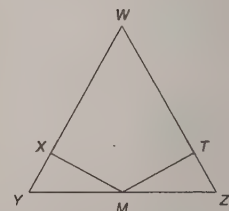
Statements	Reasons
1. ?	1. Given
2. $\angle 2 \cong \angle 1$	2. ?
3. $\angle 2$ and $\angle 6$ are supplementary; $\angle 1$ and $\angle 7$ are supplementary	3. ?
4. ?	4. If two $\angle$ s are supplementary to $\cong \angle$ s, they are $\cong$ to each other

In Exercises 31 to 33, complete each proof.

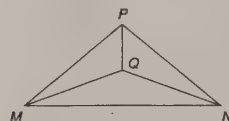
31. Given: $\angle 1 \cong \angle 3$
 $\overline{RU} \cong \overline{VU}$
 Prove: $\triangle STU$ is isosceles
 (HINT: First show that $\triangle RUS \cong \triangle VUT$.)



32. Given: $\overline{WY} \cong \overline{WZ}$
 M is the midpoint of $\overline{YZ}$
 $\overline{MX} \perp \overline{WY}$ and $\overline{MT} \perp \overline{WZ}$
 Prove: $\overline{MX} \cong \overline{MT}$

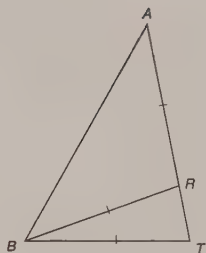


33. Given: Isosceles $\triangle MNP$ with vertex P
 isosceles $\triangle MNQ$ with vertex Q
 Prove: $\triangle MQP \cong \triangle NQP$



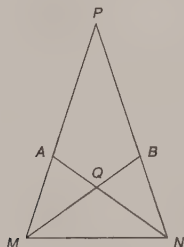
34. In isosceles triangle BAT , $\overline{AB} \cong \overline{AT}$. Also, $\overline{BR} \cong \overline{BT} \cong \overline{AR}$. If $AB = 11$ and $AR = 7$, find the perimeter of $\triangle BAT$.

35. In $\triangle BAT$, $\overline{BR} \cong \overline{BT} \cong \overline{AR}$, and $m\angle RBT = 20^\circ$. Find $m\angle A$.

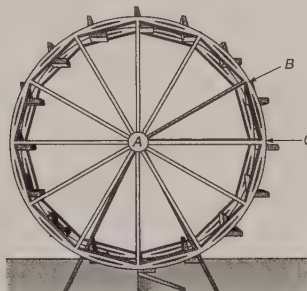


Exercises 34, 35

36. In $\triangle PMN$, $\overline{PM} \cong \overline{PN}$. $\overline{MB}$ bisects $\angle PMN$, and $\overline{NA}$ bisects $\angle PNM$. If $m\angle P = 36^\circ$, name all isosceles triangles shown in the drawing.



37. $\triangle ABC$ lies in the structural support system of the Ferris wheel. If $m\angle A = 30^\circ$ and $AB = AC = 20$ ft, find the measures of $\angle B$ and $\angle C$.



In Exercises 38 to 40, explain why each statement is true.

38. The altitude from the vertex of an isosceles triangle is also the median to the base of the triangle.
 39. The bisector of the vertex angle of an isosceles triangle bisects the base.
 40. The angle bisectors of the base angles of an isosceles triangle, together with the base, form an isosceles triangle.



BASIC CONSTRUCTIONS JUSTIFIED

In earlier sections, construction methods were introduced that appeared to achieve their goals; however, the methods were presented intuitively. In this section, we justify the construction methods and apply them in further constructions. The justification of the method is a “proof” that demonstrates that the construction accomplished its purpose. See Example 1.

EXAMPLE 1

Justify the method for constructing an angle congruent to a given angle.

Given: $\angle ABC$
 $\overline{BD} \cong \overline{BE} \cong \overline{ST} \cong \overline{SR}$ (by construction)
 $\overline{DE} \cong \overline{TR}$ (by construction)
 (See Figure 3.36.)

Prove: $\angle B \cong \angle S$

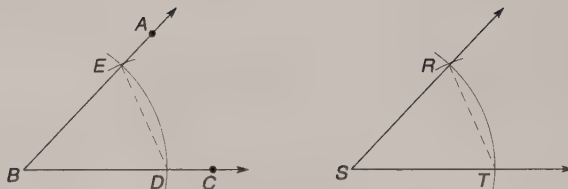


FIGURE 3.36

PROOF

Statements	Reasons
1. $\angle ABC, \overline{BD} \cong \overline{BE} \cong \overline{ST} \cong \overline{SR}$	1. Given
2. $\overline{DE} \cong \overline{TR}$	2. Given
3. $\triangle EBD \cong \triangle RST$	3. SSS
4. $\angle B \cong \angle S$	4. CPCTC

In Example 2, we will apply the construction method that was justified in Example 1. Our goal is to construct an isosceles triangle that contains an obtuse angle. It is necessary that the congruent sides include the obtuse angle.

EXAMPLE 2

Construct an isosceles triangle in which obtuse $\angle A$ is included by two sides of length a [see Figure 3.37(a)].

Solution

Construct an angle congruent to $\angle A$. Then mark off arcs of length a at points B and C as shown in Figure 3.37(b). Join B to C to complete $\triangle ABC$.

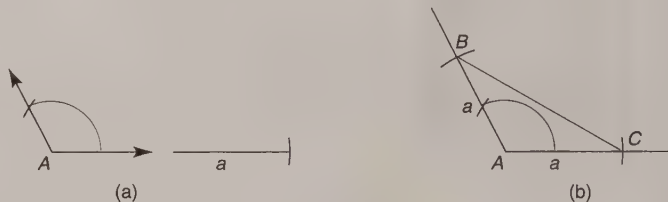
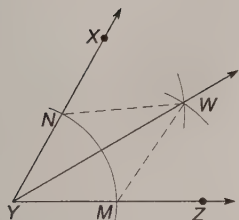


FIGURE 3.37

In Example 3, we recall the method of construction used to bisect an angle. Although the technique is illustrated, the purpose here is to justify the method.

EXAMPLE 3**FIGURE 3.38**

Justify the method for constructing the bisector of an angle. Provide the missing reasons in the proof.

Given: $\angle XYZ$
 $\overline{YM} \cong \overline{YN}$ (by construction)
 $\overline{MW} \cong \overline{NW}$ (by construction)
 (See Figure 3.38.)

Prove: $\overrightarrow{YW}$ bisects $\angle XYZ$

P R O O F

Statements	Reasons
1. $\angle XYZ$; $\overline{YM} \cong \overline{YN}$ and $\overline{MW} \cong \overline{NW}$	1. ?
2. $\overline{YW} \cong \overline{YW}$	2. ?
3. $\triangle YMW \cong \triangle YNW$	3. ?
4. $\angle MYW \cong \angle NYW$	4. ?
5. $\overrightarrow{YW}$ bisects $\angle XYZ$	5. ?

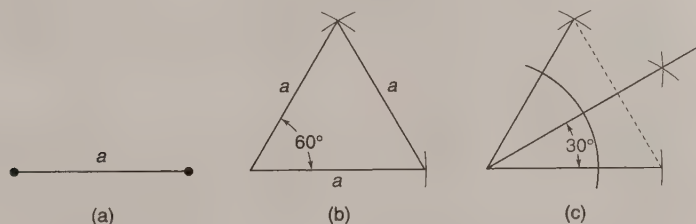
The angle bisector method can be used to construct angles of certain measures. For instance, if a right angle has been constructed, then an angle of measure 45° can be constructed by bisecting the 90° angle. In Example 4, we construct an angle of measure 30° .

EXAMPLE 4

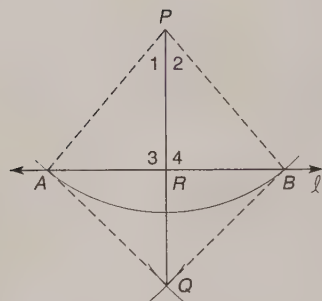
Construct an angle that measures 30° .

Solution

We begin by constructing an equilateral (and therefore equiangular) triangle. To accomplish this, mark off a line segment of length a as shown in Figure 3.39(a). From the endpoints of this line segment, mark off arcs using the same radius length a . The point of intersection determines the third vertex of this triangle, whose angles measure 60° each [see Figure 3.39(b)]. By constructing the bisector of one angle, we determine an angle that measures 30° [Figure 3.39(c)].

**FIGURE 3.39**

In Example 5, we justify the method for constructing a line perpendicular to a given line from a point not on that line. In the example, point P lies above line ℓ .

EXAMPLE 5**FIGURE 3.40**

Given: P not on ℓ
 $\overline{PA} \cong \overline{PB}$ (by construction)
 $\overline{AQ} \cong \overline{BQ}$ (by construction)
 (See Figure 3.40.)

Prove: $\overline{PQ} \perp \overline{AB}$

Provide the missing statements and reasons in the proof.

PROOF

Statements	Reasons
1. P not on ℓ $\overline{PA} \cong \overline{PB}$ and $\overline{AQ} \cong \overline{BQ}$	1. ?
2. $\overline{PQ} \cong \overline{PQ}$	2. ?
3. $\triangle PAQ \cong \triangle PBQ$	3. ?
4. $\angle 1 \cong \angle 2$	4. ?
5. $\overline{PR} \cong \overline{PR}$	5. ?
(1), (4), (5) 6. $\triangle PRA \cong \triangle PRB$	6. ?
7. $\angle 3 \cong \angle 4$	7. ?
8. ?	8. If two lines meet to form $\cong$ adjacent $\angle$ s, these lines are $\perp$

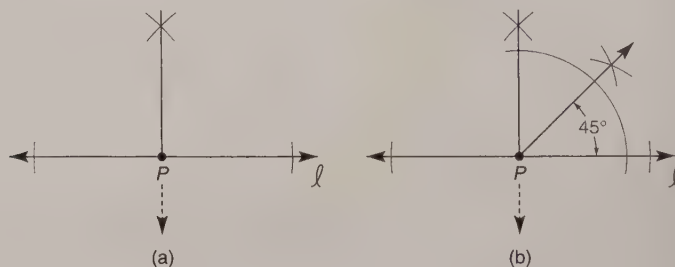
In Example 6, we recall the method for constructing the line perpendicular to a given line at a point on the line. We illustrate the technique in the example and ask that the student justify the method in Exercise 27. In Example 6, we construct an angle that measures 45° .

EXAMPLE 6

Construct an angle that measures 45° .

Solution

We begin by constructing a line segment perpendicular to line ℓ at point P [Figure 3.41(a)]. Next we bisect one of the right angles that was determined. The bisector forms an angle whose measure is 45° [Figure 3.41(b)].

**FIGURE 3.41**

As we saw in Example 4, constructing an equilateral triangle is fairly simple. It is also possible to construct other regular polygons, such as a square or a regular hexagon. In the following box, we recall some facts that will help us perform such constructions.

To construct a regular polygon:

1. Each interior angle must measure $I = \frac{(n-2)180}{n}$ degrees; alternatively, each exterior angle must measure $E = \frac{360}{n}$ degrees.
2. All sides must be congruent.

EXAMPLE 7

Construct a regular hexagon having sides of length a .

Solution

We begin by marking off a line segment of length a , as shown in Figure 3.42(a). Each exterior angle of the hexagon ($n = 6$) must measure $E = \frac{360}{6} = 60^\circ$; then each interior angle measures 120° . We construct an equilateral triangle (all sides measure a) so that a 60° exterior angle is formed, as shown in Figure 3.42(b). Again marking off an arc of length a for the second side, we construct another exterior angle of measure 60° as shown in Figure 3.42(c). This procedure is continued until the regular hexagon $ABCDEF$ is determined, as shown in Figure 3.42(d).

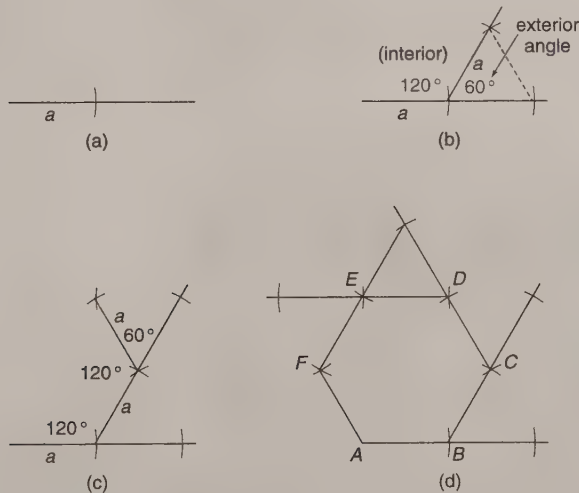
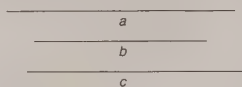


FIGURE 3.42



3.4 EXERCISES

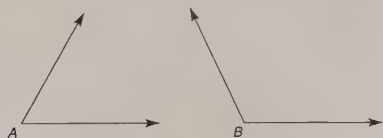
In Exercises 1 to 6, use line segments of given lengths a , b , and c to perform the constructions.



Exercises 1–6

1. Construct a line segment of length $2b$.
2. Construct a line segment of length $b + c$.
3. Construct a line segment of length $\frac{1}{2}c$.
4. Construct a line segment of length $a - b$.
5. Construct a triangle with sides of lengths a , b , and c .
6. Construct an isosceles triangle with a base of length b and legs of length a .

In Exercises 7 to 12, use the angles provided to perform the constructions.

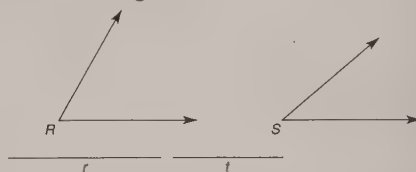


Exercises 7–12

7. Construct an angle that is congruent to acute $\angle A$.
8. Construct an angle that is congruent to obtuse $\angle B$.
9. Construct an angle that has one-half the measure of $\angle A$.
10. Construct an angle that has a measure equal to $m\angle B - m\angle A$.
11. Construct an angle that has twice the measure of $\angle A$.
12. Construct an angle whose measure averages the measures of $\angle A$ and $\angle B$.

In Exercises 13 and 14, use the angles and lengths of sides provided to construct the triangle described.

13. Construct the triangle that has sides of lengths r and t with included angle S .

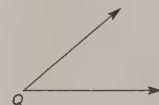


Exercises 13, 14

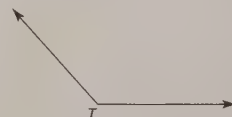
14. Construct the triangle that has a side of length t included by angles R and S .

In Exercises 15 to 18, construct angles having the given measures.

15. 90° and then 45°
16. 60° and then 30°
17. 30° and then 15°
18. 45° and then 105°
(HINT: $105^\circ = 45^\circ + 60^\circ$)
19. Describe how you would construct an angle measuring 22.5° .
20. Describe how you would construct an angle measuring 75° .
21. Construct the complement of the acute angle shown.

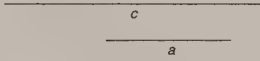


22. Construct the supplement of the obtuse angle shown.



In Exercises 23 and 24, use line segments of lengths a and c as shown.

23. Construct the right triangle with hypotenuse of length c and a leg of length a .

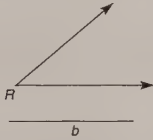


Exercises 23, 24

24. Construct an isosceles triangle with base of length c and altitude of length a .
(**HINT**: The altitude lies on the perpendicular bisector of the base.)

In Exercises 25 and 26, use the given angle and the line segment of length b .

25. Construct the right triangle in which acute angle R has a side (one leg of the triangle) of length b .

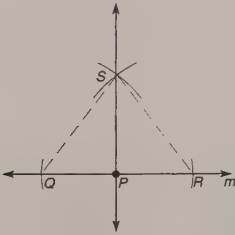


Exercises 25, 26

26. Construct an isosceles triangle with base of length b and congruent base angles having the measure of angle R .
27. Complete the justification of the construction of the line perpendicular to a given line at a point on that line.

Given: Line m , with point P on m
 $\overline{PQ} \cong \overline{PR}$ (by construction)
 $\overline{QS} \cong \overline{RS}$ (by construction)

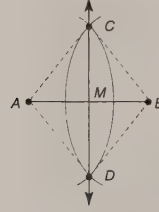
Prove: $\overleftrightarrow{SP} \perp m$



28. Complete the justification of the construction of the perpendicular bisector of a line segment.

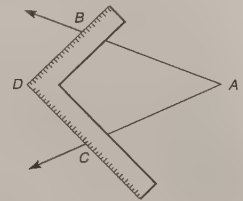
Given: $\overline{AB}$ with $\overline{AC} \cong \overline{BC}$ $\overline{AD} \cong \overline{BD}$ (by construction)

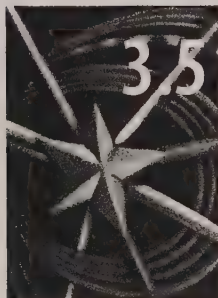
Prove: $\overline{AM} \cong \overline{MB}$ and $\overline{CD} \perp \overline{AB}$



Exercise 28

29. To construct a regular hexagon, what measure would be necessary for each interior angle? Construct an angle of that measure.
 30. To construct a regular octagon, what measure would be necessary for each interior angle? Construct an angle of that measure.
 31. To construct a regular dodecagon (12 sides), what measure would be necessary for each interior angle? Construct an angle of that measure.
 32. Draw an acute triangle and construct the three medians of the triangle. Do the medians appear to meet at a common point?
 33. Draw an obtuse triangle and construct the three altitudes of the triangle. Do the altitudes appear to meet at a common point?
 (**HINT**: In the construction of two of the altitudes, a side needs to be extended.)
 34. Draw a right triangle and construct the angle bisectors of the triangle. Do the angle bisectors appear to meet at a common point?
 35. Draw an obtuse triangle and construct the three perpendicular bisectors of its sides. Do the perpendicular bisectors of the three sides appear to meet at a common point?
 36. Construct an equilateral triangle and its three altitudes. What does intuition tell you about the three medians, the three angle bisectors, and the three perpendicular bisectors of the sides of that triangle?
 37. A carpenter has placed a square over an angle in such a manner that $\overline{AB} \cong \overline{AC}$ and $\overline{BD} \cong \overline{CD}$ (see drawing). What can you conclude about the location of point D ?





INEQUALITIES IN A TRIANGLE

Important inequality relationships exist among the measured parts of a triangle. To establish some of these, we recall and apply some facts from both algebra and geometry. A more in-depth review of inequalities can be found in Appendix A, Section A.3.

DEFINITION: Let a and b be real numbers. $a > b$ (read “ a is greater than b ”) if and only if there is a positive number p for which $a = b + p$.

For instance, $9 > 4$, because there is the positive number 5 for which $9 = 4 + 5$. Because $5 + 2 = 7$, we also know that $7 > 2$ and $7 > 5$. In geometry, let $A-B-C$ on $\overline{AC}$ so that $AB + BC = AC$; then $AC > AB$, because BC is a positive number.

LEMMAS (HELPING THEOREMS)

We will use the following theorems to help us prove the theorems found later in this section. In their role as “helping” theorems, each of the five boxed statements that follow is called a **lemma**. We will prove the first four lemmas, because their content is geometric.



FIGURE 3.43

LEMMA 3.5.1: If B is between A and C on $\overline{AC}$, then $AC > AB$ and $AC > BC$. (The measure of a line segment is greater than the measure of any of its parts. See Figure 3.43.)

Proof

By the Segment-Addition Postulate, $AC = AB + BC$. Because $BC > 0$ (meaning BC is positive), it follows that $AC > AB$. Similarly, $AC > BC$. These relationships follow logically from the definition of $a > b$.

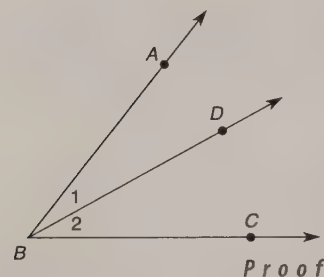


FIGURE 3.44

Proof

LEMMA 3.5.2: If $\overrightarrow{BD}$ separates $\angle ABC$ into two parts ($\angle 1$ and $\angle 2$), then $m\angle ABC > m\angle 1$ and $m\angle ABC > m\angle 2$. (The measure of an angle is greater than the measure of any of its parts. See Figure 3.44.)

By the Angle-Addition Postulate, $m\angle ABC = m\angle 1 + m\angle 2$. Because $m\angle 2 > 0$, it follows that $m\angle ABC > m\angle 1$. Similarly, $m\angle ABC > m\angle 2$.

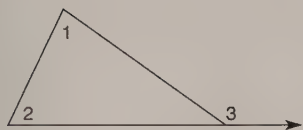


FIGURE 3.45

LEMMA 3.5.3: If $\angle 3$ is an exterior angle of a triangle and $\angle 1$ and $\angle 2$ are the nonadjacent interior angles, then $m\angle 3 > m\angle 1$ and $m\angle 3 > m\angle 2$. (The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle. See Figure 3.45.)

Proof

Because the measure of an exterior angle of a triangle equals the sum of measures of the two nonadjacent interior angles, $m\angle 3 = m\angle 1 + m\angle 2$. It follows that $m\angle 3 > m\angle 1$ and $m\angle 3 > m\angle 2$.

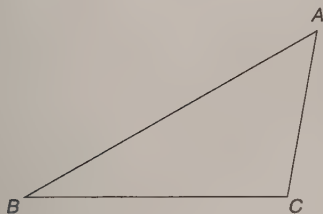


FIGURE 3.46

LEMMA 3.5.4: In $\triangle ABC$, if $\angle C$ is a right angle or an obtuse angle, then $m\angle C > m\angle A$ and $m\angle C > m\angle B$. (If a triangle contains a right or an obtuse angle, then the measure of this angle is greater than the measure of either of the remaining angles. See Figure 3.46.)

Proof

In $\triangle ABC$, $m\angle A + m\angle B + m\angle C = 180^\circ$. With $m\angle C \geq 90^\circ$, it follows that $m\angle A + m\angle B \leq 90^\circ$, and each angle ($\angle A$ and $\angle B$) must be acute. Thus $m\angle C > m\angle A$ and $m\angle C > m\angle B$.

The following theorem (also a lemma) is used in Example 1. Its proof (not given) depends on the definition of “is greater than,” which is found on the previous page.

LEMMA 3.5.5: (Addition Property of Inequality) If $a > b$ and $c > d$, then $a + c > b + d$.

EXAMPLE 1

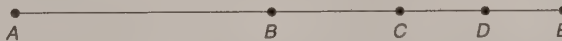
Give a paragraph proof for the following problem.

Given: $AB > CD$ and $BC > DE$

Prove: $AC > CE$

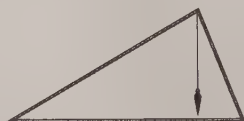
(See Figure 3.47.)

FIGURE 3.47



Proof: If $AB > CD$ and $BC > DE$, then $AB + BC > CD + DE$ by Lemma 3.5.5. But $AB + BC = AC$ and $CD + DE = CE$ by the Segment-Addition Postulate. Using substitution, it follows that $AC > CE$.

GEOMETRY IN THE REAL WORLD



A carpenter's "plumb" determines the shortest distance to a horizontal line. A vertical brace provides structural support for the roof.

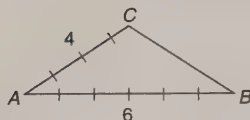


FIGURE 3.48

EXAMPLE 2

The proof in Example 1 could have been written in this more standard format.

PROOF

Statements	Reasons
1. $AB > CD$ and $BC > DE$	1. Given
2. $AB + BC > CD + DE$	2. Lemma 3.5.5
3. $AB + BC = AC$ and $CD + DE = CE$	3. Segment-Addition Postulate
4. $AC > CE$	4. Substitution

The paragraph proof and the two-column proof of Example 1 are equivalent. In either form, statements must be ordered and justified.

The remaining theorems are the "heart" of this section. Before studying the theorem and its proof, it is a good idea to visualize each theorem. Many statements of inequality are intuitive; that is, they are easy to believe even though they may not be easily proved.

Study Theorem 3.5.6 and consider Figure 3.48. It appears that $m\angle C > m\angle B$.

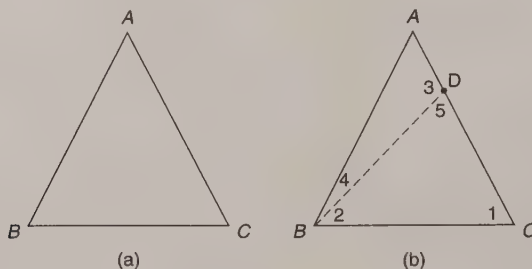
THEOREM 3.5.6: If one side of a triangle is longer than a second side, then the measure of the angle opposite the longer side is greater than the measure of the angle opposite the shorter side.

FIGURE 3.49

Provide a paragraph proof of Theorem 3.5.6.

Given: $\triangle ABC$, with $AC > BC$ [See Figure 3.49(a).]

Prove: $m\angle B > m\angle A$



Proof: Given $\triangle ABC$ with $AC > BC$, we use the Ruler Postulate to locate point D on $\overline{AC}$ so that $\overline{CD} \cong \overline{BC}$, as in Figure 3.49(b). Now $m\angle 2 = m\angle 5$ in the isosceles triangle, $\triangle BDC$. By Lemma 3.5.2, $m\angle ABC > m\angle 2$; therefore, $m\angle ABC > m\angle 5$ (*) by substitution. But $m\angle 5 > m\angle A$ (*), because $\angle 5$

is an exterior angle of $\triangle ADB$. Using the two starred statements, we can conclude by the Transitive Property of Inequality that $m\angle ABC > m\angle A$; that is, $m\angle B > m\angle A$ in Figure 3.49(a).

The relationship described in Theorem 3.5.6 extends, of course, to all sides and all angles of a triangle. That is, the largest of the three angles of a triangle is opposite the longest side, and the smallest angle is opposite the shortest side.

EXAMPLE 3

Solution

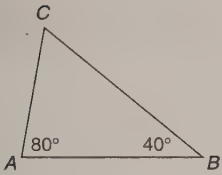


FIGURE 3.50

Given that the three sides of $\triangle ABC$ are $AB = 4$, $BC = 5$, and $AC = 6$, arrange the angles by size.

Because $AC > BC > AB$, the largest angle lies opposite $\overline{AC}$, and that is $\angle B$. The angle intermediate in size lies opposite $\overline{BC}$ ($\angle A$), and the smallest angle lies opposite $\overline{AB}$ ($\angle C$). Thus the order of the angles by size is

$$m\angle B > m\angle A > m\angle C$$

The converse of Theorem 3.5.6 is also true. It is necessary, however, to use an indirect proof to establish the converse. Recall that this method of proof begins by supposing the opposite of what we want to show. Because this assumption leads us to a contradiction, the assumption must be false and the desired claim is therefore true.

Study Theorem 3.5.7 and consider Figure 3.50, in which $m\angle A = 80^\circ$ and $m\angle B = 40^\circ$. It appears that the longer side lies opposite the larger angle; that is, $BC > AC$.

THEOREM 3.5.7: If the measure of one angle of a triangle is greater than the measure of a second angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.

The proof of Theorem 3.5.7 depends on this fact: Given real numbers a and b , only one of the following can be true.

$$a > b, a = b, \text{ or } a < b$$

EXAMPLE 4

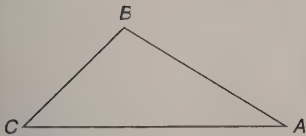


FIGURE 3.51

Prove Theorem 3.5.7 by using an indirect approach in paragraph form.

Given: $\triangle ABC$ with $m\angle B > m\angle A$ (See Figure 3.51.)

Prove: $AC > BC$

Proof: Given $\triangle ABC$ with $m\angle B > m\angle A$, assume that $AC \leq BC$. But if $AC = BC$, then $m\angle B = m\angle A$, which contradicts the hypothesis. Also, if $AC < BC$, then it follows by the previous theorem that $m\angle B < m\angle A$, which also contradicts the hypothesis. Thus the assumed statement must be false, and it follows that $AC > BC$.

EXAMPLE 5

Given $\triangle RST$ in which $m\angle R = 90^\circ$, $m\angle S = 60^\circ$, and $m\angle T = 30^\circ$, write an inequality that compares the lengths of the sides.

Solution

With $m\angle R > m\angle S > m\angle T$, it follows that the sides opposite these $\angle$ s are unequal in the same order. That is,

$$ST > RT > SR$$

The following corollary is a consequence of Theorem 3.5.7.

COROLLARY 3.5.8: The perpendicular segment from a point to a line is the shortest segment that can be drawn from the point to the line.

In Figure 3.52, $PD < PE$, $PD < PF$, and $PD < PG$. In every case, $\overline{PD}$ is opposite an acute angle of a triangle, whereas the second segment is always opposite a right angle (necessarily the largest angle of the triangle involved). With $\overline{PD} \perp \ell$, we say that PD is the *distance* from P to ℓ .

Corollary 3.5.8 can easily be extended to three dimensions.

COROLLARY 3.5.9: The perpendicular segment from a point to a plane is the shortest segment that can be drawn from the point to the plane.

In Figure 3.53, $\overline{PD}$ has a length less than that of $\overline{PE}$, $\overline{PF}$, or $\overline{PG}$; that is, $\overline{PD}$ is the shortest line segment joining point P to a point in plane R .

Our final theorem shows that no side of a triangle can have a length greater than or equal to the sum of the lengths of the other two sides. In the proof, the relationship is validated for only one of three possible inequalities. Theorem 3.5.10 is often called the Triangle Inequality. (See Figure 3.54.)

THEOREM 3.5.10: (Triangle Inequality) The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Given: $\triangle ABC$

Prove: $BA + CA > BC$

Proof: Draw $\overline{AD} \perp \overline{BC}$. Because the shortest segment from a point to $\overline{AD}$ is the perpendicular segment, $BA > BD$ and $CA > CD$. Using Lemma 3.5.5, we add the inequalities; $BA + CA > BD + CD$. By the Segment-Addition Postulate, the sum $BD + CD$ can be replaced by BC to yield $BA + CA > BC$.

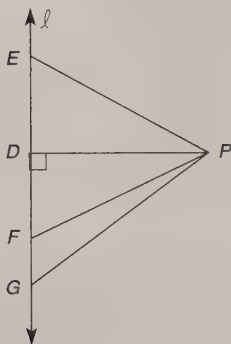


FIGURE 3.52

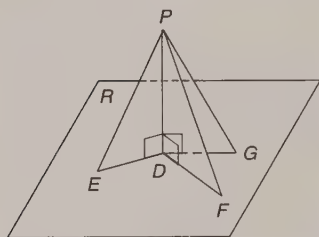


FIGURE 3.53

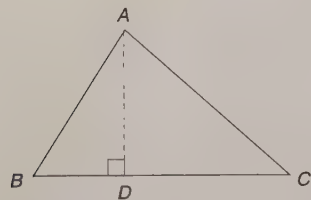


FIGURE 3.54

The following statement is an alternative form of Theorem 3.5.10. If a , b , and c are the lengths of the sides of a triangle and c is the length of any side, then $a - b < c < a + b$.

THEOREM 3.5.10: (Triangle Inequality) The length of any side of a triangle must lie between the sum and difference of the lengths of the other two sides.

EXAMPLE 6

Can a triangle have sides of the following lengths?

- a) 3, 4, and 5
- b) 3, 4, and 7
- c) 3, 4, and 8
- d) 3, 4, and x

Solution

- a) Yes, because no side has a length greater than or equal to the sum of the lengths of the other two sides
- b) No, because $7 = 3 + 4$
- c) No, because $8 > 3 + 4$
- d) Yes, if $4 - 3 < x < 4 + 3$

From Example 6, you can see that the length of one side cannot be greater than or equal to the sum of the lengths of the other two sides. Considering the alternative form of Theorem 3.5.10, we see that $4 - 3 < 5 < 4 + 3$ in part (a). When 5 is replaced by 7 [as in part (b)] or 8 [as in part (c)], this inequality becomes a false statement.

Our final example illustrates a practical application of inequality relationships in triangles.

EXAMPLE 7

On a map, firefighters are located at points A and B . A fire has broken out at point C . Which group of firefighters is nearer the location of the fire? (See Figure 3.55.)

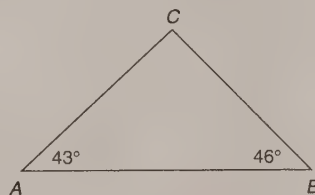


FIGURE 3.55

Solution

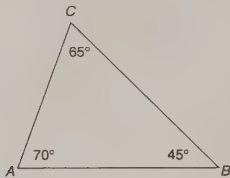
With $m\angle A = 43^\circ$ and $m\angle B = 46^\circ$, it follows that $AC > BC$. Because the distance from B to C is less than the distance from A to C , the firefighters at site B should be dispatched to the fire located at C .



EXERCISES

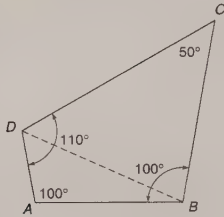
In Exercises 1 to 10, classify each statement as true or false.

1. $\overline{AB}$ is the longest side of $\triangle ABC$.



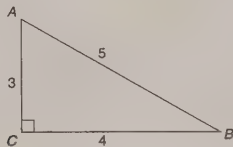
Exercises 1, 2

2. $AB < BC$
3. $DB > AB$



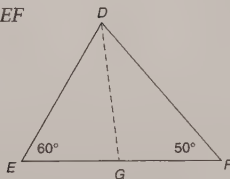
Exercises 3, 4

4. Because $m\angle A = m\angle B$, it follows that $DA = DC$.
5. $m\angle A + m\angle B = m\angle C$



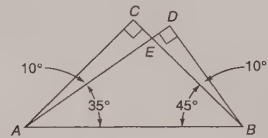
Exercises 5, 6

6. $m\angle A > m\angle B$
7. $DF > DE + EF$



Exercises 7, 8

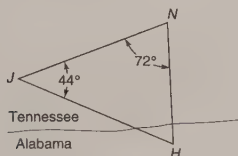
8. If $\overrightarrow{DG}$ is the bisector of $\angle EDF$, then $DG > DE$.
9. $DA > AC$



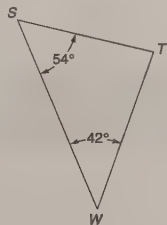
Exercises 9, 10

10. $CE = ED$
11. If possible, draw a triangle whose angles measure:
a) $100^\circ, 100^\circ$, and 60°
b) $45^\circ, 45^\circ$, and 90°
12. If possible, draw a triangle whose angles measure:
a) $80^\circ, 80^\circ$, and 50°
b) $50^\circ, 50^\circ$, and 80°
13. If possible, draw a triangle whose sides measure:
a) 8, 9, and 10
b) 8, 9, and 17
c) 8, 9, and 18
14. If possible, draw a triangle whose sides measure:
a) 7, 7, and 14
b) 6, 7, and 14
c) 6, 7, and 8
15. Two of the sides of an isosceles triangle have lengths of 10 cm and 4 cm. Which length must be the length of the base?
16. The sides of a right triangle have lengths of 6 cm, 8 cm, and 10 cm. Which length is that of the hypotenuse?
17. A triangle is both isosceles and acute. If one angle of the triangle measures 36° , what is the measure of the largest angle(s) of the triangle? What is the measure of the smallest angle(s) of the triangle?
18. One of the angles of an isosceles triangle measures 96° . What is the measure of the largest angle(s) of the triangle? What is the measure of the smallest angle(s) of the triangle?

19. NASA in Huntsville, Alabama (at point H), has called a manufacturer for parts needed as soon as possible. NASA will, in fact, send a courier for the necessary equipment. The manufacturer has two distribution centers located in nearby Tennessee—one in Nashville (at point N) and the other in Jackson (at point J). Using the angle measurements indicated on the accompanying map, determine to which town the courier should be dispatched to obtain the needed parts.

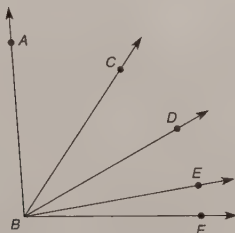


20. A tornado has just struck a small Kansas community at point T . There are Red Cross units stationed in both Salina (at point S) and Wichita (at point W). Using the angle measurements indicated on the accompanying map, determine which Red Cross unit would reach the victims first. (Assume that both units have the same mode of travel available.)



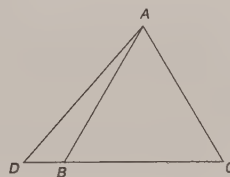
In Exercises 21 and 22, complete each proof.

21. *Given:* $m\angle ABC > m\angle DBE$
 $m\angle CBD > m\angle EBF$
Prove: $m\angle ABD > m\angle DBF$



P R O O F	
Statements	Reasons
1. ?	1. Given
2. $m\angle ABC + m\angle CBD > m\angle DBE + m\angle EBF$	2. Addition Property of Inequality
3. $m\angle ABD = m\angle ABC + m\angle CBD$ and $m\angle DBF = m\angle DBE + m\angle EBF$	3. ?
4. ?	4. Substitution

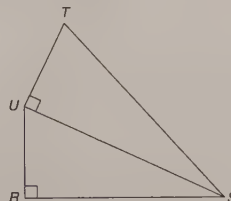
22. *Given:* Equilateral $\triangle ABC$ and D - B - C
Prove: $DA > AC$



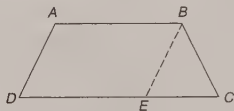
P R O O F	
Statements	Reasons
1. ?	1. Given
2. $\triangle ABC$ is equiangular, so $m\angle ABC = m\angle C$	2. ?
3. $m\angle ABC > m\angle D$ ($\angle D$ of $\triangle ABD$)	3. The measure of an ext. $\angle$ of a $\triangle$ is greater than the measure of either non-adjacent int. $\angle$
4. ?	4. Substitution
5. ?	5. ?

In Exercises 23 and 24, construct proofs.

23. *Given:* Quadrilateral $RSTU$ with diagonal $\overline{US}$
 $\angle R$ and $\angle TUS$ are right $\angle$ s
Prove: $TS > UR$



24. *Given:* Quadrilateral $ABCD$ with $\overline{AB} \cong \overline{DE}$
Prove: $DC > AB$



25. For $\triangle ABC$ and $\triangle DEF$ (not shown), suppose that $\overline{AC} \cong \overline{DF}$ and $\overline{AB} \cong \overline{DE}$ but that $m\angle A < m\angle D$. Draw a conclusion regarding $\overline{BC}$ and $\overline{EF}$.
26. In $\triangle MNP$ (not shown), $\overrightarrow{MQ}$ bisects $\angle NMP$ and $MN < MP$. Draw a conclusion about the relative lengths of $\overline{NQ}$ and $\overline{QP}$.

In Exercises 27 to 30, apply a form of Theorem 3.5.10.

27. The sides of a triangle have lengths of 4, 6, and x . Write an inequality that states the possible values of x .
28. The sides of a triangle have lengths of 7, 13, and x . As in Exercise 27, write an inequality that describes possible values of x .

29. If the lengths of two sides of a triangle are represented by $2x + 5$ and $3x + 7$ (in which x is positive), describe in terms of x the possible lengths of the third side whose length is represented by y .
30. Prove by the indirect method: "The diagonal of a square is not equal in length to the length of any of the sides of the square."
31. Prove by the indirect method:
Given: $\triangle MPN$ is not isosceles
Prove: $PM \neq PN$

In Exercises 32 and 33, prove each theorem.

32. The length of the median from the vertex of an isosceles triangle is less than the length of either of the legs.
33. The length of an altitude of a triangle that does not contain a right angle is less than the length of either side containing the same vertex as the altitude.



A Look Beyond: Historical Sketch of Archimedes

Whereas Euclid (see *A Look Beyond*, Chapter 1) was a great teacher and wrote so that the majority might understand the principles of geometry, Archimedes wrote only for the very well-educated mathematicians and scientists of his day. Archimedes (287 B.C.—212 B.C.) wrote on such topics as the measure of the circle, the quadrature of the parabola, and spirals. In his works, Archimedes found a very good approximation of π . His other geometric works included investigations of conic sections and spirals, and he also wrote about physics. He was a great inventor and is probably remembered more for his inventions than for his writing.

Several historical events concerning the life of Archimedes have been substantiated, and one account involves his detection of a dishonest goldsmith. In that story, Archimedes was called upon to determine whether the crown that had been ordered by the king was constructed entirely of gold. By applying the principle of hydrostatics (which he had discovered), Archimedes established that the goldsmith had not constructed the crown entirely of gold. (The principle of hydrostatics states that an object placed in a fluid displaces an amount of fluid equal in weight to the amount of weight the object loses while submerged.)

One of his inventions is known as Archimedes' screw. This device allows water to flow from one level to a higher level so that, for example, holds of ships can be emptied of water. Archimedes' screw was used in Egypt to drain fields when the Nile River overflowed its banks.

When Syracuse (where Archimedes lived) came under siege by the Romans, Archimedes designed a long-range catapult that was so effective that Syracuse was able to fight off the powerful Roman army for three years before being overcome.

One report concerning the inventiveness of Archimedes has been treated as false, because his result has not been duplicated. It was said that he designed a wall of mirrors that could focus and reflect the sun's heat with such intensity as to set fire to Roman ships at sea. Because recent experiments with concave mirrors have failed to produce such intense heat, this account is difficult to believe.

Archimedes eventually died at the hands of a Roman soldier, even though the Roman army had been given orders not to harm him. After his death, the Romans honored his brilliance with a tremendous monument displaying the figure of a sphere inscribed in a right circular cylinder.



Summary

■ A Look Back at Chapter 3

In this chapter, we considered several methods for proving triangles congruent. Inequality relationships for the sides and angles of a triangle were also investigated.

■ A Look Ahead to Chapter 4

In the next chapter, we use properties of triangles to develop the properties of quadrilaterals. We consider several special types of quadrilaterals, including the parallelogram, kite, rhombus, and trapezoid.

■ Important Terms and Concepts of Chapter 3

3.1 Congruent Triangles

SSS, SAS, ASA, AAS

Included Angle, Included Side

Reflexive Property of Congruence (Identity)

Symmetric, Transitive Properties of Congruence

3.2 CPCTC

Hypotenuse and Legs of a Right Triangle

HL

Pythagorean Theorem

Square Roots Property

3.3 Isosceles Triangles

Vertex, Legs, and Base of an Isosceles Triangle

Base Angles, Vertex Angle

Angle Bisector

Median

Altitude

Perpendicular Bisector

Auxiliary Line

Determined, Underdetermined,

Overdetermined Figures

Equilateral, Equiangular Triangles

Perimeter

3.4 Justifying Constructions

3.5 Lemma

Triangle Inequality

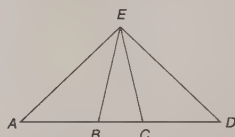
■ A Look Beyond: Historical Sketch of Archimedes



Review Exercises

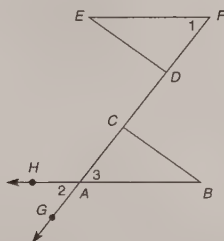
1. Given: $\angle AEB \cong \angle DEC$
 $\overline{AE} \cong \overline{ED}$

Prove: $\triangle AEB \cong \triangle DEC$



2. Given: $\overline{AB} \cong \overline{EF}$
 $\overline{AC} \cong \overline{DF}$
 $\angle 1 \cong \angle 2$

Prove: $\angle B \cong \angle E$

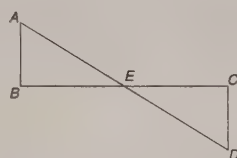


3. Given: $\overline{AD}$ bisects $\overline{BC}$

$$\overline{AB} \perp \overline{BC}$$

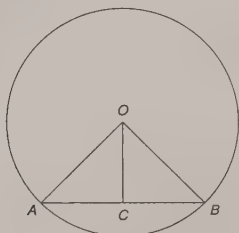
$$\overline{DC} \perp \overline{BC}$$

Prove: $\overline{AE} \cong \overline{ED}$



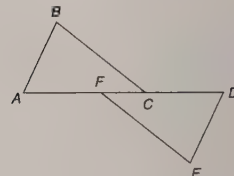
4. Given: $\overline{OA} \cong \overline{OB}$
 $\overline{OC}$ is the median to $\overline{AB}$

Prove: $\overline{OC} \perp \overline{AB}$



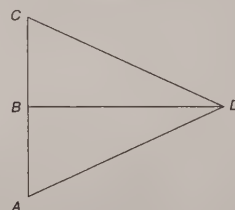
5. Given: $\overline{AB} \cong \overline{DE}$
 $\overline{AB} \parallel \overline{DE}$
 $\overline{AC} \cong \overline{DF}$

Prove: $\overline{BC} \parallel \overline{FE}$



6. Given: B is the midpoint of $\overline{AC}$
 $\overline{BD} \perp \overline{AC}$

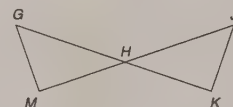
Prove: $\triangle ADC$ is isosceles



7. Given: $\overline{JM} \perp \overline{GM}$ and $\overline{GK} \perp \overline{KJ}$

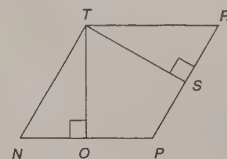
$$\overline{GH} \cong \overline{HJ}$$

Prove: $\overline{GM} \cong \overline{JK}$



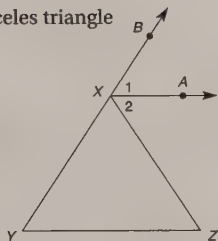
8. Given: $\overline{TN} \cong \overline{TR}$
 $\overline{TO} \perp \overline{NP}$
 $\overline{TS} \perp \overline{PR}$
 $\overline{TO} \cong \overline{TS}$

Prove: $\angle N \cong \angle R$



9. Given: $\overline{YZ}$ is the base of an isosceles triangle
 $\overline{XA} \parallel \overline{YZ}$

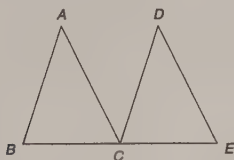
Prove: $\angle 1 \cong \angle 2$



10. Given: $\overline{AB} \parallel \overline{DC}$
 $\overline{AB} \cong \overline{DC}$

C is the midpoint of $\overline{BE}$

Prove: $\overline{AC} \parallel \overline{DE}$

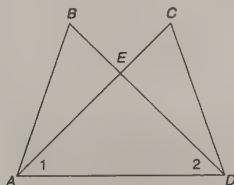


11. Given: $\angle BAD \cong \angle CDA$

$\overline{AB} \cong \overline{CD}$

Prove: $\overline{AE} \cong \overline{ED}$

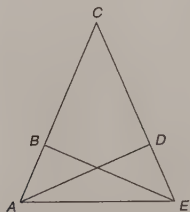
(HINT: Prove $\triangle BAD \cong \triangle CDA$ first.)



12. Given: $\overline{BE}$ is the altitude to $\overline{AC}$
 $\overline{AD}$ is the altitude to $\overline{CE}$
 $\overline{BC} \cong \overline{CD}$

Prove: $\overline{BE} \cong \overline{AD}$

(HINT: Prove $\triangle CBE \cong \triangle CDA$.)

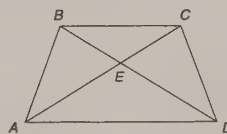


13. Given: $\overline{AB} \cong \overline{CD}$

$\angle BAD \cong \angle CDA$

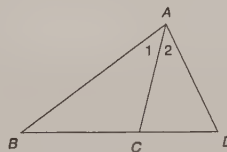
Prove: $\triangle AED$ is isosceles

(HINT: Prove $\angle CAD \cong \angle BDA$ by CPCTC.)



14. Given: $\overrightarrow{AC}$ bisects $\angle BAD$

Prove: $AD > CD$



15. In $\triangle PQR$ (not shown), $m\angle P = 67^\circ$ and $m\angle Q = 23^\circ$.

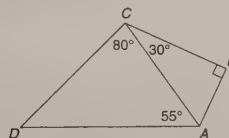
a) Name the shortest side.

b) Name the longest side.

16. In $\triangle ABC$ (not shown), $m\angle A = 40^\circ$ and $m\angle B = 65^\circ$. List the sides in order of their lengths, starting with the smallest side.

17. In $\triangle PQR$ (not shown), $PQ = 1.5$, $PR = 2$, and $QR = 2.5$. List the angles in order of size, starting with the smallest angle.

18. Name the longest line segment in the figure.



19. Which of the following can be the lengths of the sides of a triangle?

a) 3, 6, 9

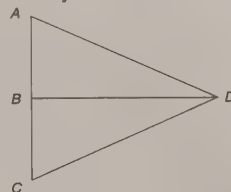
b) 4, 5, 8

c) 2, 3, 8

20. Two sides of a triangle have lengths 15 and 20. The length of the third side can be any number between $\underline{\hspace{1cm}}$ and $\underline{\hspace{1cm}}$.

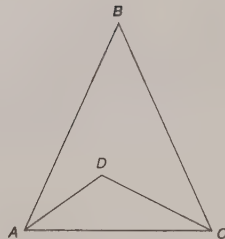
21. Given: $\overline{DB} \perp \overline{AC}$
 $\overline{AD} \cong \overline{DC}$
 $m\angle C = 70^\circ$

Find: $m\angle ADB$



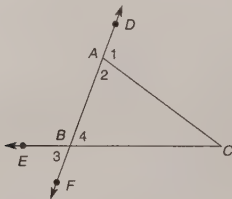
22. Given: $\overline{AB} \cong \overline{BC}$
 $\angle DAC \cong \angle BCD$
 $m\angle B = 50^\circ$

Find: $m\angle ADC$



23. Given: $\triangle ABC$ is isosceles with base $\overline{AB}$
 $m\angle 2 = 3x + 10$
 $m\angle 4 = \frac{5}{2}x + 18$

Find: $m\angle C$



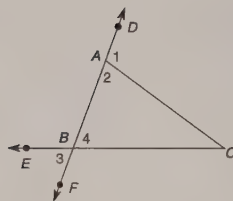
Exercises 23, 24

24. Given: $\triangle ABC$ with perimeter 40
 $AB = 10$
 $BC = x + 6$
 $AC = 2x - 3$

Find: Whether $\triangle ABC$ is scalene, isosceles, or equilateral

25. Given: $\triangle ABC$ is isosceles with base $\overline{AB}$
 $AB = y + 7$
 $BC = 3y + 5$
 $AC = 9 - y$

Find: Whether $\triangle ABC$ is also equilateral



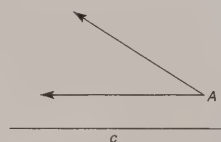
Exercises 25, 26

26. Given: $\overline{AC}$ and $\overline{BC}$ are the legs of isosceles $\triangle ABC$
 $m\angle 1 = 5x$
 $m\angle 3 = 2x + 12$

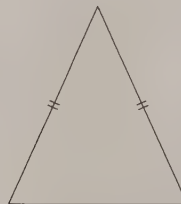
Find: $m\angle 2$

27. Construct an angle that measures 75° .

28. Construct a right triangle that has acute angle A and hypotenuse of length c .



29. Construct a second isosceles triangle in which the base angles are half as large as the base angles of the given isosceles triangle.





CHAPTER 5

SIMILAR TRIANGLES

Given two squares, one with sides of length 5 inches and the other with sides of length 3 inches, the squares have the same shape. The smaller and larger cylindrical containers shown on the grocery store shelves also illustrate figures with different sizes but the same shape. In both situations, one figure is simply an enlargement of the other. In geometry, we say that the two objects are *similar* in shape. See Section 5.2 for more information and further illustrations of similar geometric figures.

Some applications found in this chapter and later chapters lead to quadratic equations. A review of the methods for solving quadratic equations can be found in Appendix A.4 of this textbook.



CHAPTER OUTLINE

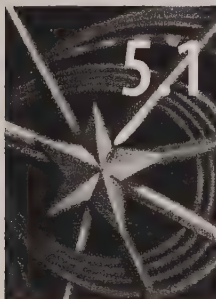
- 5.1 Ratios, Rates, and Proportions
- 5.2 Similar Triangles and Polygons
- 5.3 The Pythagorean Theorem
- 5.4 Special Right Triangles
- 5.5 Segments Divided Proportionally

A Look Beyond:

An Unusual Application of Similar Triangles

Summary

For online student resources, visit this textbook's web site at math.college.hmco.com/students.



RATIOS, RATES, AND PROPORTIONS

The concepts and techniques discussed in Section 5.1 are often necessary for the geometry applications found later in Chapter 5 and beyond.

A **ratio** is the quotient $\frac{a}{b}$ (where $b \neq 0$) that provides a comparison between numbers a and b . Because every fraction indicates a division, every fraction represents a ratio. Read “ a to b ,” the ratio is sometimes written in the form $a:b$.

It is generally preferable to provide the ratio in simplest form, so the ratio 6 to 8 would be reduced (in fraction form) from $\frac{6}{8}$ to $\frac{3}{4}$. If units of measure are involved, these units must be **commensurable** (convertible to the same unit of measure). When simplifying the ratio of two quantities that are expressed in the same unit, we eliminate the common unit in the process. If two quantities cannot be compared because no common unit of measure is possible, the quantities are **incommensurable**.

EXAMPLE 1

Find the best form of each ratio:

- 12 to 20
- 12 in. to 24 in.
- 12 in. to 3 ft (NOTE: 1 ft = 12 in.)
- 5 lb to 20 oz (NOTE: 1 lb = 16 oz)
- 5 lb to 2 ft
- 4 m to 30 cm (NOTE: 1 m = 100 cm)

Solution

$$\text{a) } \frac{12}{20} = \frac{3}{5}$$

$$\text{b) } \frac{12 \text{ in.}}{24 \text{ in.}} = \frac{12}{24} = \frac{1}{2}$$

$$\text{c) } \frac{12 \text{ in.}}{3 \text{ ft}} = \frac{12 \text{ in.}}{3(12 \text{ in.})} = \frac{12 \text{ in.}}{36 \text{ in.}} = \frac{1}{3}$$

$$\text{d) } \frac{5 \text{ lb}}{20 \text{ oz}} = \frac{5(16 \text{ oz})}{20 \text{ oz}} = \frac{80 \text{ oz}}{20 \text{ oz}} = \frac{4}{1}$$

$$\text{e) } \frac{5 \text{ lb}}{2 \text{ ft}} \text{ is incommensurable!}$$

$$\text{f) } \frac{4 \text{ m}}{30 \text{ cm}} = \frac{4(100 \text{ cm})}{30 \text{ cm}} = \frac{400 \text{ cm}}{30 \text{ cm}} = \frac{40}{3}$$

GEOMETRY IN THE REAL WORLD



At a grocery store, the cost per unit is a rate that allows the consumer to know which brand is more expensive.

A **rate** is a quotient that compares two quantities that are incommensurable. If an automobile can travel 300 miles of an interstate on a full tank of 10 gallons of gasoline, then its consumption **rate** is $\frac{300 \text{ miles}}{10 \text{ gallons}}$. In simplified form, the consumption rate is $\frac{30 \text{ mi}}{\text{gal}}$, which is read as “30 miles per gallon” and is abbreviated 30 mpg.

A **proportion** is a statement that two ratios or two rates are equal. Thus $\frac{a}{b} = \frac{c}{d}$ is a proportion and may be read as “ a is to b as c is to d .” In the order read, a is the **first term** of the proportion, b is the **second term**, c is the **third term**, and d is the **fourth term**. The first and last terms (a and d) of the proportion are the **extremes**, whereas the second and third terms (b and c) are the **means**.

The following property is convenient for solving many proportions.

PROPERTY 1: (Means-Extremes Property)

In a proportion, the product of the means equals the product of the extremes; that is, if $\frac{a}{b} = \frac{c}{d}$ (where $b \neq 0$ and $d \neq 0$), then $a \cdot d = b \cdot c$

In the false proportion $\frac{9}{12} = \frac{2}{3}$, it is obvious that $9 \cdot 3 \neq 12 \cdot 2$; on the other hand, the truth of the statement $\frac{9}{12} = \frac{3}{4}$ is evident from the fact that $9 \cdot 4 = 12 \cdot 3$. Henceforth, any proportion given in this text is intended to be a true proportion.

EXAMPLE 2

Use the Means-Extremes Property to solve each proportion for x .

$$\text{a) } \frac{x}{8} = \frac{5}{12}$$

$$\text{c) } \frac{3}{x} = \frac{x}{2}$$

$$\text{e) } \frac{x+2}{5} = \frac{4}{x-1}$$

$$\text{b) } \frac{x+1}{9} = \frac{x-3}{3}$$

$$\text{d) } \frac{x+3}{3} = \frac{9}{x-3}$$

Solution

$$\text{a) } x \cdot 12 = 8 \cdot 5$$

(Means-Extremes Property)

$$12x = 40$$

$$x = \frac{40}{12} = \frac{10}{3}$$

$$\text{b) } 3(x+1) = 9(x-3)$$

(Means-Extremes Property)

$$3x+3 = 9x-27$$

$$30 = 6x$$

$$x = 5$$

$$\text{c) } 3 \cdot 2 = x \cdot x$$

(Means-Extremes Property)

$$x^2 = 6$$

$$x = \pm\sqrt{6} \approx \pm 2.45$$

$$\text{d) } (x+3)(x-3) = 3 \cdot 9$$

(Means-Extremes Property)

$$x^2 - 9 = 27$$

$$x^2 - 36 = 0$$

$$(x+6)(x-6) = 0$$

(using factoring)

$$x+6 = 0$$

or

$$x-6 = 0$$

$$x = -6$$

or

$$x = 6$$

$$\text{e) } (x+2)(x-1) = 5 \cdot 4$$

(Means-Extremes Property)

$$x^2 + x - 2 = 20$$

$$x^2 + x - 22 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(using Quadratic Formula; see Appendix A.4)

$$= \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-22)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1+88}}{2}$$

$$= \frac{-1 \pm \sqrt{89}}{2}$$

$$\approx 4.22 \text{ or } -5.22$$

In application problems involving proportions, it is essential to order the related quantities in each ratio or rate. The following example illustrates the care that must be taken in forming the proportion for an application.

EXAMPLE 3

If an automobile can travel 90 mi on 4 gal of gasoline, how far can it travel on 6 gal of gasoline?

Solution

By form,

$$\frac{\text{number miles first trip}}{\text{number gallons first trip}} = \frac{\text{number miles second trip}}{\text{number gallons second trip}}$$

Where x represents the number of miles traveled on the second trip, we have

$$\begin{aligned}\frac{90}{4} &= \frac{x}{6} \\ 4x &= 540 \\ x &= 135\end{aligned}$$

Thus the car can travel 135 mi on 6 gal of gasoline.

In $\frac{a}{b} = \frac{b}{c}$, where the second and third terms of the proportion are identical, the value of b is known as the **geometric mean** of a and c . For example, 6 and -6 are the geometric means of 4 and 9 because $\frac{4}{6} = \frac{6}{9}$ and $\frac{4}{-6} = \frac{-6}{9}$. Because applications in geometry generally require positive solutions, we usually seek only the positive geometric mean of a and c .

EXAMPLE 4

In Figure 5.1, AD is the geometric mean of BD and DC . If $BC = 10$ and $BD = 4$, determine AD .

Solution

$\frac{BD}{AD} = \frac{AD}{DC}$. Because $DC = BC - BD$, we know that $DC = 10 - 4 = 6$. Therefore,

$$\frac{4}{x} = \frac{x}{6}$$

in which x is the length of $\overline{AD}$. Applying the Means-Extremes Property, we get

$$\begin{aligned}x^2 &= 24 \\ x &= \pm\sqrt{24} = \pm\sqrt{4 \cdot 6} = \pm\sqrt{4} \cdot \sqrt{6} = \pm 2\sqrt{6}\end{aligned}$$

To have a permissible length for $\overline{AD}$, the geometric mean is the positive solution. Thus $AD = 2\sqrt{6}$ or $AD \approx 4.90$.

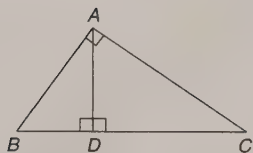


FIGURE 5.1

An **extended ratio** compares more than two quantities and must be expressed in a form such as $a:b:c$ or $d:e:f:g$. If you know that the angles of a triangle are 90° , 60° , and 30° , then the ratio that compares these measures is $90:60:30$ or $3:2:1$ (because 90, 60, and 30 have the greatest common factor of 30). Unknown quantities in the ratio $a:b:c:d: \dots$ are generally represented by variable expressions such as $ax, bx, cx, dx, \dots$

EXAMPLE 5

Suppose that the perimeter of a quadrilateral is 70 and the lengths of the sides are in the ratio 2:3:4:5. Find the measure of each side.

Solution

Let the lengths of the sides be represented by $2x$, $3x$, $4x$, and $5x$. Then

$$\begin{aligned} 2x + 3x + 4x + 5x &= 70 \\ 14x &= 70 \\ x &= 5 \end{aligned}$$

The lengths of the sides are 10, 15, 20, and 25, because

$$2x = 10, 3x = 15, 4x = 20, \text{ and } 5x = 25$$

It is possible to solve certain problems in more ways than one, as is illustrated in the next example. However, the solution is unique and is not altered by the method chosen.

EXAMPLE 6

Two complementary angles are in the ratio 2 to 3. Find the measure of each angle.

Solution

Let the first of the complementary angles have measure x ; then the second has the measure $90 - x$. Thus we have

$$\frac{x}{90 - x} = \frac{2}{3}$$

Using the Means-Extremes Property, we have

$$\begin{aligned} 3x &= 2(90 - x) \\ 3x &= 180 - 2x \\ 5x &= 180 \\ x &= 36 \\ 90 - x &= 54 \end{aligned}$$

The angles have measures of 36° and 54° .

**Alternative
Solution**

Let the measures of the two angles be $2x$ and $3x$, because they are in the ratio 2:3. Because the angles are complementary,

$$\begin{aligned} 2x + 3x &= 90 \\ 5x &= 90 \\ x &= 18 \end{aligned}$$

Now $2x = 36$ and $3x = 54$, so the measures of the two angles are 36° and 54° .

Some additional properties of proportions follow. Because they are not cited as often as the Means-Extremes Property, they are not given titles.

PROPERTY 2: In a proportion, the means or the extremes (or both the means and the extremes) may be interchanged; that is, if $\frac{a}{b} = \frac{c}{d}$ (where a, b, c , and d are nonzero), then $\frac{a}{c} = \frac{b}{d}$, $\frac{d}{b} = \frac{c}{a}$, and $\frac{d}{c} = \frac{b}{a}$.

NOTE: The last proportion is the inverted form of the given proportion. In any of the equivalent proportions, $a \cdot d = b \cdot c$ (product of means = product of extremes).

If given the proportion $\frac{2}{3} = \frac{8}{12}$, Property 2 enables us to draw conclusions such as:

1. $\frac{2}{8} = \frac{3}{12}$ (means switched)
2. $\frac{12}{3} = \frac{8}{2}$ (extremes switched)
3. $\frac{3}{2} = \frac{12}{8}$ (both sides inverted)

PROPERTY 3: If $\frac{a}{b} = \frac{c}{d}$ (where $b \neq 0$ and $d \neq 0$), then $\frac{a+b}{b} = \frac{c+d}{d}$ and $\frac{a-b}{b} = \frac{c-d}{d}$.

NOTE: If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + 1 = \frac{c}{d} + 1$. In turn, $\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}$.

When we are given the proportion $\frac{2}{3} = \frac{8}{12}$, for example, Property 3 enables us to draw such conclusions as:

1. $\frac{2+3}{3} = \frac{8+12}{12}$ (each side simplifies to $\frac{5}{3}$)
2. $\frac{2-3}{3} = \frac{8-12}{12}$ (each side equals $-\frac{1}{3}$)

Just as there are extended ratios, there are also **extended proportions**, such as

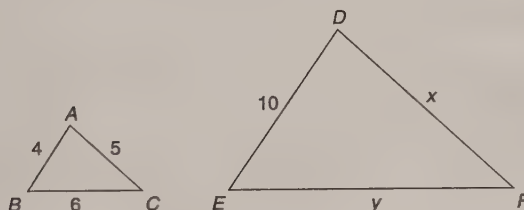
$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots$$

Suggested by different numbers of servings of a particular recipe, the statement below is an extended proportion comparing numbers of eggs to numbers of cups of milk:

$$\frac{2 \text{ eggs}}{3 \text{ cups}} = \frac{4 \text{ eggs}}{6 \text{ cups}} = \frac{6 \text{ eggs}}{9 \text{ cups}}$$

EXAMPLE 7

In the triangles shown in Figure 5.2, $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$. Find the lengths of $\overline{DF}$ and $\overline{EF}$.

**FIGURE 5.2****Solution**

Substituting, we have

$$\frac{4}{10} = \frac{5}{x} = \frac{6}{y}$$

From the equation

$$\frac{4}{10} = \frac{5}{x}$$

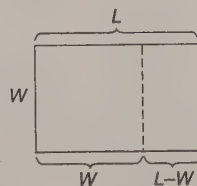
it follows that $4x = 50$ and that $x = DF = 12.5$. Using the equation

$$\frac{4}{10} = \frac{6}{y}$$

we find that $4y = 60$, so $y = EF = 15$.

DISCOVER!**THE GOLDEN RATIO**

It is believed that the “ideal” rectangle is most attractive when a square can be removed in such a way as to leave a smaller rectangle with the same shape as the original rectangle. As we shall find, the rectangles are known as *similar* in shape. Upon removal of the square, the similarity in the shapes of the rectangles requires that $\frac{W}{L} = \frac{L-W}{W}$. To discover the relationship between L and W , we choose $W = 1$ and solve the equation $\frac{1}{L} = \frac{L-1}{1}$ for L . The solution is $L = \frac{1+\sqrt{5}}{2}$. The ratio comparing length to width is known as the *golden ratio*. Because $L = \frac{1+\sqrt{5}}{2}$ when $W = 1$ and $\frac{1+\sqrt{5}}{2} \approx 1.62$, the *ideal* rectangle has a length that is approximately 1.62 times its width; that is, $L \approx 1.62W$.





EXERCISES

In Exercises 1 to 4, give the ratios in simplified form.

1. a) 12 to 15 c) 1 ft to 18 in.
b) 12 in. to 15 in. d) 1 ft to 18 oz
2. a) 20 to 36 c) 20 oz to 2 lb (1 lb = 16 oz)
b) 24 oz to 52 oz d) 2 lb to 20 oz
3. a) 15:24 c) 2 m:150 cm (1 m = 100 cm)
b) 2 ft:2 yd (1 yd = 3 ft) d) 2 m:1 lb
4. a) 24:32 c) 150 cm:2 m
b) 12 in.:2 yd d) 1 gal:24 mi

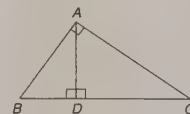
In Exercises 5 to 14, find the value of x in each proportion.

5. a) $\frac{x}{4} = \frac{9}{12}$ b) $\frac{7}{x} = \frac{21}{24}$
6. a) $\frac{x-1}{10} = \frac{3}{5}$ b) $\frac{x+1}{6} = \frac{10}{12}$
7. a) $\frac{x-3}{8} = \frac{x+3}{24}$ b) $\frac{x+1}{6} = \frac{4x-1}{18}$
8. a) $\frac{9}{x} = \frac{x}{16}$ b) $\frac{32}{x} = \frac{x}{2}$
9. a) $\frac{x}{4} = \frac{7}{x}$ b) $\frac{x}{6} = \frac{3}{x}$
10. a) $\frac{x+1}{3} = \frac{10}{x+2}$ b) $\frac{x-2}{5} = \frac{12}{x+2}$
11. a) $\frac{x+1}{x} = \frac{10}{2x}$ b) $\frac{2x+1}{x+1} = \frac{14}{3x-1}$
12. a) $\frac{x+1}{2} = \frac{7}{x-1}$ b) $\frac{x+1}{3} = \frac{5}{x-2}$
13. a) $\frac{x+1}{x} = \frac{2x}{3}$ b) $\frac{x+1}{x-1} = \frac{2x}{5}$
14. a) $\frac{x+1}{x} = \frac{x}{x-1}$ b) $\frac{x+2}{x} = \frac{2x}{x-2}$

15. Sarah ran the 300-m hurdles in 47.7 seconds. In meters per second, find the rate at which Sarah ran. Give the answer to the nearest tenth of a meter per second.
16. Fran has been hired to sew the dance troupe's dresses for the school musical. If $13\frac{1}{3}$ yd of material is needed for the four dresses, find the rate that describes the amount of material needed for each dress.

In Exercises 17 to 22, use proportions to solve each problem.

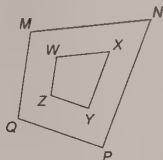
17. A recipe calls for 4 eggs and 3 cups of milk. To prepare for a larger number of guests, a cook uses 14 eggs. How many cups of milk are needed?
18. If a school secretary copies 168 worksheets for a class of 28 students, how many must be prepared for a class of 32 students?
19. An electrician installs 20 electrical outlets in a new six-room house. Assuming proportionality, how many outlets should be installed in a new construction having seven rooms? (Round up to an integer.)
20. The secretarial pool (15 secretaries in all) on one floor of a corporate complex has access to four copy machines. If there are 23 secretaries on a different floor, approximately what number of copy machines should be available? (Assume a proportionality.)
21. Assume that AD is the geometric mean of BD and DC in $\triangle ABC$ shown in the accompanying drawing.
 - a) Find AD if $BD = 6$ and $DC = 8$.
 - b) Find BD if $AD = 6$ and $DC = 8$.



Exercises 21, 22

22. In the drawing, assume that AB is the geometric mean of BD and BC .
 - a) Find AB if $BD = 6$ and $DC = 10$.
 - b) Find DC if $AB = 10$ and $BC = 15$.
23. The salaries of a secretary, a salesperson, and a vice president for a retail sales company are in the ratio 2:3:5. If their combined annual salaries amount to \$92,500, what is the annual salary of each?
24. If the angles of a quadrilateral are in the ratio of 2:3:4:6, find the measure of each angle.

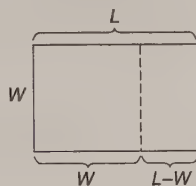
25. The measures of two complementary angles are in the ratio 4:5. Find the measure of each angle, using the two methods shown in Example 6.
26. The measures of two supplementary angles are in the ratio of 2:7. Find the measure of each angle, using the two methods of Example 6.
27. If 1 inch equals 2.54 centimeters, use a proportion to convert 12 inches to centimeters.
(HINT: $\frac{2.54 \text{ cm}}{1 \text{ in.}} = \frac{x \text{ cm}}{12 \text{ in.}}$)
28. If 1 kilogram equals 2.2 pounds, use a proportion to convert 12 pounds to kilograms.
29. For the quadrilaterals shown, $\frac{MN}{WX} = \frac{NP}{XY} = \frac{PQ}{YZ} = \frac{MQ}{WZ}$. If $MN = 7$, $WX = 3$, and $PQ = 6$, find YZ .



Exercises 29, 30

30. For this exercise, use the drawing and extended ratio of Exercise 29. If $NP = 2 \cdot XY$ and $WZ = 3\frac{1}{2}$, find MQ .

31. Two numbers a and b are in the ratio 3:4. If the first number is decreased by 2 and the second is decreased by 1, they are in the ratio 2:3. Find a and b .
32. If the ratio of the measure of the complement of an angle to the measure of its supplement is 1:4, what is the measure of the angle?
33. On a blueprint, a 1-in. scale corresponds to 3 ft. To show a room with actual dimensions 12 ft wide by 14 ft long, what dimensions should be shown on the blueprint?
34. To find the golden ratio (see Discover! on page 203), solve the equation $\frac{1}{L} = \frac{L-1}{1}$ for L .
(HINT: You will need the Quadratic Formula.)



35. Find:
- the exact length of an ideal rectangle with width $W = 5$ by solving $\frac{5}{L} = \frac{L-5}{5}$.
 - the approximate length of an ideal rectangle with width $W = 5$ by using $L \approx 1.62W$.



SIMILAR TRIANGLES AND POLYGONS

When two geometric figures have exactly the same shape, they are **similar**; the symbol for “is similar to” is $\sim$. When two figures have the same shape ($\sim$) and all corresponding parts have equal ($=$) measures, the two figures are **congruent** ($\cong$). Note that the symbol for congruence combines the symbols for similarity and equality.

Two-dimensional figures can be similar, as $\triangle ABC$ and $\triangle DEF$ are in Figure 5.3 on the following page, but it is also possible for three-dimensional figures to be similar. Similar orange juice containers are shown in Figures 5.4(a) and (b) on the next page. Informally, two figures are termed “similar” if one is an enlargement of the other. Thus a tuna fish can and an orange juice can are *not* similar, even if both are right-circular cylinders [see Figures 5.4(b) and (c)]. We will consider cylinders in greater detail in Chapter 8.

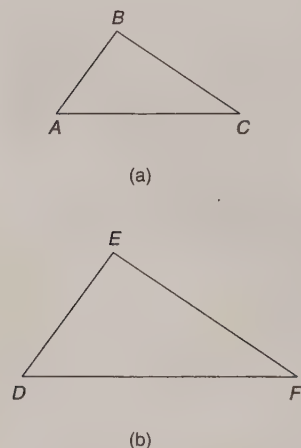


FIGURE 5.3

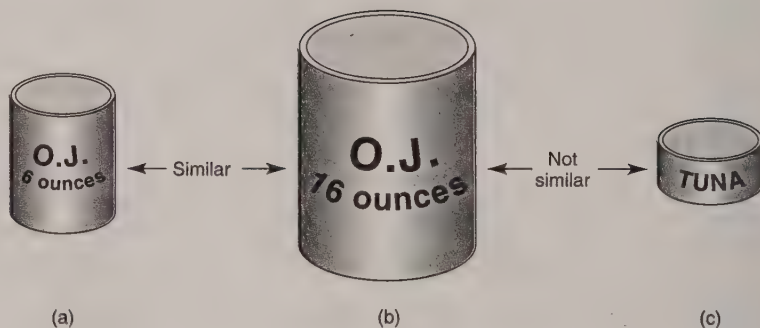


FIGURE 5.4

Generally our discussion of similarity will be limited to plane figures.

For two polygons to be similar, one requirement is that each angle of one polygon must be congruent to the corresponding angle of the other. Although this congruence of angles is necessary for similarity of polygons, it alone is not sufficient to establish similarity. The vertices of the congruent angles are **corresponding vertices** of the similar polygons. If $\angle A$ in one polygon is congruent to $\angle M$ in the second polygon, then vertex A corresponds to vertex M , and this is symbolized $A \leftrightarrow M$; we can indicate that $\angle A$ corresponds to $\angle M$ by writing $\angle A \leftrightarrow \angle M$. A pair of angles like $\angle A$ and $\angle M$ are **corresponding angles**, and the sides determined by consecutive and corresponding vertices are **corresponding sides** of the similar polygons.

EXAMPLE 1

Given that quadrilaterals $ABCD$ and $HJKL$ are similar, with congruent angles indicated in Figure 5.5, name the vertices, angles, and sides that correspond to each other.

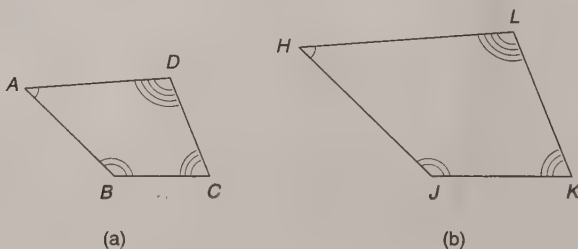


FIGURE 5.5

Solution

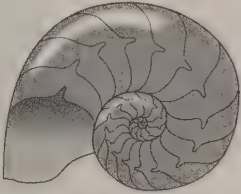
Because $\angle A \cong \angle H$, it follows that

$$A \leftrightarrow H \quad \text{and} \quad \angle A \leftrightarrow \angle H$$

Similarly,

$$\begin{array}{lll} B \leftrightarrow J & \text{and} & \angle B \leftrightarrow \angle J \\ C \leftrightarrow K & \text{and} & \angle C \leftrightarrow \angle K \\ D \leftrightarrow L & \text{and} & \angle D \leftrightarrow \angle L \end{array}$$

GEOMETRY IN NATURE



The segments of the chambered nautilus are similar (not congruent) in shape.

When pairs of consecutive and corresponding vertices are associated, the corresponding sides are included between the corresponding angles.

$$\overline{AB} \leftrightarrow \overline{HJ}, \quad \overline{BC} \leftrightarrow \overline{JK}, \quad \overline{CD} \leftrightarrow \overline{KL}, \quad \overline{AD} \leftrightarrow \overline{HL}$$

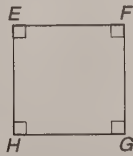
DEFINITION: Two polygons are **similar** if and only if two conditions are satisfied:

1. All pairs of corresponding angles are congruent.
2. All pairs of corresponding sides are proportional.

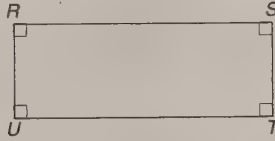
The second condition for similarity requires that the following extended proportion exists for the sides of the similar quadrilaterals of Example 1.

$$\frac{AB}{HJ} = \frac{BC}{JK} = \frac{CD}{KL} = \frac{AD}{HL}$$

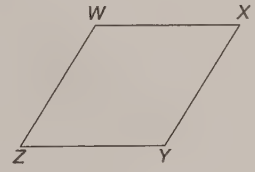
Note that *both* conditions for similarity are necessary! Although condition 1 is satisfied for square $EFGH$ and rectangle $RSTU$ [see Figures 5.6(a) and (b)], the figures are not similar—that is, one is not an enlargement of the other—because the extended proportion is not true. On the other hand, condition 2 is satisfied for square $EFGH$ and rhombus $WXYZ$ [see Figures 5.6(a) and (c)], but the figures are not similar because the angles are not congruent.



(a)



(b)



(c)

FIGURE 5.6

EXAMPLE 2

Which figures must be similar?

- | | |
|---------------------------------|------------------------|
| a) Any two isosceles triangles? | c) Any two rectangles? |
| b) Any two regular pentagons? | d) Any two squares? |

Solution

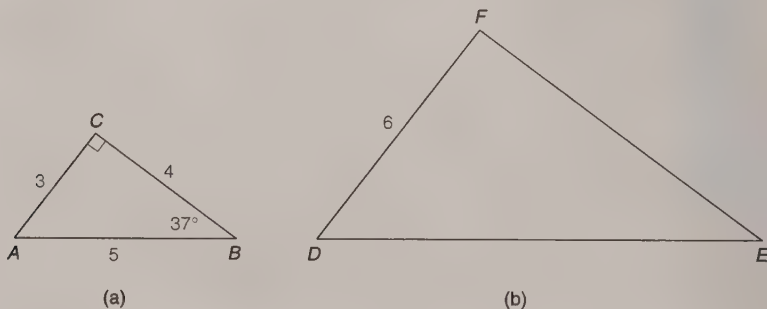
- No; $\angle$ pairs need not be $\cong$, nor do the pairs of sides need to be proportional.
- Yes; all angles are congruent (measure 108° each), and all pairs of sides are proportional.
- No; all angles measure 90° , but the pairs of sides are not necessarily proportional.
- Yes; all angles measure 90° , and all pairs of sides are proportional.

The practice of naming corresponding vertices in consecutive order for the two polygons is most convenient! For instance, if pentagon $ABCDE$ is similar to pentagon $MNPQR$, then we know that $A \leftrightarrow M$, $B \leftrightarrow N$, $C \leftrightarrow P$, $D \leftrightarrow Q$, $E \leftrightarrow R$, $\angle A \cong \angle M$, $\angle B \cong \angle N$, $\angle C \cong \angle P$, $\angle D \cong \angle Q$, and $\angle E \cong \angle R$. Because of the indicated correspondence of vertices, we also know that

$$\frac{AB}{MN} = \frac{BC}{NP} = \frac{CD}{PQ} = \frac{DE}{QR} = \frac{EA}{RM}$$

EXAMPLE 3

If $\triangle ABC \sim \triangle DEF$ in Figure 5.7, use the indicated measures to find the measures of the remaining parts of each of the triangles.

**FIGURE 5.7****Solution**

Because the sum of the measures of the angles of a triangle is 180° ,

$$m\angle A = 180 - (90 + 37) = 53^\circ$$

And because of the similarity and the corresponding vertices,

$$m\angle D = 53^\circ, \quad m\angle E = 37^\circ, \quad m\angle F = 90^\circ$$

The proportion that relates the lengths of the sides is

$$\frac{AC}{DF} = \frac{CB}{FE} = \frac{AB}{DE} \quad \text{so} \quad \frac{3}{6} = \frac{4}{FE} = \frac{5}{DE}$$

From $\frac{3}{6} = \frac{4}{FE}$, we see that

$$\begin{aligned} 3 \cdot FE &= 6 \cdot 4 = 24 \\ FE &= 8 \end{aligned}$$

From $\frac{3}{6} = \frac{5}{DE}$, we see that

$$\begin{aligned} 3 \cdot DE &= 6 \cdot 5 = 30 \\ DE &= 10 \end{aligned}$$

In a proportion, the ratios can *all* be inverted; thus Example 3 could have been solved by using the ratio

$$\frac{DF}{AC} = \frac{FE}{CB} = \frac{DE}{AB}$$

In an extended proportion, the equal ratios must all be equal to the same constant value. By designating this number (which is often called the “constant of proportionality”) by k , we see that

$$\frac{DF}{AC} = k \quad \frac{FE}{CB} = k \quad \frac{DE}{AB} = k$$

It follows that $DF = k \cdot AC$, $FE = k \cdot CB$, and $DE = k \cdot AB$. In Example 3, this constant of proportionality had the value $k = 2$, which means that each side of the larger triangle was twice the length of each side of the smaller triangle.

Our definition of similar polygons (and therefore of similar triangles) is almost impossible to use as a method of proof. Fortunately, some easier methods are available for proving triangles similar. If two triangles are carefully sketched or constructed so that their angles are congruent, they will appear to be similar, as shown in Figure 5.8.

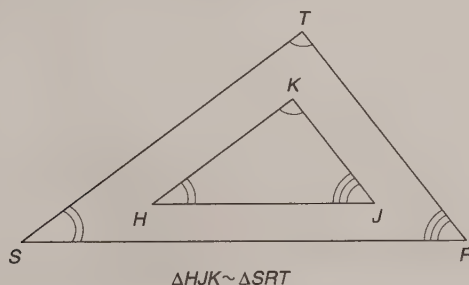


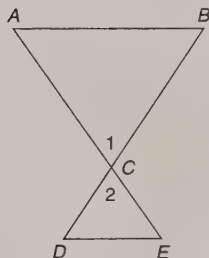
FIGURE 5.8

POSTULATE 15:

If the three angles of one triangle are congruent to the three angles of a second triangle, then the triangles are similar (AAA).

Corollary 5.2.1 follows from knowing that if two angles of one triangle are congruent to two angles of another triangle, then the third angles *must* also be congruent. See Corollary 2.4.4.

COROLLARY 5.2.1: If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar (AA).

EXAMPLE 4**FIGURE 5.9**

Provide a two-column proof of the following problem.

Given: $\overline{AB} \parallel \overline{DE}$ in Figure 5.9.

Prove: $\triangle ABC \sim \triangle EDC$

P R O O F

Statements	Reasons
1. $\overline{AB} \parallel \overline{DE}$	1. Given
2. $\angle A \cong \angle E$	2. If two $\parallel$ lines are cut by a transversal, the alternate interior angles are $\cong$
3. $\angle 1 \cong \angle 2$	3. Vertical angles are $\cong$
4. $\triangle ABC \sim \triangle EDC$	4. AA

In many instances, we wish to prove some relationship beyond the similarity of triangles. This is possible through the definition of similarity. The following part of the definition of similarity is often cited as a reason in a proof. This fact, abbreviated CSSTP, is used in Example 5. Although the statement involves triangles, we realize that the corresponding sides of any two similar polygons are proportional. That is, the ratio of any pair of corresponding sides equals the ratio of another pair of corresponding sides.

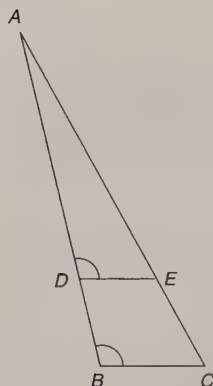
CSSTP: Corresponding sides of similar triangles are proportional.

EXAMPLE 5

Complete the following two-column proof.

Given: $\angle ADE \cong \angle B$ in Figure 5.10

Prove: $\frac{DE}{BC} = \frac{AE}{AC}$

**FIGURE 5.10****P R O O F**

Statements	Reasons
1. $\angle ADE \cong \angle B$	1. Given
2. $\angle A \cong \angle A$	2. Identity
3. $\triangle ADE \sim \triangle ABC$	3. AA
4. $\frac{DE}{BC} = \frac{AE}{AC}$	4. CSSTP

NOTE: In this proof, DE appears above BC because the sides with these names lie opposite $\angle A$ in the two similar triangles. AE and AC are the lengths of the sides opposite the congruent and corresponding angles $\angle ADE$ and $\angle B$. That is, corresponding sides of similar triangles always lie opposite corresponding angles.

THEOREM 5.2.2: The lengths of the corresponding altitudes of similar triangles have the same ratio as the lengths of any pair of corresponding sides.

The proof of this theorem is left to the student; see Exercise 25. Note that this proof also requires the use of CSSTP.

EXAMPLE 6

Solution

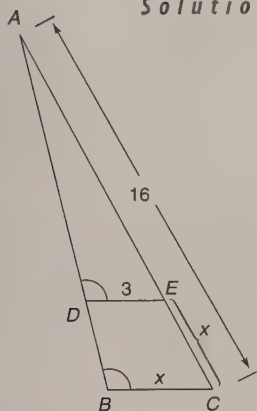


FIGURE 5.11

$\angle ADE \cong \angle B$ in Figure 5.11. If $DE = 3$, $AC = 16$, and $EC = BC$, find the length BC .

From the similar triangles, we proved (in Example 5) that $\frac{DE}{BC} = \frac{AE}{AC}$. With $AC = AE + EC$ and representing the lengths of the congruent segments (EC and BC) by x , we have

$$16 = AE + x \quad \text{so} \quad AE = 16 - x$$

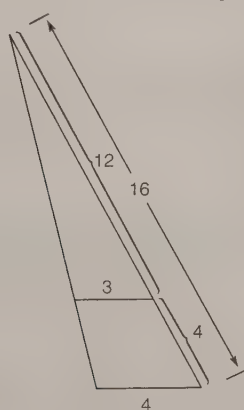
Substituting into the proportion, we have

$$\frac{3}{x} = \frac{16 - x}{16}$$

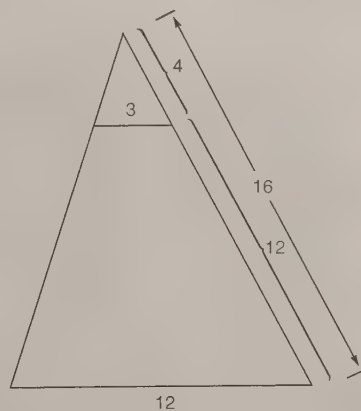
It follows that

$$\begin{aligned} x(16 - x) &= 3 \cdot 16 \\ 16x - x^2 &= 48 \\ x^2 - 16x + 48 &= 0 \\ (x - 4)(x - 12) &= 0 \end{aligned}$$

Now x (or BC) equals 4 or 12. Each length is acceptable, but the scaled drawings differ, as illustrated in Figure 5.12.



(a)



(b)

FIGURE 5.12

In Example 7, you are asked to prove that the product of lengths of two segments equals the product of lengths of two other segments. Here is a plan for establishing such a relationship:

1. Use AA to show that two triangles are similar, like $\triangle ABC \sim \triangle DEF$.
2. Use CSSTP to form a proportion by choosing the ratios from

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

3. Use the Means-Extremes Property to obtain equal products, such as

$$AB \cdot EF = DE \cdot BC$$

when $\frac{AB}{DE} = \frac{BC}{EF}$ is the proportion chosen in step 2.

EXAMPLE 7

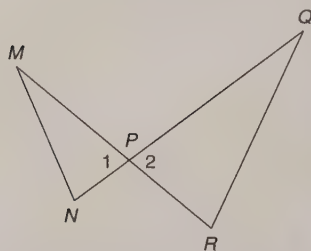


FIGURE 5.13

Use a paragraph proof for this problem.

Given: $\angle M \cong \angle Q$ in Figure 5.13.

Prove: $NP \cdot QR = RP \cdot MN$

Proof: By hypothesis, $\angle M \cong \angle Q$. Also, $\angle 1 \cong \angle 2$ by the fact that vertical angles are congruent. Now $\triangle MPN \sim \triangle QPR$ by AA. Using CSSTP, $\frac{NP}{RP} = \frac{MN}{QR}$. Then $NP \cdot QR = RP \cdot MN$ by the Means-Extremes Property.

NOTE: In the proof, the sides selected for the proportion were carefully chosen. The statement to be proved suggested that we include NP , QR , RP , and MN in the proportion.

In addition to AA, there are other methods that can be used to establish similar triangles. To distinguish the following techniques for showing triangles similar from methods for proving triangles congruent, we use $SAS\sim$ and $SSS\sim$ to identify these theorems. They are stated without proof.

THEOREM 5.2.3 ($SAS\sim$): If an angle of one triangle is congruent to an angle of a second triangle and the pairs of sides including the angles are proportional, then the triangles are similar.

THEOREM 5.2.4 ($SSS\sim$): If the three sides of one triangle are proportional to the three corresponding sides of a second triangle, then the triangles are similar.

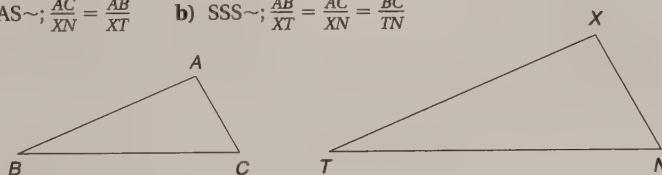
EXAMPLE 8

Which method (AA, SAS~, or SSS~) establishes that $\triangle ABC \sim \triangle XTN$? See Figure 5.14.

- a) $\angle A \cong \angle X$, $AC = 6$, $XN = 9$, $AB = 8$, and $XT = 12$
 b) $AB = 6$, $AC = 4$, $BC = 8$, $XT = 9$, $XN = 6$, and $TN = 12$

Solution

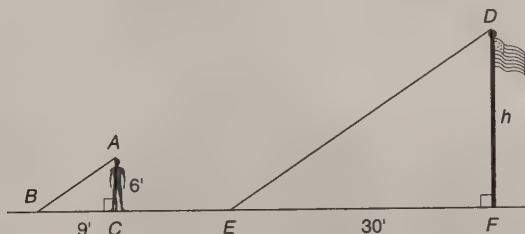
- a) SAS~; $\frac{AC}{XN} = \frac{AB}{XT}$ b) SSS~; $\frac{AB}{XT} = \frac{AC}{XN} = \frac{BC}{TN}$

**FIGURE 5.14**

The following example uses a method called *shadow reckoning*. Not a new technique, this method of calculating a length dates back more than 2500 years. It was used by Thales to estimate the height of the pyramids in Egypt. In an application, the method assumes (correctly) that angles such as $\angle A$ and $\angle D$ of Figure 5.15 are congruent. With $\angle C \cong \angle F$ as well, $\triangle ABC \sim \triangle DEF$ by AA.

EXAMPLE 9

Darnell is curious about the height of a flagpole that stands in front of his school. Darnell, who is 6 ft tall, casts a shadow that he paces off at 9 ft. He walks the length of the shadow of the flagpole, a distance of 30 ft. How tall is the flagpole?

**FIGURE 5.15****Solution**

In Figure 5.15, $\triangle ABC \sim \triangle DEF$. From similar triangles, we know that $\frac{AC}{BC} = \frac{DF}{EF}$ or $\frac{AC}{BC} = \frac{DF}{EF}$ by interchanging the means.

Where h is the height of the flagpole, substitution into the second proportion leads to

$$\frac{6}{9} = \frac{h}{30} \rightarrow 9h = 180 \rightarrow h = 20$$

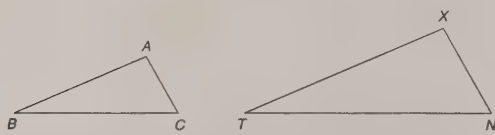
The height of the flagpole is 20 ft.



5.2 EXERCISES

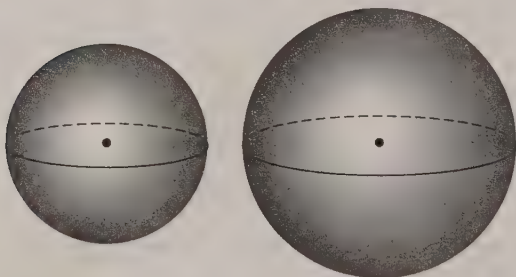
In Exercises 1 and 2, refer to the drawing.

1. a) Given that $A \leftrightarrow X$, $B \leftrightarrow T$, and $C \leftrightarrow N$, write a statement claiming that the triangles shown are similar.
- b) Given that $A \leftrightarrow N$, $C \leftrightarrow X$, and $B \leftrightarrow T$, write a statement claiming that the triangles shown are similar.

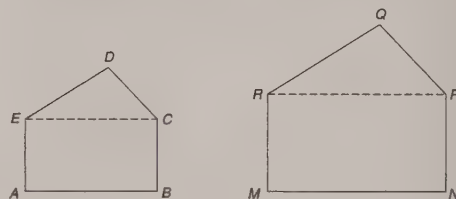


Exercises 1, 2

2. a) If $\triangle ABC \sim \triangle XTN$, which angle of $\triangle ABC$ corresponds to $\angle N$ of $\triangle XTN$?
- b) If $\triangle ABC \sim \triangle XTN$, which side of $\triangle XTN$ corresponds to side AC of $\triangle ABC$?
3. A **sphere** is the three-dimensional surface that contains all points in space lying at a fixed distance from a point known as the center of the sphere. Consider the two spheres shown. Are these two spheres similar? Are any two spheres similar? Explain.

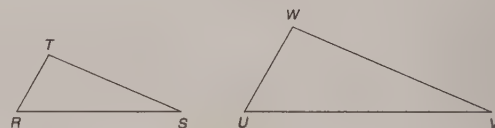


4. Given that rectangle $ABCE$ is similar to rectangle $MNPR$ and that $\triangle CDE \sim \triangle PQR$, what can you conclude regarding pentagon $ABCDE$ and pentagon $MNPQR$?



In Exercises 5 to 8, name the method (AA, SSS~, or SAS~) that is used to show that the triangles are similar.

5. $WU = \frac{3}{2} \cdot TR$, $WV = \frac{3}{2} \cdot TS$, and $UV = \frac{3}{2} \cdot RS$



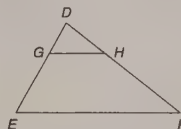
Exercises 5–8

6. $\angle T \cong \angle W$ and $\angle R \cong \angle U$
7. $\angle T \cong \angle W$ and $\frac{TR}{WU} = \frac{TS}{WV}$
8. $\frac{TR}{WU} = \frac{TS}{WV} = \frac{RS}{UV}$

In Exercises 9 and 10, name the method that explains why $\triangle DGH \sim \triangle DEF$.

9. $\frac{DG}{DE} = \frac{DH}{DF}$

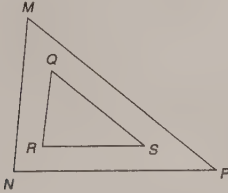
10. $DE = 3 \cdot DG$ and $DF = 3 \cdot DH$



Exercises 9, 10

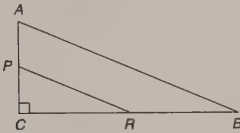
11. Given: $\triangle MNP \sim \triangle QRS$, $m\angle M = 56^\circ$, $m\angle R = 82^\circ$,
 $MN = 9$, $QR = 6$, $RS = 7$, $MP = 12$

Find: a) $m\angle N$ c) NP
b) $m\angle P$ d) QS



12. Given: $\triangle ABC \sim \triangle PRC$, $m\angle A = 67^\circ$, $PC = 5$, $CR = 12$,
 $PR = 13$, $AB = 26$

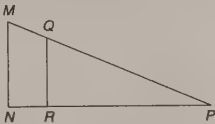
Find: a) $m\angle B$ c) AC
b) $m\angle RPC$ d) CB



13. a) Does the similarity relationship have a **reflexive** property for triangles (and polygons in general)?
b) Is there a **symmetric** property for the similarity of triangles (and polygons)?
c) Is there a **transitive** property for the similarity of triangles (and polygons)?
14. Using the names of properties from Exercise 13, identify the property illustrated by each statement:
a) If $\triangle 1 \sim \triangle 2$, then $\triangle 2 \sim \triangle 1$
b) If $\triangle 1 \sim \triangle 2$, $\triangle 2 \sim \triangle 3$, and $\triangle 3 \sim \triangle 4$, then $\triangle 1 \sim \triangle 4$
c) $\triangle 1 \sim \triangle 1$

In Exercises 15 to 18, complete each proof.

15. Given: $\overline{MN} \perp \overline{NP}$, $\overline{QR} \perp \overline{RP}$
Prove: $\triangle MNP \sim \triangle QRP$



Exercises 15, 16

PROOF

Statements	Reasons
1. ?	1. Given
2. $\angle s$ N and QRP are right $\angle s$	2. ?
3. ?	3. All right $\angle s$ are $\cong$
4. $\angle P \cong \angle P$	4. ?
5. ?	5. ?

16. Given: $\overline{MN} \parallel \overline{QR}$ (See figure for Exercise 15.)
Prove: $\triangle MNP \sim \triangle QRP$

PROOF

Statements	Reasons
1. ?	1. Given
2. $\angle M \cong \angle RQP$	2. ?
3. ?	3. If two $\parallel$ lines are cut by a transversal, the corresponding $\angle s$ are $\cong$
4. ?	4. ?

17. Given: $\angle H \cong \angle F$
Prove: $\triangle HJK \sim \triangle FGK$



Exercises 17, 18

PROOF

Statements	Reasons
1. ?	1. Given
2. $\angle HKJ \cong \angle FKG$	2. ?
3. ?	3. ?

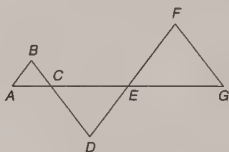
18. Given: $\overline{HJ} \perp \overline{JF}$, $\overline{HG} \perp \overline{FG}$ (See figure for Exercise 17.)
 Prove: $\triangle HJK \sim \triangle FGK$

PROOF

Statements	Reasons
1. ?	1. Given
2. $\angle G$ and $\angle J$ are right $\angle$ s	2. ?
3. $\angle G \cong \angle J$	3. ?
4. $\angle HKJ \cong \angle GKF$	4. ?
5. ?	5. ?

In Exercises 19 and 20, provide two-column proofs.

19. Given: $\overline{AB} \parallel \overline{DF}$, $\overline{BD} \parallel \overline{FG}$
 Prove: $\triangle ABC \sim \triangle EFG$

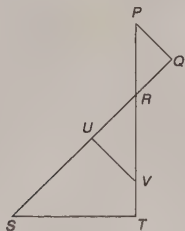


Exercises 19, 20

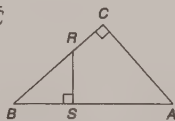
20. Given: $\overline{AB} \cong \overline{BC}$, $\overline{CD} \cong \overline{DE}$, and $\overline{EF} \cong \overline{FG}$
 Prove: $\triangle ABC \sim \triangle EFG$

In Exercises 21 to 24, provide paragraph proofs.

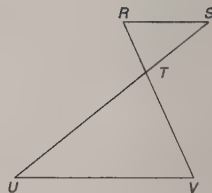
21. Given: $\angle RVU \cong \angle S$, $\angle RUV \cong \angle Q$
 Prove: $\triangle PQR \sim \triangle STR$



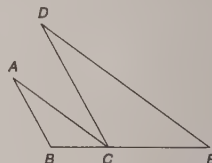
22. Given: $\overline{RS} \perp \overline{AB}$, $\overline{CB} \perp \overline{AC}$
 Prove: $\triangle BSR \sim \triangle BCA$



23. Given: $\overline{RS} \parallel \overline{UV}$
 Prove: $\frac{RT}{VT} = \frac{RS}{VU}$



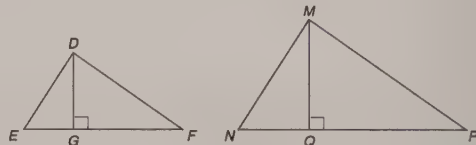
24. Given: $\overline{AB} \parallel \overline{DC}$, $\overline{AC} \parallel \overline{DE}$
 Prove: $\frac{AB}{DC} = \frac{BC}{CE}$



25. Use a two-column proof to prove the following theorem:
 "The lengths of the corresponding altitudes of similar triangles have the same ratio as the lengths of any pair of corresponding sides."

Given: $\triangle DEF \sim \triangle MNP$; $\overline{DG}$ and $\overline{MQ}$ are altitudes

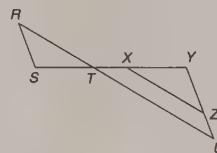
Prove: $\frac{DG}{MQ} = \frac{DE}{MN}$



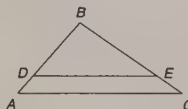
26. Provide a paragraph proof for the following problem.

Given: $\overline{RS} \parallel \overline{YZ}$, $\overline{RU} \parallel \overline{XZ}$

Prove: $RS \cdot ZX = ZY \cdot RT$



In Exercises 27 to 30, $\triangle ABC \sim \triangle DBE$.



Exercises 27-30

27. Given: $AC = 8$, $DE = 6$, $CB = 6$
 Find: EB
 (HINT: Let $EB = x$, and solve an equation.)

28. Given: $AC = 10$, $CB = 12$
 E the midpoint of $\overline{CB}$

Find: DE

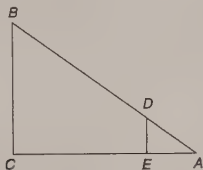
29. Given: $AC = 10$, $DE = 8$, $AD = 4$

Find: DB

30. Given: $CB = 12$, $CE = 4$, $AD = 5$

Find: DB

In Exercises 31 to 34, $\triangle ADE \sim \triangle ABC$.



Exercises 31–34

31. Given: $DE = 4$, $AE = 6$, $EC = BC$

Find: BC

32. Given: $DE = 5$, $AD = 8$, $DB = BC$

Find: AB

(HINT: Find DB first.)

33. Given: $DE = 4$, $AC = 20$, $EC = BC$

Find: BC

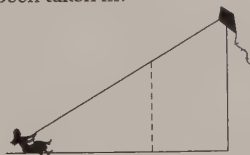
34. Given: $AD = 4$, $AC = 18$, $DB = AE$

Find: AE

35. A person who is walking away from a 10-ft lamppost casts a shadow 6 ft long. If the person is at a distance of 10 ft from the lamppost at that moment, what is the person's height?

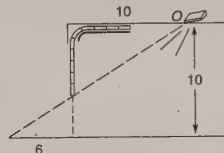


36. With 100 ft of string out, a kite is 64 ft above ground level. When the girl flying the kite pulls in 40 ft of string, the angle formed by the string and the ground does not change. What is the height of the kite above the ground after the 40 ft of string have been taken in?

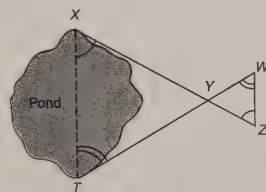


37. While admiring a rather tall tree, Fred notes that the shadow of his 6-ft frame has a length of 3 paces. On the level ground, he walks off the complete shadow of the tree in 37 paces. How tall is the tree?

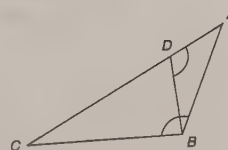
38. As a garage door closes, light is cast 6 ft beyond the base of the door (as shown in the accompanying drawing) by a light fixture that is set in the garage ceiling 10 ft back from the door. If the ceiling of the garage is 10 ft above the floor, how far is the garage door above the floor at the time that light is cast 6 ft beyond the door?



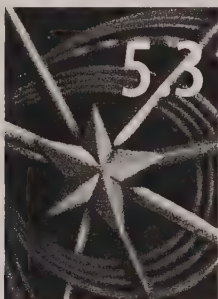
39. The distance across a pond is to be measured indirectly by using similar triangles. If $XY = 160$ ft, $YW = 40$ ft, $TY = 120$ ft, and $WZ = 50$ ft, find XT .



40. In the figure,
 $\angle ABC \cong \angle ADB$. Find
 AB if $AD = 2$ and $DC = 6$.



41. Prove that the altitude drawn to the hypotenuse of a right triangle separates the right triangle into two right triangles that are similar to each other and to the original right triangle.
42. Prove that the line segment joining the midpoints of two sides of a triangle determine a triangle that is similar to the original triangle.



THE PYTHAGOREAN THEOREM

The following theorem, which was proved in Exercise 41 of Section 5.2, will enable us to prove the well-known Pythagorean Theorem.

THEOREM 5.3.1: The altitude drawn to the hypotenuse of a right triangle separates the right triangle into two right triangles that are similar to each other and to the original right triangle.

Theorem 5.3.1 is illustrated by Figure 5.16, in which the right triangle $\triangle ABC$ has its right angle at vertex C so that $\overline{CD}$ is the altitude to hypotenuse $\overline{AB}$. The smaller triangles are shown in Figures 5.16(b) and (c), and the original triangle is shown in Figure 5.16(d). Note the matched arcs indicating congruent angles.

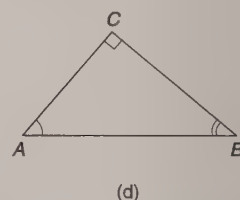
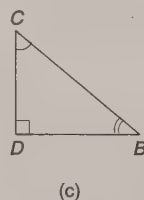
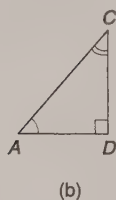
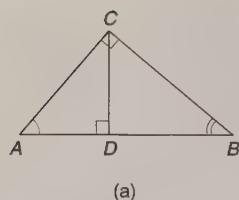


FIGURE 5.16

REMINDER

CSSTP means "corresponding sides of similar triangles are proportional."

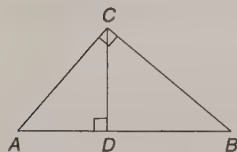


FIGURE 5.17

Two less obvious claims are now stated as theorems, and proofs of these are provided. In Figure 5.16(a), $\overline{AD}$ and $\overline{DB}$ are known as segments (parts) of the hypotenuse $\overline{AB}$. Furthermore, $\overline{AD}$ is the segment of the hypotenuse *adjacent* to leg $\overline{AC}$, and $\overline{BD}$ is the segment of the hypotenuse *adjacent* to leg $\overline{BC}$.

THEOREM 5.3.2: The length of the altitude to the hypotenuse of a right triangle is the geometric mean of the lengths of the segments of the hypotenuse.

Given: $\triangle ABC$ in Figure 5.17, with right $\angle ACB$, $\overline{CD} \perp \overline{AB}$

Prove: $\frac{AD}{CD} = \frac{CD}{DB}$

Proof: With right $\triangle ABC$ and $\overline{CD} \perp \overline{AB}$, we can use the fact that the altitude drawn to the hypotenuse of a right triangle separates it into two right triangles that are similar to each other. Because $\triangle ADC \sim \triangle CDB$, it follows by CSSTP that

$$\frac{AD}{CD} = \frac{CD}{DB} \quad [\text{See Figures 5.16(b) and (c).}]$$

In general, the phrase *segments of the hypotenuse* refers to the parts of the hypotenuse determined by the altitude drawn from the vertex of a right triangle to the hypotenuse. See Theorem 5.3.3 and its proof, in which $\overline{AD}$ and $\overline{DB}$ are the segments of hypotenuse $\overline{AB}$ determined by altitude $\overline{CD}$ of right triangle ABC .

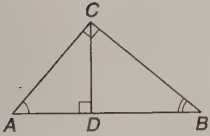


FIGURE 5.18

THEOREM 5.3.3: The length of each leg of a right triangle is the geometric mean of the length of the hypotenuse and the length of the segment of the hypotenuse adjacent to that leg.

Given: $\triangle ABC$ with right $\angle ACB$; $\overline{CD} \perp \overline{AB}$. (See Figure 5.18.)

Prove: $\frac{AB}{AC} = \frac{AC}{AD}$

Proof: With right $\triangle ABC$ and $\overline{CD} \perp \overline{AB}$, we know that $\triangle ADC \sim \triangle ACB$ because the altitude drawn to the hypotenuse of a right triangle forms two right triangles that are similar to the original right triangle. (See Figure 5.19.) In turn, $\frac{AB}{AC} = \frac{AC}{AD}$. (Similarly, it can be shown that $\frac{AB}{CB} = \frac{CB}{DB}$.)

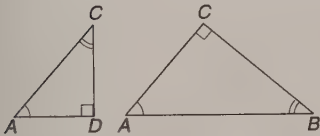


FIGURE 5.19

NOTE: Although $\overline{AD}$ and $\overline{DB}$ are both segments of the hypotenuse, $\overline{AD}$ is the segment adjacent to (next to) $\overline{AC}$.

The preceding theorem opens the doors to a proof of the famous Pythagorean Theorem, one of the most often applied relationships in geometry. Although the theorem's title gives credit to the Greek geometer Pythagoras, many other proofs are known, and the ancient Chinese were aware of the relationship before the time of Pythagoras.

THEOREM 5.3.4: (Pythagorean Theorem) The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs.

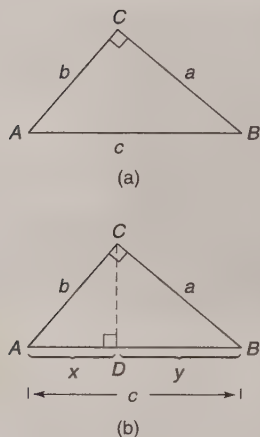


FIGURE 5.20

EXAMPLE 1

Thus, where c is the length of the hypotenuse and a and b are the lengths of the legs, $c^2 = a^2 + b^2$.

Given: In Figure 5.20(a), $\triangle ABC$ with right $\angle C$

Prove: $c^2 = a^2 + b^2$

Proof: Draw $\overline{CD} \perp \overline{AB}$, as shown in Figure 5.20(b).

Denote $AD = x$ and $DB = y$. By Theorem 5.3.3,

$$\frac{c}{b} = \frac{b}{x} \quad \text{and} \quad \frac{c}{a} = \frac{a}{y}$$

Therefore, $b^2 = cx$ and $a^2 = cy$

Using the Addition Property of Equality, we have

$$a^2 + b^2 = cy + cx = c(y + x)$$

But $y + x = x + y = AD + DB = AB = c$. Thus

$$a^2 + b^2 = c(c) = c^2$$

Given $\triangle RST$ with right $\angle S$ in Figure 5.21, find:

- RT if $RS = 3$ and $ST = 4$
- RT if $RS = 4$ and $ST = 6$
- RS if $RT = 13$ and $ST = 12$
- ST if $RS = 6$ and $RT = 9$

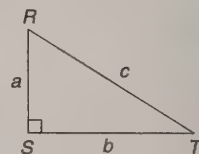


FIGURE 5.21

Solution

With right $\angle S$, the hypotenuse is $\overline{RT}$. Then $RT = c$, $RS = a$, and $ST = b$.

a) $3^2 + 4^2 = c^2 \rightarrow 9 + 16 = c^2$

$$c^2 = 25$$

$$c = 5; RT = 5$$

b) $4^2 + 6^2 = c^2 \rightarrow 16 + 36 = c^2$

$$c^2 = 52$$

$$c = \sqrt{52} = \sqrt{4 \cdot 13} = \sqrt{4} \cdot \sqrt{13} = 2\sqrt{13}$$

$$RT = 2\sqrt{13} \approx 7.21$$

c) $a^2 + 12^2 = 13^2 \rightarrow a^2 + 144 = 169$

$$a^2 = 25$$

$$a = 5; RS = 5$$

d) $6^2 + b^2 = 9^2 \rightarrow 36 + b^2 = 81$

$$b^2 = 45$$

$$b = \sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$$

$$ST = 3\sqrt{5} \approx 6.71$$

DISCOVER!

A video titled "The Rule of Pythagoras" is available through Project Mathematics at Cal Tech University in Pasadena, CA. It is well worth watching!

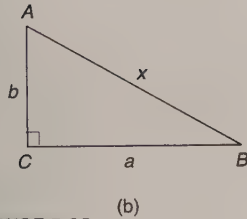
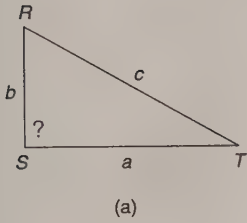


FIGURE 5.22

EXAMPLE 2

The converse of the Pythagorean Theorem is also true.

THEOREM 5.3.5: (Converse of Pythagorean Theorem) If a , b , and c are the lengths of the three sides of a triangle, with c the length of the longest side, and if $c^2 = a^2 + b^2$, then the triangle is a right triangle with the right angle opposite the side of length c .

Given: $\triangle RST$ [Figure 5.22(a)] with sides a , b , and c so that $c^2 = a^2 + b^2$

Prove: $\triangle RST$ is a right triangle

Proof: We are given $\triangle RST$ for which $c^2 = a^2 + b^2$. Construct the right $\triangle ABC$, which has legs of lengths a and b and a hypotenuse of length x . [See Figure 5.22(b).] By the Pythagorean Theorem, $x^2 = a^2 + b^2$. By substitution, $x^2 = c^2$ and $x = c$. Thus $\triangle RTS \cong \triangle ABC$ by SSS. Then $\angle S$, opposite the side of length c , must be $\cong$ to $\angle C$, the right $\angle$ of $\triangle ABC$. Then $\angle S$ is a right $\angle$, and $\triangle RST$ is a right triangle.

Which of the following can be the lengths of the sides of a right triangle?

- a) $a = 5$, $b = 12$, $c = 13$
- b) $a = 15$, $b = 8$, $c = 17$
- c) $a = 7$, $b = 9$, $c = 10$
- d) $a = \sqrt{2}$, $b = \sqrt{3}$, $c = \sqrt{5}$

Solution

- a) Because $5^2 + 12^2 = 13^2$ (that is, $25 + 144 = 169$), this triangle is a right triangle.
- b) Because $15^2 + 8^2 = 17^2$ (that is, $225 + 64 = 289$), this triangle is a right triangle.
- c) $7^2 + 9^2 = 49 + 81 = 130$, which is not 10^2 (that is, 100), so this triangle is not a right triangle.
- d) Because $(\sqrt{2})^2 + (\sqrt{3})^2 = (\sqrt{5})^2$ (that is, $2 + 3 = 5$), this triangle is a right triangle.

EXAMPLE 3

A ladder 12 ft long is leaning against a wall so that its base is 4 ft from the wall at ground level (see Figure 5.23). How far up the wall does the ladder reach?

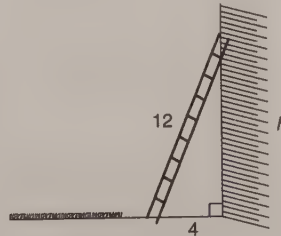


FIGURE 5.23

Solution**REMINDER**

The diagonals of a rhombus are perpendicular bisectors of each other.

The desired height is represented by h , so we have

$$4^2 + h^2 = 12^2$$

$$16 + h^2 = 144$$

$$h^2 = 128$$

$$h = \sqrt{128} = \sqrt{64 \cdot 2} = \sqrt{64} \cdot \sqrt{2} = 8\sqrt{2}$$

The height is represented exactly by $h = 8\sqrt{2}$, which is approximately 11.31 ft.

EXAMPLE 4**Solution**

One diagonal of a rhombus has the same length, 10 cm, as each side (see Figure 5.24). How long is the other diagonal?

Because the diagonals are perpendicular bisectors of each other, four right $\triangle$ s are formed. For each right $\triangle$, a side of the rhombus is the hypotenuse. Half of each diagonal is a leg of each right triangle. Therefore,

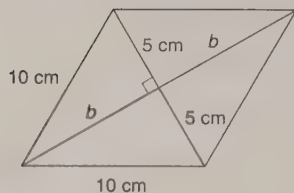
$$5^2 + b^2 = 10^2$$

$$25 + b^2 = 100$$

$$b^2 = 75$$

$$b = \sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$$

Thus the length of the whole diagonal is $10\sqrt{3}$ cm ≈ 17.32 cm.

**FIGURE 5.24**

Example 5 also uses the Pythagorean Theorem, but it is considerably more complicated than Example 4. Indeed, it is one of those situations that may require some insight to solve. Note that the triangle described in Example 5 is *not* a right triangle, because $4^2 + 5^2 \neq 6^2$.

EXAMPLE 5**Solution**

A triangle has sides of lengths 4, 5, and 6, as shown in Figure 5.25. Find the length of the altitude to the side of length 6.

The altitude to the side of length 6 separates that side into two parts whose lengths are given by x and $6 - x$. Using the two right triangles formed, we have, by the Pythagorean Theorem,

$$x^2 + h^2 = 4^2 \quad \text{and} \quad (6 - x)^2 + h^2 = 5^2$$

Subtracting the first equation from the second, we can calculate x .

$$36 - 12x + x^2 + h^2 = 25$$

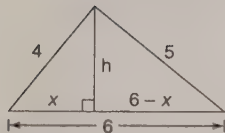
$$x^2 + h^2 = 16$$

$$36 - 12x = 9$$

$$-12x = -27$$

(subtraction)

$$x = \frac{27}{12} = \frac{9}{4}$$

**FIGURE 5.25**

Now we use $x = \frac{9}{4}$ to find h :

$$x^2 + h^2 = 4^2$$

$$\left(\frac{9}{4}\right)^2 + h^2 = 4^2$$

$$\frac{81}{16} + h^2 = 16$$

$$\frac{81}{16} + h^2 = \frac{256}{16}$$

$$h^2 = \frac{175}{16}$$

$$h = \frac{\sqrt{175}}{4} = \frac{\sqrt{25 \cdot 7}}{4} = \frac{\sqrt{25} \cdot \sqrt{7}}{4} = \frac{5\sqrt{7}}{4} \approx 3.31$$

It is now possible to prove the HL method for the congruence of triangles.

THEOREM 5.3.6: If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of a second right triangle, then the triangles are congruent (HL).

Given: Right $\triangle ABC$ with right $\angle C$ and right $\triangle DEF$ with right $\angle F$ (see Figure 5.26); $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{EF}$

Prove: $\triangle ABC \cong \triangle EDF$

Proof: With right $\angle C$, the hypotenuse of $\triangle ABC$ is $\overline{AB}$; similarly, $\overline{DE}$ is the hypotenuse of right $\triangle EDF$. Because $\overline{AB} \cong \overline{DE}$, we denote the common length by c ; that is, $AB = DE = c$. Because $\overline{AC} \cong \overline{EF}$, we also have $AC = EF = a$. Then

$$a^2 + (BC)^2 = c^2 \quad \text{and} \quad a^2 + (DF)^2 = c^2$$

which leads to

$$BC = \sqrt{c^2 - a^2} \quad \text{and} \quad DF = \sqrt{c^2 - a^2}$$

Then $BC = DF$ so that $\overline{BC} \cong \overline{DF}$. Hence, $\triangle ABC \cong \triangle EDF$ by SSS.

Our work with the Pythagorean Theorem would be incomplete if we did not address two concerns. The first, Pythagorean triples, involves natural (or counting) numbers as possible choices of a , b , and c . The second leads to a classification of triangles according to the lengths of their sides; see Theorem 5.3.7.

DEFINITION: A **Pythagorean triple** is a set of three natural numbers (a, b, c) for which $a^2 + b^2 = c^2$.

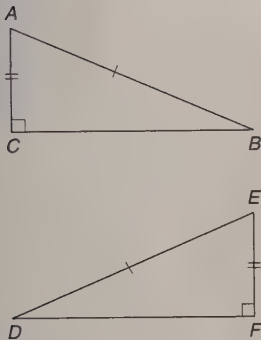


FIGURE 5.26

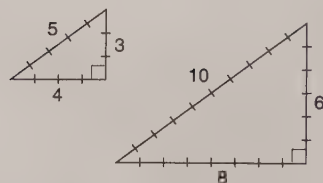


FIGURE 5.27

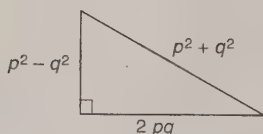


FIGURE 5.28

Three sets of Pythagorean triples encountered in this section are (3, 4, 5), (5, 12, 13), and (8, 15, 17). These numbers will always fit the sides of a right triangle.

Natural-number multiples of any of these triples will also constitute Pythagorean triples. For example, doubling (3, 4, 5) yields (6, 8, 10), which is also a Pythagorean triple. In Figure 5.27, the triangles are similar by SSS.

The Pythagorean triple (3, 4, 5) also leads to (9, 12, 15), (12, 16, 20), and (15, 20, 25). The Pythagorean triple (5, 12, 13) leads to triples such as (10, 24, 26) and (15, 36, 39). Basic Pythagorean triples that are used less frequently include (7, 24, 25), (9, 40, 41), and (20, 21, 29).

Pythagorean triples can be generated by using any of several formulas. One formula uses $2pq$ for one leg, $p^2 - q^2$ for the other leg, and $p^2 + q^2$ for the hypotenuse, where p and q are natural numbers and $p > q$. (See Figure 5.28.)

The following table lists some Pythagorean triples corresponding to choices for p and q . The triples printed in boldface type are basic triples, also known as primitive triples. Even though no exercises in this section include the phrase *Pythagorean triple*, using these triples and their multiples will save you considerable time and effort. In the final column, the resulting triple is provided, in order, from a (small) to c (large).

p	q	a (or b) $p^2 - q^2$	b (or a) $2pq$	c $p^2 + q^2$	(a, b, c)
2	1	3	4	5	(3, 4, 5)
3	1	8	6	10	(6, 8, 10)
3	2	5	12	13	(5, 12, 13)
4	1	15	8	17	(8, 15, 17)
4	3	7	24	25	(7, 24, 25)
5	1	24	10	26	(10, 24, 26)
5	2	21	20	29	(20, 21, 29)
5	3	16	30	34	(16, 30, 34)
5	4	9	40	41	(9, 40, 41)

The Converse of the Pythagorean Theorem allows us to recognize a right triangle by knowing the lengths of its sides. A variation on the Converse allows us to recognize whether a triangle is acute or obtuse. This theorem is stated without proof.

THEOREM 5.3.7: Let a , b , and c represent the lengths of the three sides of a triangle, with c the length of the longest side.

1. If $c^2 > a^2 + b^2$, then the triangle is obtuse and the obtuse angle lies opposite the side of length c .
2. If $c^2 < a^2 + b^2$, then the triangle is acute.

EXAMPLE 6

Determine the type of triangle represented if the lengths of its sides are as follows:

- a) 4, 5, 7 b) 6, 7, 8 c) 9, 12, 15 d) 3, 4, 9

Solution

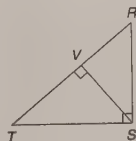
- a) Because c is the longest side, $c = 7$, and we have $7^2 > 4^2 + 5^2$, or $49 > 16 + 25$; the triangle is obtuse.
 b) Choosing $c = 8$, we have $8^2 < 6^2 + 7^2$, or $64 < 36 + 49$; the triangle is acute.
 c) Choosing $c = 15$, we have $15^2 = 9^2 + 12^2$, or $225 = 81 + 144$; the triangle is a right triangle.
 d) Because $9 > 3 + 4$, no triangle is possible. (Remember that the sum of the lengths of two sides of a triangle must be greater than the length of the third side.)

5.3 EXERCISES

1. By naming the vertices in order, state three different triangles that are similar to each other.

2. Write a proportion in which

- a) SV is used as a geometric mean.
 b) RS is a geometric mean.
 c) TS is a geometric mean.



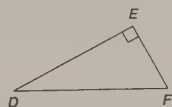
Exercises 1-4

3. Use Theorem 5.3.2 to find RV if $SV = 6$ and $VT = 8$.

4. Use Theorem 5.3.3 to find RT if $RS = 6$ and $VR = 4$.

5. Find the length of $\overline{DF}$ if:

- a) $DE = 8$ and $EF = 6$
 b) $DE = 5$ and $EF = 3$



Exercises 5-8

6. Find the length of $\overline{DE}$ if:

- a) $DF = 13$ and $EF = 5$
 b) $DF = 12$ and $EF = 6\sqrt{3}$

7. Find EF if:

- a) $DF = 17$ and $DE = 15$
 b) $DF = 12$ and $DE = 8\sqrt{2}$

8. Find DF if:

- a) $DE = 12$ and $EF = 5$
 b) $DE = 12$ and $EF = 6$

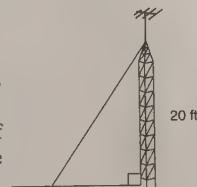
9. Determine the type of triangle represented if the lengths of its sides are:

- a) $a = 4$, $b = 3$, and $c = 5$
 b) $a = 4$, $b = 5$, and $c = 6$
 c) $a = 2$, $b = \sqrt{3}$, and $c = \sqrt{7}$
 d) $a = 3$, $b = 8$, and $c = 15$

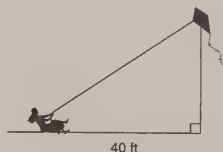
10. Determine the type of triangle represented if the lengths of its sides are:

- a) $a = 1.5$, $b = 2$, and $c = 2.5$
 b) $a = 20$, $b = 21$, and $c = 29$
 c) $a = 10$, $b = 12$, and $c = 16$
 d) $a = 5$, $b = 7$, and $c = 9$

11. A guy wire 25 ft long supports an antenna at a point that is 20 ft above the base of the antenna. How far from the base of the antenna is the guy wire secured?



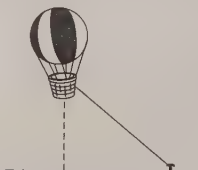
12. A strong wind holds a kite 30 ft above the earth in a position 40 ft across the ground. How much string does the girl have out (to the kite)?



13. A boat is 6 m below the level of a pier and 12 m from the pier as measured across the water. How much rope is needed to reach the boat?



14. A hot-air balloon is held in place by the ground crew at a point that is 21 ft from a point directly beneath the balloon. If the rope is of length 29 ft, how far above ground level is the balloon?



15. A rectangle has a width of 16 cm and a diagonal of length 20 cm. How long is the rectangle?
16. A right triangle has legs of lengths x and $2x + 2$ and a hypotenuse of length $2x + 3$. What are the lengths of its sides?
17. A rectangle has base $x + 3$, altitude $x + 1$, and diagonals $2x$ each. What are the lengths of its base, altitude, and diagonals?
18. The diagonals of a rhombus measure 6 m and 8 m. How long are each of the congruent sides?
19. Each side of a rhombus measures 12 in. If one diagonal is 18 in. long, how long is the other diagonal?
20. An isosceles right triangle has a hypotenuse of length 10 cm. How long is each leg?
21. Each leg of an isosceles right triangle has a length of $6\sqrt{2}$ in. What is the length of the hypotenuse?
22. In right $\triangle ABC$ with right $\angle C$, $AB = 10$ and $BC = 8$. Find the length of $\overline{MB}$ if M is the midpoint of $\overline{AC}$.

23. In right $\triangle ABC$ with right $\angle C$, $AB = 17$ and $BC = 15$. Find the length of $\overline{MN}$ if M and N are the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively.

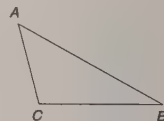
24. Find the length of the altitude to the 10-in. side of a triangle whose sides are 6, 8, and 10 in. in length.
25. Find the length of the altitude to the 26-in. side of a triangle whose sides are 10, 24, and 26 in. in length.
26. In quadrilateral $ABCD$, $\overline{BC} \perp \overline{AB}$ and $\overline{DC} \perp$ diagonal $\overline{AC}$. If $AB = 4$, $BC = 3$, and $DC = 12$, determine DA .



27. In quadrilateral $RSTU$, $\overline{RS} \perp \overline{ST}$ and $\overline{UT} \perp$ diagonal $\overline{RT}$. If $RS = 6$, $ST = 8$, and $RU = 15$, find UT .

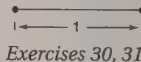


28. Given: $\triangle ABC$ is not a right $\triangle$
Prove: $a^2 + b^2 \neq c^2$

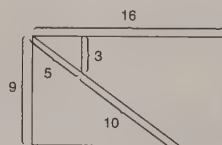


29. $\Rightarrow$ If $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$, show that $c^2 = a^2 + b^2$.

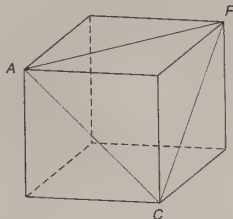
30. Given that the line segment shown has length 1, construct a line segment whose length is $\sqrt{2}$.



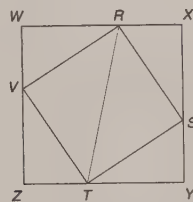
31. Using the same line segment as in Exercise 30, construct a segment of length 2 and then a second segment of length $\sqrt{5}$.
32. When the rectangle in the accompanying drawing (whose dimensions are 16 by 9) is cut into pieces and rearranged, a square can be formed. What is the perimeter of this square?



33. A , C , and F are three of the vertices of the cube shown in the accompanying figure. Given that each face of the cube is a square, what is the measure of angle ACF ?

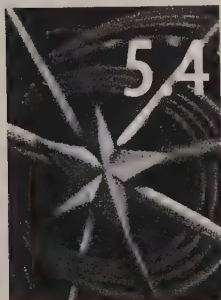


35. In the figure, square $RSTV$ has its vertices on the sides of square $WXYZ$ as shown. If $ZT = 5$ and $TY = 12$, find TS . Also find RT .



34. ➤ Find the length of the altitude to the 8-in. side of a triangle whose sides are 4, 6, and 8 in. long. (HINT: See Example 5.)

36. Prove that if (a, b, c) is a Pythagorean triple and n is a natural number, then (na, nb, nc) is also a Pythagorean triple.



SPECIAL RIGHT TRIANGLES

Many of the calculations that we do in this section involve square root radicals. To understand some of these calculations better, it may be necessary to review the Properties of Square Roots in Appendix A.4.

Certain right triangles occur so often that they deserve more attention than others. The special right triangles we will consider have angle measures of 45° , 45° , and 90° or of 30° , 60° , and 90° .

THE 45° - 45° - 90° RIGHT TRIANGLE

In the 45 - 45 - 90 triangle, the legs are opposite the congruent angles and are also congruent. Rather than using a and b to represent the lengths of the legs, we use a for both lengths, as shown in Figure 5.29. It then follows by the Pythagorean Theorem that

$$\begin{aligned} c^2 &= a^2 + a^2 \\ c^2 &= 2a^2 \\ c &= \sqrt{2a^2} \\ c &= \sqrt{2} \cdot \sqrt{a^2} \\ c &= a\sqrt{2} \end{aligned}$$

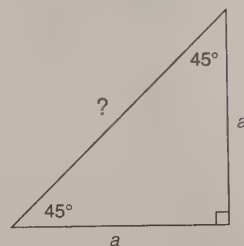


FIGURE 5.29

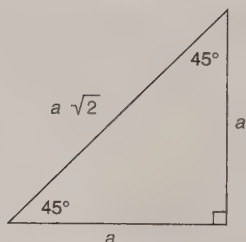


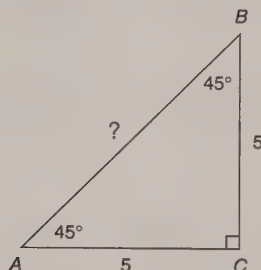
FIGURE 5.30

THEOREM 5.4.1: (45-45-90 Theorem) In a triangle whose angles measure 45° , 45° , and 90° , the hypotenuse has a length equal to the product of $\sqrt{2}$ and the length of either leg.

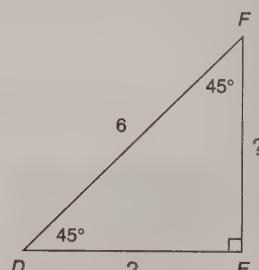
It is better to memorize the sketch in Figure 5.30 than to repeat the steps preceding the 45-45-90 Theorem.

EXAMPLE 1

Find the lengths of the missing sides in each triangle in Figure 5.31.



(a)



(b)

FIGURE 5.31**Solution**

- a) The length of hypotenuse $\overline{AB}$ is $5\sqrt{2}$, the product of $\sqrt{2}$ and the length of either of the equal legs.
 b) Let a denote the length of $\overline{DE}$ and of $\overline{EF}$. The length of hypotenuse $\overline{DF}$ is $a\sqrt{2}$. Then $a\sqrt{2} = 6$, so $a = \frac{6}{\sqrt{2}}$. Simplifying,

$$\begin{aligned} a &= \frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{6\sqrt{2}}{2} \\ &= 3\sqrt{2} \end{aligned}$$

Therefore, $DE = EF = 3\sqrt{2} \approx 4.24$.

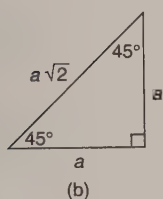
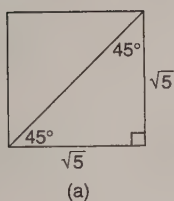
NOTE: If we use the Pythagorean Theorem to solve Example 1, the solution in part (a) can be found by solving the equation $5^2 + 5^2 = c^2$ and the solution in part (b) can be found by solving $a^2 + a^2 = 6^2$.

EXAMPLE 2

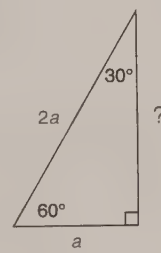
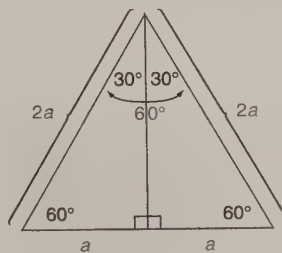
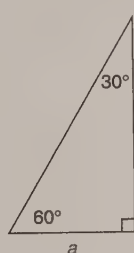
Each side of a square has a length of $\sqrt{5}$. Find the length of a diagonal.

Solution

The square shown in Figure 5.32(a) is separated into two 45° - 45° - 90° triangles. With each of the congruent legs represented by a in Figure 5.32(b), we see that $a = \sqrt{5}$ and the diagonal (hypotenuse length) is $a \cdot \sqrt{2} = \sqrt{5} \cdot \sqrt{2}$, so $a = \sqrt{10} \approx 3.16$.

**FIGURE 5.32****THE 30° - 60° - 90° RIGHT TRIANGLE**

The second special triangle is the 30° - 60° - 90° triangle, shown in Figure 5.33(a). Its sides are related to each other in a way that can be seen by reflecting (as a mirror might) the triangle across the side included by the 30° and 90° angles. As shown in Figure 5.33(b), the reflection produces an equiangular and therefore equilateral triangle in which the length of the side that was the hypotenuse of the 30° - 60° - 90° triangle is twice the length of the leg opposite the 30° angle. Extending the shorter leg while allowing for the 30° angle ensures that the two smaller triangles are congruent.

**FIGURE 5.33**

Take another look at the 30° - 60° - 90° triangle in Figure 5.33(c). Because we know that the length of the hypotenuse is twice the length of the shorter leg, we can again use the Pythagorean Theorem to represent the length of the longer leg. Where b is the length of the longer leg, we have

$$\begin{aligned} c^2 &= a^2 + b^2 \\ (2a)^2 &= a^2 + b^2 \\ 4a^2 &= a^2 + b^2 \\ 3a^2 &= b^2, \text{ so } b^2 = 3a^2 \end{aligned}$$

Thus

$$\begin{aligned} b &= \sqrt{3a^2} \\ b &= \sqrt{3} \cdot \sqrt{a^2} \\ b &= a\sqrt{3} \end{aligned}$$

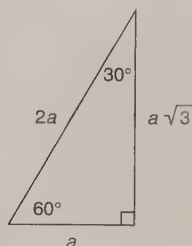


FIGURE 5.34

THEOREM 5.4.2: (30-60-90 Theorem) In a triangle whose angles measure 30° , 60° , and 90° , the hypotenuse has a length equal to twice the length of the shorter leg, and the length of the longer leg is the product of $\sqrt{3}$ and the length of the shorter leg.

It would be best to memorize the sketch in Figure 5.34. So that you will more easily recall which expression is used for each side, remember that the lengths of the sides follow the same order as the angles opposite them. Thus:

opposite the $30^\circ \angle$ (smallest angle) is a (shortest side)
 opposite the $60^\circ \angle$ (middle angle) is $a\sqrt{3}$ (middle side)
 opposite the $90^\circ \angle$ (largest angle) is $2a$ (longest side)

EXAMPLE 3

Find the lengths of the missing sides of each triangle in Figure 5.35.

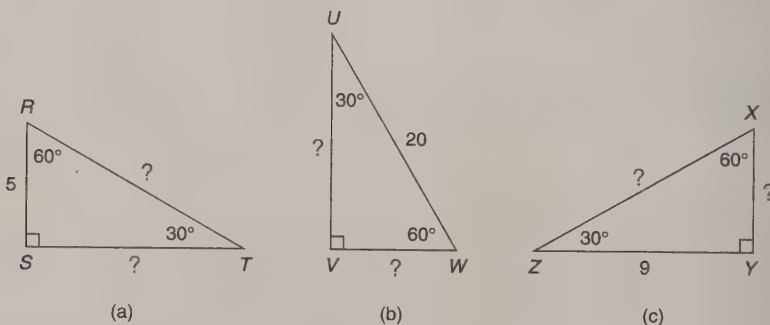


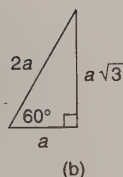
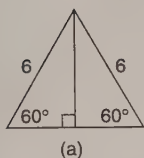
FIGURE 5.35

Solution

- a) $RT = 2 \cdot RS = 2 \cdot 5 = 10$
 $ST = RS\sqrt{3} = 5\sqrt{3} \approx 8.66$
- b) $UW = 2 \cdot VW \rightarrow 20 = 2 \cdot VW \rightarrow VW = 10$
 $UV = VW\sqrt{3} = 10\sqrt{3} \approx 17.32$
- c) $ZY = XY\sqrt{3} \rightarrow 9 = XY \cdot \sqrt{3} \rightarrow XY = \frac{9}{\sqrt{3}} = \frac{9}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$
 $= \frac{9\sqrt{3}}{3} = 3\sqrt{3} \approx 5.20$
 $XZ = 2 \cdot XY = 2 \cdot 3\sqrt{3} = 6\sqrt{3} \approx 10.39$

EXAMPLE 4

Each side of an equilateral triangle measures 6 in. Find the length of an altitude of the triangle.

Solution

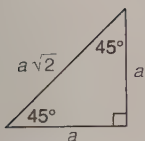
The equilateral triangle shown in Figure 5.36(a) is separated into two 30°-60°-90° triangles by the altitude. In the 30°-60°-90° triangle in Figure 5.36(b), the side of the equilateral triangle becomes the hypotenuse, so $2a = 6$ and $a = 3$. The altitude lies opposite the 60° angle of the 30°-60°-90° triangle, so its length is $a\sqrt{3}$ or $3\sqrt{3}$ in. ≈ 5.20 in.

The converse of Theorem 5.4.1 is true and is described in the following theorem.

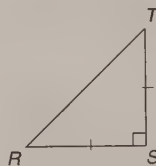
THEOREM 5.4.3: If the length of the hypotenuse of a right triangle equals the product of $\sqrt{2}$ and the length of either leg, then the angles of the triangle measure 45°, 45°, and 90°.

FIGURE 5.36*Proof*

In Figure 5.37, we represent the length of the hypotenuse by $h = a\sqrt{2}$, where a is the length of either leg. In a right triangle, the angles that lie opposite the congruent legs are also congruent. In a right triangle, the acute angles are complementary, so each of the congruent acute angles measures 45°.

EXAMPLE 5**FIGURE 5.37**

In right $\triangle RST$, $RS = ST$. (See Figure 5.38.) What are the measures of the angles of the triangle? If $RT = 12\sqrt{2}$, what is the length of RS (or ST)?

**FIGURE 5.38***Solution*

The longest side is the hypotenuse $\overline{RT}$, so the right angle is $\angle S$ and $m\angle S = 90^\circ$. Because $\overline{RS} \cong \overline{ST}$, the congruent acute angles are $\angle s R$ and T and $m\angle R = m\angle T = 45^\circ$. Because $RT = 12\sqrt{2}$, $RS = ST = 12$.

The converse of Theorem 5.4.2 is also true and can be proved by the indirect method. Rather than construct the proof, we state and apply this theorem. See Figure 5.39.

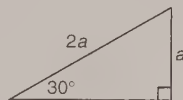


FIGURE 5.39

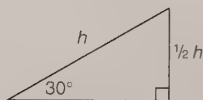


FIGURE 5.40

THEOREM 5.4.4: If the length of the hypotenuse of a right triangle is twice the length of one leg of the triangle, then the angle of the triangle opposite that leg measures 30° .

An equivalent form of this theorem is stated as follows:

If one leg of a right triangle has a length equal to one-half the length of the hypotenuse, then the angle of the triangle opposite that leg measures 30° (see Figure 5.40).

EXAMPLE 6

In right $\triangle ABC$ with right $\angle C$, $AB = 24.6$ and $BC = 12.3$ (see Figure 5.41). What are the measures of the angles of the triangle? Also, what is the length of AC ?

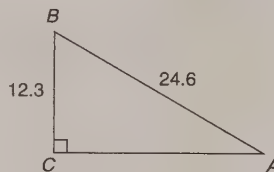


FIGURE 5.41

Solution

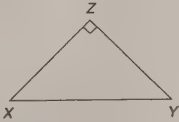
Because $\angle C$ is a right angle, $m\angle C = 90^\circ$ and $\overline{AB}$ is the hypotenuse. Because $BC = \frac{1}{2}(AB)$, the angle opposite $\overline{BC}$ measures 30° . Thus $m\angle A = 30^\circ$ and $m\angle B = 60^\circ$.

Because $\overline{AC}$ lies opposite the 60° angle, $AC = (12.3)\sqrt{3} \approx 21.3$.

5.4 EXERCISES

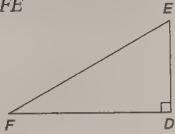
In Exercises 1 to 18, find the missing lengths. Give your answers in both simplest radical form and as approximations to two decimal places.

1. Given: Right $\triangle XYZ$ with $m\angle X = 45^\circ$ and $XZ = 8$
Find: YZ and XY



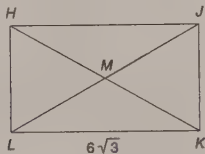
Exercises 1–4

2. Given: Right $\triangle XYZ$ with $\overline{XZ} \cong \overline{YZ}$ and $XY = 10$
Find: XZ and YZ
3. Given: Right $\triangle XYZ$ with $\overline{XZ} \cong \overline{YZ}$ and $XY = 10\sqrt{2}$
Find: XZ and YZ
4. Given: Right $\triangle XYZ$ with $m\angle X = 45^\circ$ and $XY = 12\sqrt{2}$
Find: XZ and YZ
5. Given: Right $\triangle DEF$ with $m\angle E = 60^\circ$ and $DE = 5$
Find: DF and FE

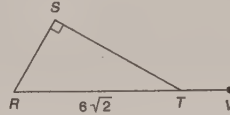


Exercises 5–8

6. Given: Right $\triangle DEF$ with $m\angle F = 30^\circ$ and $FE = 12$
Find: DF and DE
7. Given: Right $\triangle DEF$ with $m\angle E = 60^\circ$ and $FD = 12\sqrt{3}$
Find: DE and FE
8. Given: Right $\triangle DEF$ with $m\angle E = 2 \cdot m\angle F$ and $EF = 12\sqrt{3}$
Find: DE and DF
9. Given: Rectangle $HJKL$ with diagonals $\overline{HK}$ and $\overline{JL}$
 $m\angle HKL = 30^\circ$
Find: HL , HK , and MK

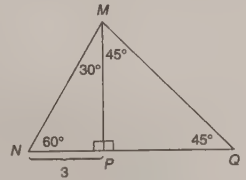


10. Given: Right $\triangle RST$ with $RT = 6\sqrt{2}$ and $m\angle STV = 150^\circ$
Find: RS and ST

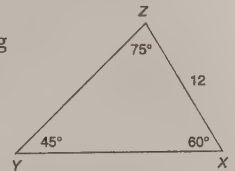


In Exercises 11–15, create drawings as needed.

11. Given: $\triangle ABC$ with $m\angle A = m\angle B = 45^\circ$ and $BC = 6$
Find: AC and AB
12. Given: Right $\triangle MNP$ with $MP = PN$ and $MN = 10\sqrt{2}$
Find: PM and PN
13. Given: $\triangle RST$ with $m\angle T = 30^\circ$, $m\angle S = 60^\circ$, and $ST = 12$
Find: RS and RT
14. Given: $\triangle XYZ$ with $\overline{XY} \cong \overline{XZ} \cong \overline{YZ}$
 $\overline{ZW} \perp \overline{XY}$
 $YZ = 6$
Find: ZW
15. Given: Square $ABCD$ with diagonals $\overline{DB}$ and $\overline{AC}$ intersecting at E
 $DC = 5\sqrt{3}$
Find: DB
16. Given: $\triangle NQM$ with angles as shown in the drawing
 $\overline{MP} \perp \overline{NQ}$
Find: NM , MP , MQ , PQ , and NQ

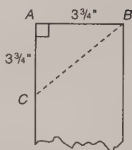


17. Given: $\triangle XYZ$ with angles as shown in the drawing
Find: XY
(HINT: Compare this drawing to the one for Exercise 16.)

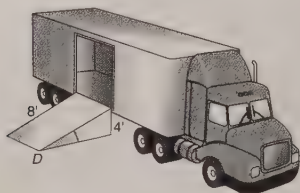


18. Given: Rhombus $ABCD$ in which diagonals $\overline{AC}$ and $\overline{DB}$ intersect at E
 $DB = AB = 8$
Find: AC

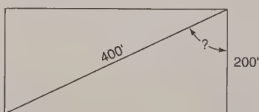
19. A carpenter is working with a board that is $3\frac{3}{4}$ in. wide. After marking off a point down the side of length $3\frac{3}{4}$ in., the carpenter makes a cut along $\overline{BC}$ with a saw. What is the measure of the angle ($\angle ACB$) that is formed?



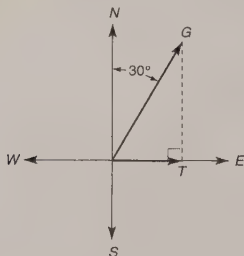
20. To unload groceries from a delivery truck at the Piggly Wiggly Market, an 8-ft ramp that rises 4 ft to the door of the trailer is used. What is the measure of the indicated angle ($\angle D$)?



21. A jogger runs along two sides of an open rectangular lot. If the first side of the lot is 200 ft long and the diagonal distance across the lot is 400 ft, what is the measure of the angle formed by the 200-ft and 400-ft dimensions? To the nearest foot, how much farther does the jogger run by traveling the two sides of the block rather than the diagonal distance across the lot?



22. Thelma's boat leaves the dock at the same time that Gina's boat leaves the dock. Thelma's boat travels due east at 12 mph. Gina's boat travels at 24 mph in the direction N 30° E. To the nearest tenth of a mile, how far apart will the boats be in half an hour?



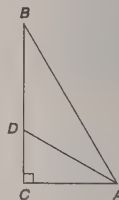
In Exercises 23–29, give both exact solutions and approximate solutions to two decimal places.

23. Given: In $\triangle ABC$, $\overrightarrow{AD}$ bisects $\angle BAC$
 $m\angle B = 30^\circ$ and $AB = 12$

Find: DC and DB

24. Given: In $\triangle ABC$, $\overrightarrow{AD}$ bisects $\angle BAC$
 $AB = 20$ and $AC = 10$

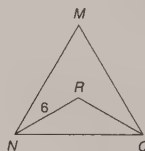
Find: DC and DB



Exercises 23, 24

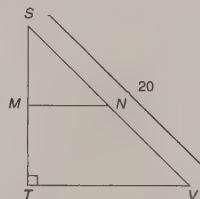
25. Given: $\triangle MNQ$ is equilateral and $NR = 6$
 $\overrightarrow{NR}$ bisects $\angle MNQ$
 $\overrightarrow{QR}$ bisects $\angle MQN$

Find: NQ



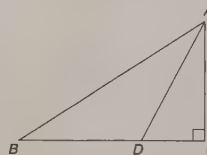
26. Given: $\triangle STV$ is an isosceles right triangle
 M and N are midpoints of $\overline{ST}$ and $\overline{SV}$

Find: MN



27. Given: Right $\triangle ABC$ with $m\angle C = 90^\circ$ and $m\angle BAC = 60^\circ$; point D on $\overline{BC}$; $\overrightarrow{AD}$ bisects $\angle BAC$ and $AB = 12$

Find: BD



Exercises 27, 28

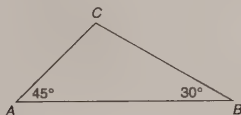
28. Given: Right $\triangle ABC$ with $m\angle C = 90^\circ$ and $m\angle BAC = 60^\circ$; point D on $\overline{BC}$; $\overrightarrow{AD}$ bisects $\angle BAC$ and $AC = 2\sqrt{3}$

Find: BD

29. **Given:** $\triangle ABC$ with $m\angle A = 45^\circ$, $m\angle B = 30^\circ$, and $BC = 12$

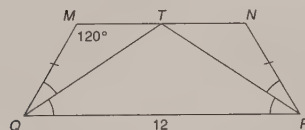
Find: AB

(**HINT:** Use altitude $\overline{CD}$ from C to $\overline{AB}$ as an auxiliary line.)



30. **Given:** Isosceles trapezoid $MNPQ$ with $QP = 12$ and $m\angle M = 120^\circ$; the bisectors of $\angle$ s MQP and NPQ meet at point T on $\overline{MN}$

Find: The perimeter of $MNPQ$



SEGMENTS DIVIDED PROPORTIONALLY

In this section, we begin with an informal description of the phrase *divided proportionally*. Suppose that three children have been provided with a joint savings account by their parents. Equal monthly deposits have been made to the account for each child since birth. If the ages of the children are 2, 4, and 6 (assume exactness of ages for simplicity) and the total in the account is \$7200, then the amount that each child should receive can be found by solving the equation

$$2x + 4x + 6x = 7200$$

Solving this equation leads to the solution \$1200 for the 2-year-old, \$2400 for the 4-year-old, and \$3600 for the 6-year-old. We say that the amount has been divided proportionally. Expressed as a proportion, this is

$$\frac{1200}{2} = \frac{2400}{4} = \frac{3600}{6}$$

In Figure 5.42, $\overline{AC}$ and $\overline{DF}$ are divided proportionally at points B and E if

$$\frac{AB}{DE} = \frac{BC}{EF} \left(\text{or } \frac{AB}{BC} = \frac{DE}{EF} \right)$$

Of course, a pair of segments may be divided proportionally by several points, as shown in Figure 5.43. In this case, $\overline{RW}$ and $\overline{HM}$ are divided proportionally when

$$\frac{RS}{HJ} = \frac{ST}{JK} = \frac{TV}{KL} = \frac{VW}{LM}$$

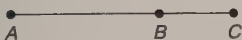


FIGURE 5.42

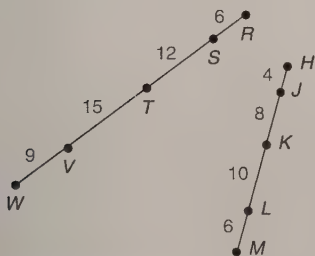


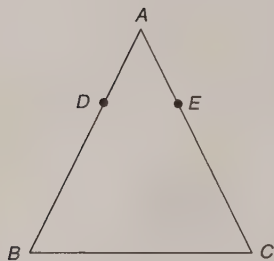
FIGURE 5.43

EXAMPLE 1

In Figure 5.44, points D and E divide $\overline{AB}$ and $\overline{AC}$ proportionally. If $AD = 4$, $DB = 7$, and $EC = 6$, find AE .

Solution

$\frac{AD}{AE} = \frac{DB}{EC}$, so $\frac{4}{x} = \frac{7}{6}$, where $x = AE$. Then $7x = 24$, so $x = AE = \frac{24}{7} = 3\frac{3}{7}$.

**FIGURE 5.44**

A property that will be proved in Exercise 25 of this section is

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a+c}{b+d} = \frac{a}{b} = \frac{c}{d}$$

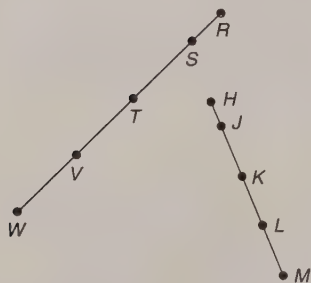
In words, we may restate this property as follows:

The fraction whose numerator and denominator are determined, respectively, by adding numerators and denominators of equal fractions is equal to each of those equal fractions.

Here is a numerical example of this claim:

$$\text{If } \frac{2}{3} = \frac{4}{6}, \text{ then } \frac{2+4}{3+6} = \frac{2}{3} = \frac{4}{6}$$

In Example 2, the preceding property is necessary as a reason.

EXAMPLE 2**FIGURE 5.45**

Given: $\overline{RW}$ and $\overline{HM}$ are divided proportionally at the points shown in Figure 5.45.

Prove: $\frac{RT}{HK} = \frac{TW}{KM}$

Proof: $\overline{RW}$ and $\overline{HM}$ are divided proportionally so that

$$\frac{RS}{HJ} = \frac{ST}{JK} = \frac{TV}{KL} = \frac{VW}{LM}$$

Using the property that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+c}{b+d} = \frac{a}{b} = \frac{c}{d}$, we have

$$\frac{RS}{HJ} = \frac{RS+ST}{HJ+JK} = \frac{TV+VW}{KL+LM} = \frac{TW}{KM}$$

Because $RS + ST = RT$, $HJ + JK = HK$, $TV + VW = TW$, and $KL + LM = KM$,

$$\frac{RT}{HK} = \frac{TW}{KM}$$

Two properties that were introduced earlier (Property 3 of Section 5.1) are now recalled.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a \pm b}{b} = \frac{c \pm d}{d}$$

The subtraction part of the property is needed for the proof of Theorem 5.5.1.

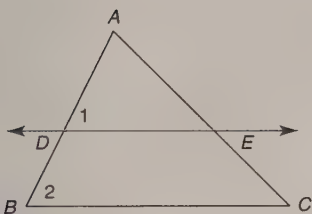


FIGURE 5.46

THEOREM 5.5.1: If a line is parallel to one side of a triangle and intersects the other two sides, then it divides these sides proportionally.

Given: In Figure 5.46, $\triangle ABC$ with $\overleftrightarrow{DE} \parallel \overline{BC}$ and with $\overleftrightarrow{DE}$ intersecting $\overline{AB}$ at D and $\overline{AC}$ at E

Prove: $\frac{AD}{DB} = \frac{AE}{EC}$

Proof: Because $\overleftrightarrow{DE} \parallel \overline{BC}$, $\angle 1 \cong \angle 2$. With $\angle A$ as a common angle for $\triangle ADE$ and $\triangle ABC$, it follows by AA that these triangles are similar. Now

$$\frac{AB}{AD} = \frac{AC}{AE} \quad \text{by CSSTP}$$

By Property 3 of Section 5.1,

$$\frac{AB - AD}{AD} = \frac{AC - AE}{AE}$$

Because $AB - AD = DB$ and $AC - AE = EC$, the proportion becomes

$$\frac{DB}{AD} = \frac{EC}{AE}$$

Inverting both fractions gives the desired conclusion:

$$\frac{AD}{DB} = \frac{AE}{EC}$$

COROLLARY 5.5.2: When three (or more) parallel lines are cut by a pair of transversals, the transversals are divided proportionally by the parallel lines.

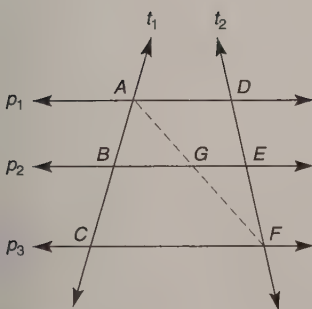


FIGURE 5.47

Corollary 5.5.2 is now illustrated, and its proof is briefly described. This corollary claims, for example, that $\frac{AB}{BC} = \frac{DE}{EF}$ as shown in Figure 5.47, where $p_1 \parallel p_2 \parallel p_3$. The proof requires the use of an auxiliary segment $\overline{AF}$. Then

$$\frac{AB}{BC} = \frac{AG}{GF} \quad \text{and} \quad \frac{AG}{GF} = \frac{DE}{EF} \quad \text{by Theorem 5.5.1}$$

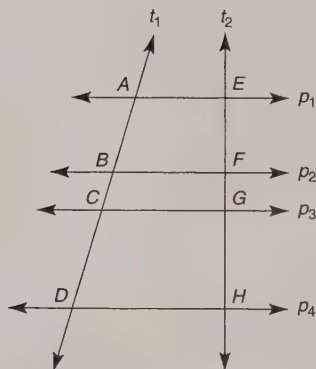
Consequently,

$$\frac{AB}{BC} = \frac{DE}{EF} \quad \text{Transitive Property of Equality}$$

NOTE: By interchanging the means, we can write this proportion in the form $\frac{AB}{DE} = \frac{BC}{EF}$.

EXAMPLE 3

Given parallel lines $p_1, p_2, p_3,$ and p_4 cut by t_1 and t_2 so that $AB = 4$, $EF = 3$, $BC = 2$, and $GH = 5$, find FG and CD . (See Figure 5.48.)

Solution**FIGURE 5.48**

Because the transversals are divided proportionally,

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH}$$

so

$$\frac{4}{3} = \frac{2}{FG} = \frac{CD}{5}$$

Then

$$4 \cdot FG = 6$$

$$\text{and } 3 \cdot CD = 20$$

$$FG = \frac{3}{2} = 1\frac{1}{2}$$

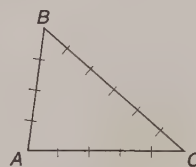
and

$$CD = \frac{20}{3} = 6\frac{2}{3}$$

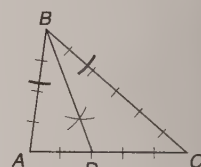
The following activity leads us to the relationship described in Theorem 5.5.3.

DISCOVER!

On a piece of paper, draw or construct $\triangle ABC$ whose sides measure $AB = 4$, $BC = 6$, and $AC = 5$. Then construct the angle bisector $\overrightarrow{BD}$ of $\angle B$. How does $\frac{AB}{AD}$ compare to $\frac{BC}{DC}$?



(a)

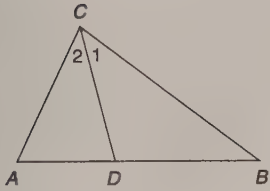


(b)

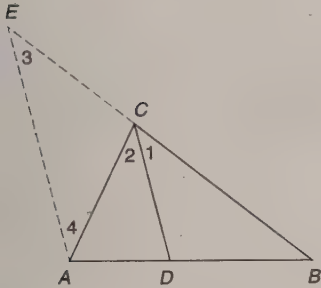
ANSWER
Though not by chance, it may come as a surprise that $\frac{AB}{AD} = \frac{BC}{DC}$ (that is, $\frac{4}{2} = \frac{6}{3}$) and $\frac{AB}{AC} = \frac{BC}{AC}$ (that is, $\frac{4}{5} = \frac{6}{5}$). It seems that the bisector of an angle included by two sides of a triangle separates the third side into segments whose lengths are proportional to the lengths of the two sides forming the angle.

The proof of Theorem 5.5.3 requires the use of Theorem 5.5.1.

THEOREM 5.5.3: If a ray bisects one angle of a triangle, then it divides the opposite side into segments whose lengths are proportional to the lengths of the two sides that form the bisected angle.



(a)



(b)

Given: $\triangle ABC$ in Figure 5.49(a), in which $\overrightarrow{CD}$ bisects $\angle ACB$

Prove: $\frac{AD}{AC} = \frac{DB}{CB}$

Proof: We begin by extending $\overline{BC}$ beyond C (there is only one line through B and C) to meet the line through A drawn parallel to $\overline{DC}$. [See Figure 5.49(b).] Let E be the point of intersection (these lines must intersect; otherwise $\overline{AE}$ would have two parallels, $\overline{BC}$ and $\overline{CD}$, through point C).

Because $\overline{CD} \parallel \overline{EA}$, we have

$$\frac{EC}{AD} = \frac{CB}{DB}$$

by Theorem 5.5.1. Now $\angle 1 \cong \angle 2$ (because of the angle bisector), $\angle 1 \cong \angle 3$ (corresponding angles for parallel lines), and $\angle 2 \cong \angle 4$ (alternate interior angles for parallel lines). By the Transitive Property, $\angle 3 \cong \angle 4$, so $\triangle ACE$ is isosceles with $EC \cong AC$. Using substitution, the proportion becomes

$$\frac{AC}{AD} = \frac{CB}{DB} \quad \text{or} \quad \frac{AD}{AC} = \frac{DB}{CB} \quad \text{by inversion}$$

The “Prove” of the preceding theorem indicates that one form of the proportionality described is given informally by

$$\frac{\text{segment at left}}{\text{side at left}} = \frac{\text{segment at right}}{\text{side at right}}$$

Equivalently, the proportion could state

$$\frac{\text{segment at left}}{\text{segment at right}} = \frac{\text{side at left}}{\text{side at right}}$$

Other forms of the proportion are also possible!

EXAMPLE 4

For $\triangle XYZ$ in Figure 5.50, $XY = 3$ and $YZ = 5$. If $\overrightarrow{YW}$ bisects $\angle XYZ$ and $XW = 2$, find XZ .

Solution

Let $WZ = x$. We know that $\frac{YX}{XW} = \frac{YZ}{WZ}$, so $\frac{3}{2} = \frac{5}{x}$.

Therefore,

$$\begin{aligned} 3x &= 10 \\ x &= \frac{10}{3} = 3\frac{1}{3} \end{aligned}$$

Then $WZ = 3\frac{1}{3}$.

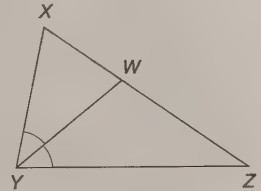


FIGURE 5.50

Because $XZ = XW + WZ$, we have $XZ = 2 + 3\frac{1}{3} = 5\frac{1}{3}$.

EXAMPLE 5

In Figure 5.50, $\triangle XYZ$ has sides of lengths $XY = 3$, $YZ = 4$, and $XZ = 5$. If $\overrightarrow{YW}$ bisects $\angle XYZ$, find XW and WZ .

Solution

Let $XW = y$; then $WZ = 5 - y$, and $\frac{XY}{YZ} = \frac{XW}{WZ}$ becomes $\frac{3}{4} = \frac{y}{5 - y}$. From this proportion, we can find y .

$$\begin{aligned} 3(5 - y) &= 4y \\ 15 - 3y &= 4y \\ 15 &= 7y \\ y &= \frac{15}{7} \end{aligned}$$

$$\text{Then } XW = \frac{15}{7} = 2\frac{1}{7} \text{ and } WZ = 5 - 2\frac{1}{7} = 2\frac{6}{7}.$$

**5.5 EXERCISES**

- In preparing a certain recipe, a chef uses 5 oz of ingredient A, 4 oz of ingredient B, and 6 oz of ingredient C. If 90 oz of this dish are needed, how many ounces of each ingredient should be used?
- In a chemical mixture, 2 g of chemical A are used for each gram of chemical B, and 3 g of chemical C are needed for each gram of B. If 72 g of the mixture are prepared, what amount (in grams) of each chemical is needed?

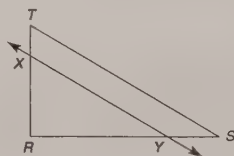
- Given that $\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH}$, do the following proportions hold?

- $\frac{AC}{EG} = \frac{CD}{GH}$
- $\frac{AB}{EF} = \frac{BD}{FH}$

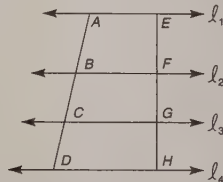


- Given that $\overrightarrow{XY} \parallel \overrightarrow{TS}$, do the following proportions hold?

- $\frac{TX}{XR} = \frac{RY}{YS}$
- $\frac{TR}{XR} = \frac{SR}{YR}$



- Given: $\ell_1 \parallel \ell_2 \parallel \ell_3 \parallel \ell_4$, $AB = 5$, $BC = 4$, $CD = 3$, $EH = 10$

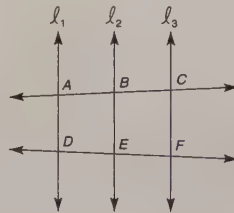
Find: EF , FG , and GH 

Exercises 5, 6

- Given: $\ell_1 \parallel \ell_2 \parallel \ell_3 \parallel \ell_4$, $AB = 7$, $BC = 5$, $CD = 4$, $EF = 6$

Find: FG , GH , EH

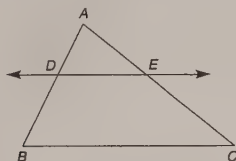
- Given: $\ell_1 \parallel \ell_2 \parallel \ell_3$, $AB = 4$, $BC = 5$, $DE = x$, $EF = 12 - x$

Find: x , DE , EF 

Exercises 7, 8

8. Given: $\ell_1 \parallel \ell_2 \parallel \ell_3$, $AB = 5$, $BC = x$, $DE = x - 2$, $EF = 7$
Find: x , BC , DE (See figure for Exercise 7.)

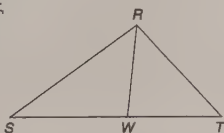
9. Given: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$, $AD = 5$, $DB = 12$, $AE = 7$
Find: EC



Exercises 9–12

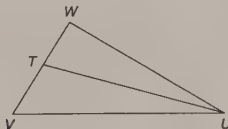
10. Given: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$, $AD = 6$, $DB = 10$, $AC = 20$
Find: EC
11. Given: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$, $AD = a - 1$, $DB = 2a + 2$, $AE = a$,
 $EC = 4a - 5$
Find: a and AD
12. Given: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$, $AD = 5$, $DB = a + 3$, $AE = a + 1$,
 $EC = 3(a - 1)$
Find: a and EC
13. Given: $\overleftrightarrow{RW}$ bisects $\angle SRT$
Do the following equalities hold?

- a) $SW = WT$
b) $\frac{RS}{RT} = \frac{SW}{WT}$



Exercises 13, 14

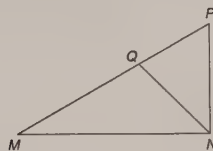
14. Given: $\overleftrightarrow{RW}$ bisects $\angle SRT$
Do the following equalities hold?
- a) $\frac{RS}{SW} = \frac{RT}{WT}$
b) $m\angle S = m\angle T$
15. Given: $\overleftrightarrow{UT}$ bisects $\angle WUV$, $WU = 8$, $UV = 12$, $WT = 6$
Find: TV



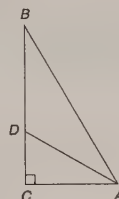
Exercises 15, 16

16. Given: $\overleftrightarrow{UT}$ bisects $\angle WUV$, $WU = 9$, $UV = 12$, $WV = 9$
Find: WT

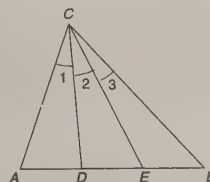
17. Given: $\overleftrightarrow{NQ}$ bisects $\angle MNP$, $NP = MQ$, $QP = 8$, $MN = 12$
Find: NP



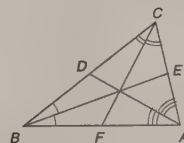
18. Given: In $\triangle ABC$, $\overleftrightarrow{AD}$ bisects $\angle BAC$
 $AB = 20$ and $AC = 16$
Find: DC and DB



19. In $\triangle ABC$, $\angle ACB$ is trisected by $\overleftrightarrow{CD}$ and $\overleftrightarrow{CE}$ so that $\angle 1 \cong \angle 2 \cong \angle 3$. Write two different proportions that follow from this information.



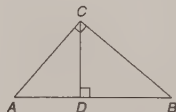
20. In $\triangle ABC$, $m\angle CAB = 80^\circ$, $m\angle ACB = 60^\circ$, and $m\angle ABC = 40^\circ$. With the angle bisectors as shown, which line segment is longer?
- a) $\overline{AE}$ or $\overline{EC}$? b) $\overline{CD}$ or $\overline{DB}$? c) $\overline{AF}$ or $\overline{FB}$?



21. In right $\triangle RST$ (not shown) with right $\angle S$, $\overleftrightarrow{RV}$ bisects $\angle SRT$ so that V lies on side $\overline{ST}$. If $RS = 6$, $ST = 6\sqrt{3}$, and $RT = 12$, find SV and VT .

22. *Given:* AC is the geometric mean between AD and AB .
 $AD = 4$, $DB = 6$

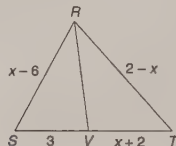
Find: AC



23. *Given:* $\overrightarrow{RV}$ bisects $\angle SRT$, $RS = x - 6$, $SV = 3$,
 $RT = 2 - x$, and $VT = x + 2$.

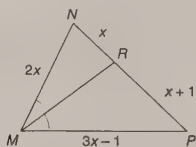
Find: x

(**HINT:** You will need to apply the Quadratic Formula.)



24. *Given:* $\overrightarrow{MR}$ bisects $\angle NMP$, $MN = 2x$, $NR = x$,
 $RP = x + 1$, and $MP = 3x - 1$.

Find: x



25. Complete the proof of this property:

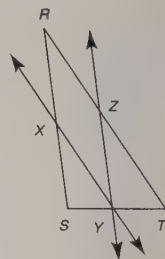
$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a+c}{b+d} = \frac{a}{b} \text{ and } \frac{a+c}{b+d} = \frac{c}{d}$$

PROOF

Statements	Reasons
1. $\frac{a}{b} = \frac{c}{d}$	1. ?
2. $b \cdot c = a \cdot d$	2. ?
3. $ab + bc = ab + ad$	3. ?
4. $b(a + c) = a(b + d)$	4. ?
5. $\frac{a+c}{b+d} = \frac{a}{b}$	5. Means-Extremes Property (symmetric form)
6. $\frac{a+c}{b+d} = \frac{c}{d}$	6. ?

26. *Given:* $\triangle RST$, with $\overrightarrow{XY} \parallel \overrightarrow{RT}$, $\overrightarrow{YZ} \parallel \overrightarrow{RS}$

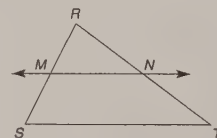
$$\text{Prove: } \frac{RX}{XS} = \frac{ZT}{RZ}$$



27. Use Theorem 5.5.1 and the drawing to complete the proof of this theorem: "If a line is parallel to one side of a triangle and passes through the midpoint of a second side, then it will pass through the midpoint of the third side."

Given: $\triangle RST$ with M the midpoint of RS ; $\overrightarrow{MN} \parallel \overrightarrow{ST}$

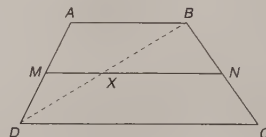
Prove: N is the midpoint of RT



28. Use Exercise 27 and the following drawing to complete the proof of this theorem: "The length of the median of a trapezoid is one-half the sum of the lengths of the two bases."

Given: Trapezoid $ABCD$ with median $\overline{MN}$

Prove: $MN = \frac{1}{2}(AB + CD)$

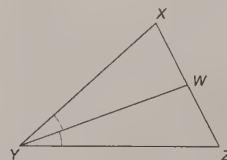


29. Use Theorem 5.5.3 to complete the proof of this theorem: "If the bisector of an angle of a triangle also bisects the opposite side, then the triangle is an isosceles triangle."

Given: $\triangle XYZ$; $\overrightarrow{YW}$ bisects $\angle XYZ$; $\overline{WX} \cong \overline{WZ}$

Prove: $\triangle XYZ$ is isosceles

(**HINT:** Use a proportion to show that $YX = YZ$.)



30. ➤ In right $\triangle ABC$ (not shown) with right $\angle C$, $\overrightarrow{AD}$ bisects $\angle BAC$ so that D lies on side $\overline{CB}$. If $AC = 6$ and $DC = 3$, find BD and AB .

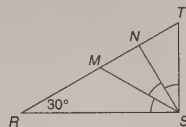
(HINT: Let $BD = x$ and $AB = 2x$. Then use the Pythagorean Theorem.)

31. ➤ Given: $\triangle ABC$ (not shown) is isosceles with $m\angle ABC = m\angle C = 72^\circ$; $\overrightarrow{BD}$ bisects $\angle ABC$ and $AB = 1$

Find: BC

32. ➤ Given: $\triangle RST$ with right $\angle RST$; $m\angle R = 30^\circ$ and $ST = 6$; $\angle RST$ is trisected by $\overrightarrow{SM}$ and $\overrightarrow{SN}$

Find: TN , NM , and MR



A Look Beyond: An Unusual Application of Similar Triangles

The following problem is one that can be solved in many ways. If methods of calculus are applied, the solution is found through many complicated and tedious calculations. The simplest solution, which follows, utilizes geometry and similar triangles.

Problem: A hiker is at a location 450 ft downstream from his campsite. He is 200 ft away from the straight stream, and his tent is 100 ft away, as shown in Figure 5.51(a). Across the flat field, he sees that a spark from his campfire has ignited the tent. Taking the empty bucket he is carrying, he runs to the river to get water and then on to the tent. To what point on the river should he run to minimize the distance he travels?

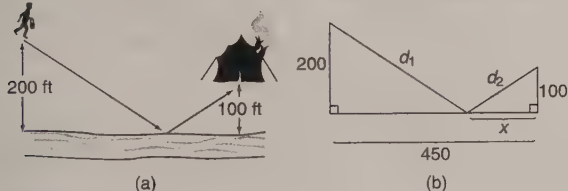


FIGURE 5.51

We wish to determine x in Figure 5.51(b) so that the total distance $D = d_1 + d_2$ is as small as possible. Consider three possible choices of this point on the river. These are suggested by dashed, dotted, and solid lines in Figure 5.52(a). Also consider the reflections of the triangles across the river. [See Figure 5.52(b).]

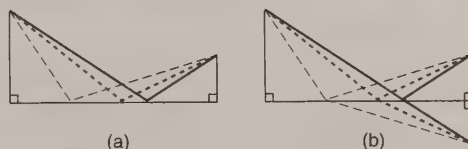


FIGURE 5.52

The minimum distance D occurs where the segments of lengths d_1 and d_2 form a straight line. That is, the configuration with the solid line segments minimizes the distance. In that case, the triangle at left and the reflected triangle at right are similar. (See Figure 5.53.)

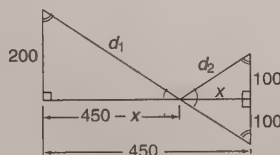


FIGURE 5.53

Thus

$$\frac{200}{100} = \frac{450 - x}{x}$$

$$\begin{aligned} 200x &= 100(450 - x) \\ 200x &= 45,000 - 100x \\ 300x &= 45,000 \\ x &= 150 \end{aligned}$$

Accordingly, the desired point on the river is 300 ft (determined by $450 - x$) upstream from the hiker's location.



Summary

■ A Look Back at Chapter 5

The goal of this chapter has been to define similarity for all polygons. We postulated a method for proving triangles similar and showed that proportions are a consequence of similar triangles, a line parallel to one side of a triangle, and a ray bisecting one angle of a triangle. The Pythagorean Theorem and its converse were proved. We discussed the 30-60-90 triangle, the 45-45-90 triangle, and other special right triangles with sides forming Pythagorean triples.

■ A Look Ahead to Chapter 6

In the next chapter, we will begin our work with the circle. Segments and lines of the circle will be defined, as will special angles in a circle. Several theorems dealing with the measurements of these angles and line segments will be proved. Our work with constructions will enable us to deal with the locus of points and the concurrence of lines.

■ Important Terms and Concepts of Chapter 5

- 5.1 Ratio, Rate, Proportion
Commensurable, Incommensurable
Extremes, Means of a Proportion
Means-Extremes Property
Geometric Mean
Extended Ratio, Extended Proportion
- 5.2 Similar Polygons
Congruent Polygons
Corresponding Vertices, Angles, and Sides
AAA, AA, and CSSTP
SAS~ and SSS~
- 5.3 Pythagorean Theorem and Converse
Pythagorean Triple
- 5.4 45-45-90 Triangle
30-60-90 Triangle
- 5.5 Segments Divided Proportionally

■ A Look Beyond: An Unusual Application of Similar Triangles



Review Exercises

Answer true or false for Review Exercises 1 to 7.

1. The ratio of 12 hr to 1 day is 2 to 1.
2. If the numerator and the denominator of a ratio are multiplied by 4, the new ratio equals the given ratio.
3. The value of a ratio must be less than 1.
4. The three numbers 6, 14, and 22 are in a ratio of 3:7:11.
5. To express a ratio correctly, the terms must have the same unit of measure.
6. The ratio 3:4 is the same as the ratio 4:3.
7. If the second and third terms of a proportion are equal, then either is the geometric mean of the first and fourth terms.

8. Find the value(s) of x in each proportion:

a) $\frac{x}{6} = \frac{3}{x}$

b) $\frac{x-5}{3} = \frac{2x-3}{7}$

c) $\frac{6}{x+4} = \frac{2}{x+2}$

d) $\frac{x+3}{5} = \frac{x+5}{7}$

e) $\frac{x-2}{x-5} = \frac{2x+1}{x-1}$

f) $\frac{x(x+5)}{4x+4} = \frac{9}{5}$

g) $\frac{x-1}{x+2} = \frac{10}{3x-2}$

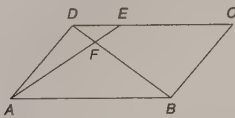
h) $\frac{x+7}{2} = \frac{x+2}{x-2}$

Use proportions to solve Review Exercises 9 to 11.

9. Four containers of fruit juice cost \$2.52. How much do six containers cost?
10. Two packages of M&Ms cost 69¢. How many packages can you buy for \$2.25?
11. A rug measuring 20 square meters costs \$132. How much would a 12 square-meter rug of the same material cost?
12. The ratio of the measures of the sides of a quadrilateral is 2:3:5:7. If the perimeter is 68, find the length of each side.
13. The length and width of a rectangle are 18 and 12, respectively. A similar rectangle has length 27. What is its width?
14. The sides of a triangle are 6, 8, and 9. The shortest side of a similar triangle is 15. How long are its other sides?
15. The ratio of the measure of the supplement of an angle to that of the complement of the angle is 5:2. Find the measure of the supplement.

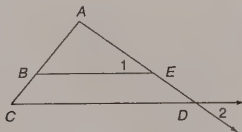
16. *Given:* $ABCD$ is a parallelogram
 $\overline{DB}$ intersects $\overline{AE}$ at point F

Prove: $\frac{AF}{EF} = \frac{AB}{DE}$



17. *Given:* $\angle 1 \cong \angle 2$

Prove: $\frac{AB}{AC} = \frac{BE}{CD}$



18. *Given:* $\triangle ABC \sim \triangle DEF$ (not shown)

$m\angle A = 50^\circ$, $m\angle E = 33^\circ$

$m\angle D = 2x + 40$

Find: x , $m\angle F$

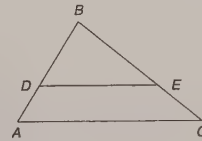
19. *Given:* In $\triangle ABC$ and $\triangle DEF$ (not shown)

$\angle B \cong \angle F$ and $\angle C \cong \angle E$

$AC = 9$, $DE = 3$, $DF = 2$, $FE = 4$

Find: AB , BC

For Review Exercises 20 to 22, $\overline{DE} \parallel \overline{AC}$.



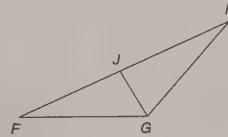
Exercises 20–22

20. $BD = 6$, $BE = 8$, $EC = 4$, $AD = ?$

21. $AD = 4$, $BD = 8$, $DE = 3$, $AC = ?$

22. $AD = 2$, $AB = 10$, $BE = 5$, $BC = ?$

For Review Exercises 23 to 25, $\vec{GJ}$ bisects $\angle FGH$.



Exercises 23–25

23. *Given:* $FG = 10$, $GH = 8$, $FJ = 7$

Find: JH

24. *Given:* $GF:GH = 1:2$, $FJ = 5$

Find: JH

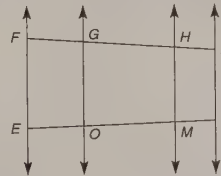
25. *Given:* $FG = 8$, $HG = 12$, $FH = 15$

Find: FJ

26. *Given:* $\vec{EF} \parallel \vec{GO} \parallel \vec{HM} \parallel \vec{JK}$, with transversals $\overline{FJ}$ and $\overline{EK}$

$FG = 2$, $GH = 8$, $HJ = 5$, $EM = 6$

Find: EO , EK

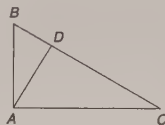


27. Prove that if a line bisects one side of a triangle and is parallel to a second side, then it bisects the third side.

28. Prove that the diagonals of a trapezoid divide themselves proportionally.

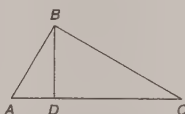
29. Given: $\triangle ABC$ with right $\angle BAC$
 $AD \perp \overline{BC}$

- a) $BD = 3$, $AD = 5$, $DC = ?$
b) $AC = 10$, $DC = 4$, $BD = ?$
c) $BD = 2$, $BC = 6$, $BA = ?$
d) $BD = 3$, $AC = 3\sqrt{2}$, $DC = ?$

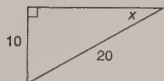


30. Given: $\triangle ABC$ with right $\angle ABC$
 $\overline{BD} \perp \overline{AC}$

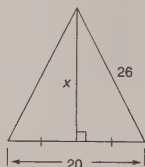
- a) $BD = 12$, $AD = 9$, $DC = ?$
b) $DC = 5$, $BC = 15$, $AD = ?$
c) $AD = 2$, $DC = 8$, $AB = ?$
d) $AB = 2\sqrt{6}$, $DC = 2$, $AD = ?$



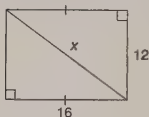
31. In the drawings shown, find x .



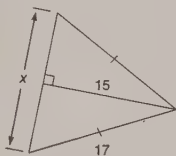
(a)



(b)



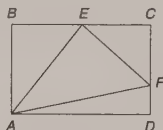
(c)



(d)

32. Given: $ABCD$ is a rectangle
 E is the midpoint of $\overline{BC}$
 $AB = 16$, $CF = 9$, $AD = 24$

Find: AE , EF , AF



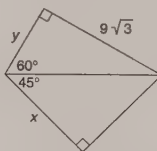
33. Find the length of a diagonal of a square whose side is 4 in. long.
34. Find the length of a side of a square whose diagonal is 6 cm long.
35. Find the length of a side of a rhombus whose diagonals are 48 cm and 14 cm long.

36. Find the length of an altitude of an equilateral triangle whose side is 10 in. long.

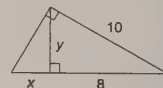
37. Find the length of a side of an equilateral triangle if an altitude is 6 in. long.

38. The lengths of three sides of a triangle are 13 cm, 14 cm, and 15 cm. Find the length of the altitude to the 14-cm side.

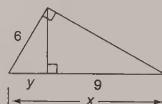
39. In the drawings, find x and y .



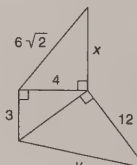
(a)



(b)

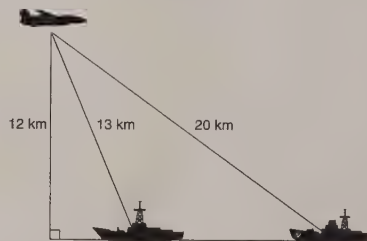


(c)



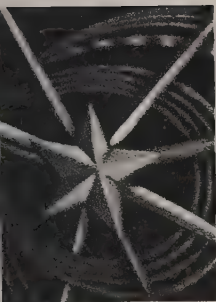
(d)

40. An observation aircraft flying at a height of 12 km has detected a Brazilian ship at a distance of 20 km from the aircraft and in line with an American ship that is 13 km from the aircraft. How far apart are the U.S. and Brazilian ships?



41. Tell whether each set of numbers represents the lengths of the sides of an acute triangle, of an obtuse triangle, of a right triangle, or of no triangle:

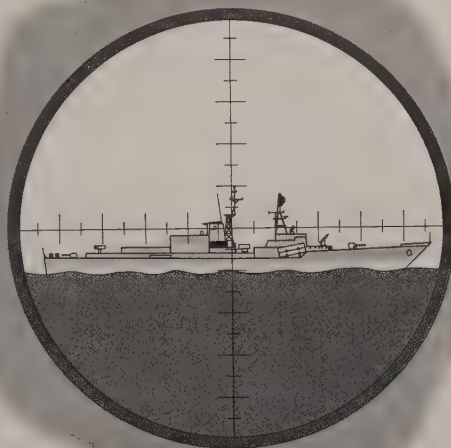
- a) 12, 13, 14
b) 11, 5, 18
c) 9, 15, 18
d) 6, 8, 10
e) 8, 7, 16
f) 8, 7, 6
g) 9, 13, 8
h) 4, 2, 3



ANALYTIC GEOMETRY

The French mathematician René Descartes is considered the father of analytic geometry. His inspiration relating algebra and geometry, the Cartesian coordinate system, was a major breakthrough in the development of much of mathematics. Some have written that Descartes was inspired while watching a spider weave its web in the corner of his room. Others say that the brilliant idea came to Descartes as he was dreaming. In any event, the branch of mathematics known as calculus would not have grown as rapidly without analytic geometry.

The periscope reminds us of the coordinate system for which Descartes is well known.



CHAPTER OUTLINE

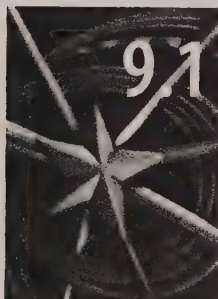
- 9.1 The Rectangular Coordinate System
- 9.2 Graphs of Linear Equations and Slope
- 9.3 Preparing to Do Analytic Proofs
- 9.4 Analytic Proofs
- 9.5 Equations of Lines

A Look Beyond:

The Banach-Tarski Paradox

Summary

For online student resources, visit this textbook's web site at math.college.hmco.com/students.



THE RECTANGULAR COORDINATE SYSTEM

Graphing the solution sets for $3x - 2 = 7$ and $3x - 2 > 7$ required a single number line to indicate the value of x . (See Appendices A.2 and A.3 for further information.) In this chapter, we deal with equations containing two variables; to relate algebra to plane geometry, we will need two number lines.

The study of the relationships between number pairs and points is usually referred to as **analytic geometry**. The **Cartesian coordinate system** or **rectangular coordinate system** is the plane that results when two number lines intersect perpendicularly at the origin (the point corresponding to the number 0) of each line. The horizontal number line is known as the **x axis**, and its numerical coordinates increase from left to right. On the vertical number line, the **y axis**, values increase from bottom to top; see Figure 9.1. The two axes separate the plane into four sections known as **quadrants**; the quadrants are numbered I, II, III, and IV, as shown. The point that marks the common origin of the two number lines is the **origin** of the rectangular coordinate system. It is convenient to identify the origin as $(0, 0)$; this notation indicates that the **x coordinate** (listed first) is 0 and also that the **y coordinate** (listed second) is 0.

In the coordinate system in Figure 9.2, the point $(3, -2)$ is shown. For each point, we have the order (x, y) ; these pairs are referred to as **ordered pairs** because x must precede y . To plot (or locate) this point, we see that $x = 3$ and that $y = -2$. Thus the point is located by moving 3 units to the right of the origin and then 2 units down from the x axis. The dashed lines shown are used to emphasize the reason why it is called the *rectangular* coordinate system. Note that this point $(3, -2)$ could also have been located by first moving down 2 units and then moving 3 units to the right of the y axis. This point is located in Quadrant IV. In Figure 9.2, ordered pairs of plus and minus signs characterize the signs of the coordinates of a point in each quadrant.

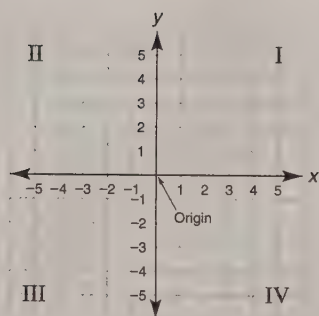


FIGURE 9.1

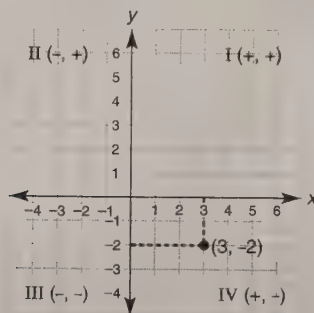


FIGURE 9.2

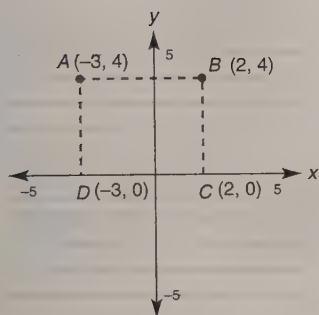
EXAMPLE 1

Plot points $A(-3, 4)$ and $B(2, 4)$, and find the distance between them.

Solution

Point A is located by moving 3 units to the left of the origin and then 4 units up from the x axis. Point B is located by moving 2 units to the right of the origin and then 4 units up from the x axis. In Figure 9.3, $\overline{AB}$ is a horizontal segment.

In the rectangular coordinate system, $ABCD$ is a rectangle in which $DC = 5$; $\overline{DC}$ is easily measured because it lies on the x axis. Because the opposite sides of a rectangle are congruent, it follows that $AB = 5$.

**FIGURE 9.3**

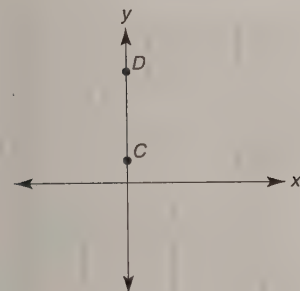
DEFINITION: Given points $A(x_1, y_1)$ and $B(x_2, y_1)$ on a horizontal line segment $\overline{AB}$, the **distance** between these points is

$$AB = x_2 - x_1 \text{ if } x_2 > x_1 \quad \text{or} \quad AB = x_1 - x_2 \text{ if } x_1 > x_2$$

In the preceding definition, repeated y coordinates characterize a horizontal line segment. In the following definition, repeated x coordinates determine a vertical line segment. In each definition, the distance is found by subtracting the smaller from the larger of the two unequal coordinates.

DEFINITION: Given points $C(x_1, y_1)$ and $D(x_1, y_2)$ on a vertical line segment $\overline{CD}$, the **distance** between these points is

$$CD = y_2 - y_1 \text{ if } y_2 > y_1 \quad \text{or} \quad CD = y_1 - y_2 \text{ if } y_1 > y_2$$

**FIGURE 9.4****EXAMPLE 2**

In Figure 9.4, name the coordinates of points C and D , and find the distance between them.

Solution

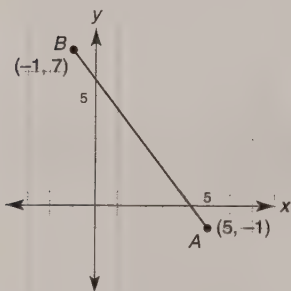
C is the point $(0, 1)$ because C is 1 unit above the origin; similarly, D is the point $(0, 5)$. We designate the coordinates of point C by $x_1 = 0$ and $y_1 = 1$ and the coordinates of point D by $x_1 = 0$ and $y_2 = 5$. Using the preceding definition,

$$CD = y_2 - y_1 = 5 - 1 = 4$$

We now turn our attention to the more general problem of finding the distance between any two points.

*Proof***DISCOVER!**

Plot the points $A(0, 0)$ and $B(4, 3)$. Now find AB by using the Pythagorean Theorem. To accomplish this, you will need to form a path from A to B along horizontal and vertical line segments.

ANSWER
S**FIGURE 9.6****EXAMPLE 3***Solution*

Using the Distance Formula and choosing $x_1 = 5$ and $y_1 = -1$ (from point A) and $x_2 = -1$ and $y_2 = 7$ (from point B), we obtain

$$\begin{aligned} d &= \sqrt{(-1 - 5)^2 + [7 - (-1)]^2} \\ &= \sqrt{(-6)^2 + (8)^2} \\ &= \sqrt{100} = 10 \end{aligned}$$

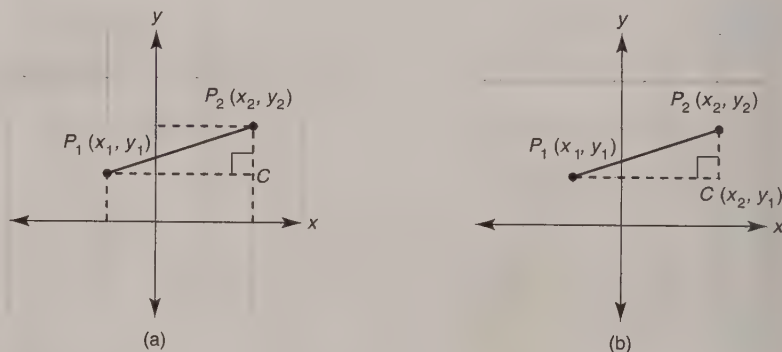
THE DISTANCE FORMULA

THEOREM 9.1.1: (Distance Formula) The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In the coordinate system in Figure 9.5 are points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. In addition to drawing the segment joining these points, we draw an auxiliary horizontal segment through P_1 and an auxiliary vertical segment through P_2 ; these meet at point C in Figure 9.5(a). Using Figure 9.5(b) and the definitions for lengths of horizontal and vertical segments,

$$P_1C = x_2 - x_1 \quad \text{and} \quad P_2C = y_2 - y_1$$

**FIGURE 9.5**

In right triangle P_1P_2C in Figure 9.5(b), let $d = P_1P_2$. By the Pythagorean Theorem,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Taking the positive square root for length,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

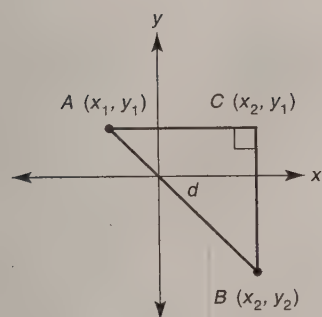


FIGURE 9.7

NOTE: If the coordinates of point A were designated as $x_2 = 5$ and $y_2 = -1$ and those of point B were designated as $x_1 = -1$ and $y_1 = 7$, the distance would remain the same.

Looking back at the proof of the Distance Formula, Figure 9.5 shows only one of several possible placements of points. If the placement had been as shown in Figure 9.7, then we would have had $AC = x_2 - x_1$ because $x_2 > x_1$, and $BC = y_1 - y_2$ because $y_1 > y_2$. The Pythagorean Theorem leads to a different result:

$$d^2 = (x_2 - x_1)^2 + (y_1 - y_2)^2$$

but this can be converted to the earlier formula by using the fact that

$$(y_1 - y_2)^2 = (y_2 - y_1)^2$$

This follows from the fact that $(-a)^2 = a^2$ for any real number a .

The following example reminds us of the form of a **linear equation**, an equation whose graph is a straight line. In general, this form is $Ax + By = C$ for constants A , B , and C (where A and B do not both equal 0). We will consider the graphing of linear equations in Section 9.2.

EXAMPLE 4

Find the equation that describes all points (x, y) that are equidistant from $A(5, -1)$ and $B(-1, 7)$.

Solution

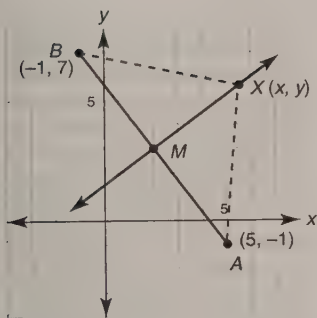


FIGURE 9.8

In Chapter 6, we saw that the locus of points equidistant from two fixed points is a line. This line, $\overleftrightarrow{MX}$ in Figure 9.8, is the perpendicular bisector of $\overline{AB}$.

If X is on the locus, then $AX = BX$. By the Distance Formula, we have

$$\sqrt{(x - 5)^2 + [y - (-1)]^2} = \sqrt{[x - (-1)]^2 + (y - 7)^2}$$

or

$$(x - 5)^2 + (y + 1)^2 = (x + 1)^2 + (y - 7)^2$$

after simplifying and squaring. Then

$$x^2 - 10x + 25 + y^2 + 2y + 1 = x^2 + 2x + 1 + y^2 - 14y + 49$$

Eliminating x^2 and y^2 terms by subtraction leads to the equation

$$-12x + 16y = 24$$

When we divide by 4, the equation of the line becomes

$$-3x + 4y = 6$$

If we divide the equation $-12x + 16y = 24$ by -4 , an equivalent solution is

$$3x - 4y = -6$$

In Figure 9.8, point M was the midpoint of $\overline{AB}$. It will be shown in Example 5(a) that M is the point $(2, 3)$.

DISCOVER!

On a number line, x_2 lies to the right of x_1 . Then $x_2 > x_1$ and the distance between points is $(x_2 - x_1)$. To find the number m that is midway between x_1 and x_2 , we add one-half the distance $(x_2 - x_1)$ to x_1 . That is,

$$m = x_1 + \frac{1}{2}(x_2 - x_1).$$

Complete the simplification of m .

$$\begin{array}{l} \text{ANSWER} \\ \frac{2}{2x + 1x} = u \\ \text{so } (2x + 1x)\frac{2}{2} = u \end{array}$$

THE MIDPOINT FORMULA

A generalized midpoint formula is given in Theorem 9.1.2. The result shows that the coordinates of the midpoint of a line segment are the averages of the coordinates of the endpoints. See the Discover! box at left

To prove the Midpoint Formula, we need to establish two things:

1. $BM + MA = BA$, which establishes that the three points A , M , and B are collinear; and
2. $BM = MA$, which establishes that point M is midway between A and B .

THEOREM 9.1.2: (Midpoint Formula) The midpoint M of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ has coordinates x_M and y_M , where

$$(x_M, y_M) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

that is,

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

EXAMPLE 5

Use the Midpoint Formula to find the midpoint of the line segment joining:

- a) $(5, -1)$ and $(-1, 7)$ b) (a, b) and (c, d)

Solution

- a) Using the Midpoint Formula and setting $x_1 = 5$, $y_1 = -1$, $x_2 = -1$, and $y_2 = 7$, we have

$$M = \left(\frac{5 + (-1)}{2}, \frac{-1 + 7}{2} \right) \quad \text{or} \quad M = (2, 3)$$

- b) Using the Midpoint Formula and setting $x_1 = a$, $y_1 = b$, $x_2 = c$, and $y_2 = d$, we have

$$M = \left(\frac{a + c}{2}, \frac{b + d}{2} \right)$$

In part (a) of Example 5, it may be helpful to make a sketch of the segment; this will enable you to test whether your solution is reasonable! In part (b), we are generalizing the coordinates in preparation for the analytic geometry proofs that appear later in the chapter. In those sections, we will choose the x and y values of each point in such a way as to be as general as possible. When the Midpoint Formula is used, it is a good idea to select coordinates such as $(2a, 2b)$ for a point so that division by 2 will not introduce fractions.

PROOF OF THE MIDPOINT FORMULA (OPTIONAL)

For the segment joining P_1 and P_2 , we designate the midpoint by M , as shown in Figure 9.9. Let the coordinates of M be designated by (x_M, y_M) . Now construct horizontal segments through P_1 and M and vertical segments through M and P_2 to intersect at points A and B , as shown in Figure 9.9(b). Because $\angle A$ and $\angle B$ are right angles, $\angle A \cong \angle B$.

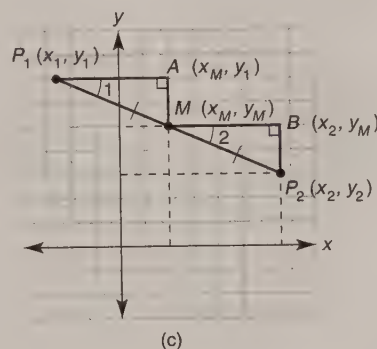
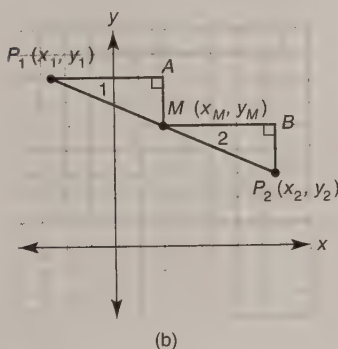
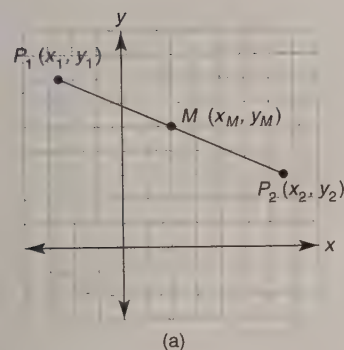


FIGURE 9.9

Because $\overline{P_1A}$ and $\overline{MB}$ are both horizontal, these segments are parallel. Then $\angle 1 \cong \angle 2$. With $\overline{P_1M} \cong \overline{MP_2}$ by the definition of a midpoint, we have $\triangle P_1AM \cong \triangle MBP_2$ by AAS. Because A is the point (x_M, y_1) , we have $P_1A = x_M - x_1$. Likewise, the coordinates of B are (x_2, y_M) , so $MB = x_2 - x_M$. Because $\overline{P_1A} \cong \overline{MB}$ by CPCTC, we represent the common length of the segments $\overline{P_1A}$ and $\overline{MB}$ by a . From the first equation, $x_M - x_1 = a$, so $x_M = x_1 + a$. From the second equation, $x_2 - x_M = a$, so $x_2 = x_M + a$. Substituting $x_1 + a$ for x_M , we have

$$\begin{aligned}(x_1 + a) + a &= x_2 \\ x_1 + 2a &= x_2\end{aligned}$$

Then $2a = x_2 - x_1$

so $a = \frac{x_2 - x_1}{2}$

It follows that

$$\begin{aligned}x_M &= x_1 + a \\ &= x_1 + \frac{x_2 - x_1}{2} \\ &= \frac{2x_1}{2} + \frac{x_2 - x_1}{2} \\ &= \frac{x_1 + x_2}{2}\end{aligned}$$

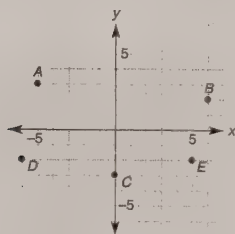
The y coordinate of the midpoint can be determined in a similar manner, to show that $y_M = \frac{y_1 + y_2}{2}$. Then

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$



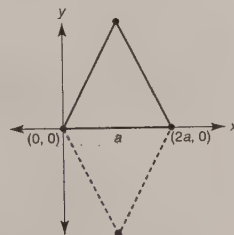
9.1 EXERCISES

- Plot and then label the points $A(0, -3)$, $B(3, -4)$, $C(5, 6)$, $D(-2, -5)$, and $E(-3, 5)$.
- Give the coordinates of each point A , B , C , D , and E . Also name the quadrant in which each point lies.

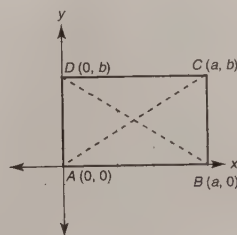


- Find the distance between each pair of points:
 - $(5, -3)$ and $(5, 1)$
 - $(-3, 4)$ and $(5, 4)$
 - $(0, 2)$ and $(0, -3)$
 - $(-2, 0)$ and $(7, 0)$
- If the distance between $(-2, 3)$ and $(-2, a)$ is 5 units, find all possible values of a .
- If the distance between $(b, 3)$ and $(7, 3)$ is 3.5 units, find all possible values of b .
- Find an expression for the distance between (a, b) and (a, c) if $b > c$.
- Find the distance between each pair of points:
 - $(0, -3)$ and $(4, 0)$
 - $(-2, 5)$ and $(4, -3)$
 - $(3, 2)$ and $(5, -2)$
 - $(a, 0)$ and $(0, b)$
- Find the distance between each pair of points:
 - $(-3, -7)$ and $(2, 5)$
 - $(0, 0)$ and $(-2, 6)$
 - $(-a, -b)$ and (a, b)
 - $(2a, 2b)$ and $(2c, 2d)$
- Find the midpoint of the line segment that joins each pair of points:
 - $(0, -3)$ and $(4, 0)$
 - $(-2, 5)$ and $(4, -3)$
 - $(3, 2)$ and $(5, -2)$
 - $(a, 0)$ and $(0, b)$
- Find the midpoint of the line segment that joins each pair of points:
 - $(-3, -7)$ and $(2, 5)$
 - $(0, 0)$ and $(-2, 6)$
 - $(-a, -b)$ and (a, b)
 - $(2a, 2b)$ and $(2c, 2d)$
- Points A and B have symmetry with respect to the origin O . That is, $A-O-B$ and $AO = OB$, so it follows that the origin $(0, 0)$ is the midpoint of $\overline{AB}$. Find the coordinates of B if A is the point:
 - $(3, -4)$
 - $(0, 2)$
 - $(a, 0)$
 - (b, c)
- Points A and B have symmetry with respect to point $C(2, 3)$. That is, $A-C-B$ and $AC = CB$, so it follows that C is the midpoint of $\overline{AB}$. Find the coordinates of B if A is the point:
 - $(3, -4)$
 - $(0, 2)$
 - $(5, 0)$
 - (a, b)
- Points A and B have symmetry with respect to point C . That is, $A-C-B$ and $AC = CB$, so it follows that C is the midpoint of $\overline{AB}$. Find the coordinates of C given the points:
 - $A(3, -4)$ and $B(5, -1)$
 - $A(0, 2)$ and $B(0, 6)$
 - $A(5, -3)$ and $B(2, 1)$
 - $A(2a, 0)$ and $B(0, 2b)$

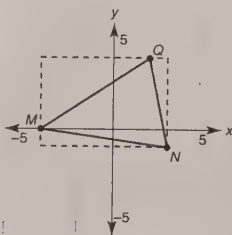
14. Points A and B have *symmetry with respect to line ℓ* if ℓ is the perpendicular bisector of $\overline{AB}$. The x axis is the perpendicular bisector of $\overline{AB}$. Find the coordinates of B if A is the point:
- a) $(3, -4)$ c) $(0, a)$
b) $(0, 2)$ d) (b, c)
15. Points A and B have *symmetry with respect to line ℓ* if ℓ is the perpendicular bisector of $\overline{AB}$. Points A and B are located so that they have symmetry with respect to the x axis. Find the coordinates of A if B is the point:
- a) $(5, 1)$ c) $(2, a)$
b) $(0, 5)$ d) (b, c)
16. Points A and B have *symmetry with respect to the vertical line where $x = 2$* . That is, this vertical line is the perpendicular bisector of $\overline{AB}$. Find the coordinates of A if B is the point:
- a) $(5, 1)$ c) $(-6, a)$
b) $(0, 5)$ d) (b, c)
17. Points A and B have *symmetry with respect to line ℓ* if ℓ is the perpendicular bisector of $\overline{AB}$. Points A and B are located so that they have symmetry with respect to the y axis. Find the coordinates of A if B is the point:
- a) $(3, -4)$ c) $(a, 0)$
b) $(2, 0)$ d) (b, c)
18. Points A and B have *symmetry with respect to line ℓ* if ℓ is the perpendicular bisector of $\overline{AB}$. Points A and B are located so that they have symmetry with respect to either the x axis or the y axis. Name the axis of symmetry for:
- a) $A(3, -4)$ and $B(3, 4)$ c) $A(3, -4)$ and $B(-3, -4)$
b) $A(2, 0)$ and $B(-2, 0)$ d) $A(a, b)$ and $B(a, -b)$
19. Points A and B have *symmetry with respect to line ℓ* if ℓ is the perpendicular bisector of $\overline{AB}$. Points A and B are located so that they have symmetry with respect to a vertical line ($x = a$) or a horizontal line ($y = b$). Give an equation like $x = 3$ for the axis of symmetry for:
- a) $A(3, -4)$ and $B(5, -4)$ c) $A(7, -4)$ and $B(-3, -4)$
b) $A(a, 0)$ and $B(a, 2b)$ d) $A(a, 7)$ and $B(a, -1)$
20. $M(3, -4)$ is the midpoint of $\overline{AB}$, in which A is the point $(-5, 7)$. Find the coordinates of B .
(NOTE: Algebra can be used!)
21. $M(2.1, -5.7)$ is the midpoint of $\overline{AB}$, in which A is the point $(1.7, 2.3)$. Find the coordinates of B .
22. A circle has its center at the point $(-2, 3)$. If one endpoint of the diameter is at $(3, -5)$, find the other endpoint.
23. A rectangle $ABCD$ has three of its vertices at $A(2, -1)$, $B(6, -1)$, and $C(6, 3)$. Find the fourth vertex D and the area of rectangle $ABCD$.
24. A rectangle $MNPQ$ has three of its vertices at $M(0, 0)$, $N(a, 0)$, and $Q(0, b)$. Find the fourth vertex P and the area of the rectangle $MNPQ$.
25. Use the Distance Formula to determine the type of triangle that has these vertices:
- a) $A(0, 0)$, $B(4, 0)$, and $C(2, 5)$
b) $D(0, 0)$, $E(4, 0)$, and $F(2, 2\sqrt{3})$
c) $G(-5, 2)$, $H(-2, 6)$, and $K(2, 3)$
26. Use the method of Example 4 to find the equation of the line that describes all points equidistant from the points $(-3, 4)$ and $(3, 2)$.
27. Use the method of Example 4 to find the equation of the line that describes all points equidistant from the points $(1, 2)$ and $(4, 5)$.
28. For coplanar points A , B , and C , suppose that you have used the Distance Formula to show that $AB = 5$, $BC = 10$, and $AC = 15$. What can you conclude regarding points A , B , and C ?
29. If two vertices of an equilateral triangle are at $(0, 0)$ and $(2a, 0)$, what is the third vertex?



30. The rectangle whose vertices are $A(0, 0)$, $B(a, 0)$, $C(a, b)$, and $D(0, b)$ is shown. Use the Distance Formula to draw a conclusion concerning the lengths of the diagonals $\overline{AC}$ and $\overline{BD}$.



31. ➤ There are two points on the y axis at a distance of 6 units from the point $(3, 1)$. Determine the coordinates of each point.
32. ➤ There are two points on the x axis at a distance of 6 units from the point $(3, 1)$. Determine the coordinates of each point.
33. The triangle that has vertices at $M(-4, 0)$, $N(3, -1)$, and $Q(2, 4)$ has been boxed in as shown. Find the area of $\triangle MNQ$.



Exercises 33, 34

34. Use the method suggested in Exercise 33 to find the area of $\triangle RST$, with $R(-2, 4)$, $S(-1, -2)$, and $T(6, 5)$.
35. Determine the area of $\triangle ABC$ if $A = (2, 1)$, $B = (5, 3)$, and C is the "reflection" of B across the x axis.
36. Find the area of $\triangle ABC$ in Exercise 35, but assume that C is the "reflection" of B across the y axis.

For Exercises 37 to 42, refer to Examples 5 and 6 of Section 8.3.

37. Find the exact volume of the right circular cone that results when the triangular region with vertices at $(0, 0)$, $(5, 0)$, and $(0, 9)$ is rotated about the
- a) x axis. b) y axis.

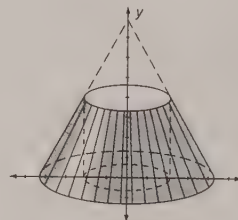
38. Find the exact volume of the solid that results when the triangular region with vertices at $(0, 0)$, $(6, 0)$, and $(6, 4)$ is rotated about the
- a) x axis. b) y axis.

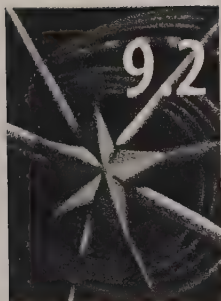
39. Find the exact volume of the solid formed when the rectangular region with vertices at $(0, 0)$, $(6, 0)$, $(6, 4)$, and $(0, 4)$ is revolved about the
- a) x axis. b) y axis.

40. Find the exact volume of the solid formed when the region bounded in Quadrant I by the lines $x = 9$ and $y = 5$ is revolved about the
- a) x axis. b) y axis.

41. Find the exact lateral area of each solid in Exercise 40.

42. ➤ Find the volume of the solid formed when the triangular region having vertices at $(2, 0)$, $(4, 0)$, and $(2, 4)$ is rotated about the y axis.





GRAPHS OF LINEAR EQUATIONS AND SLOPE

In Section 9.1, we were reminded that the general form of the equation of a line is $Ax + By = C$ (where A and B do not both equal 0). Some examples of linear equations are $2x + 3y = 12$, $3x - 4y = 12$, and $3x = -6$; as we shall see, the graph of each of these equations is a line.

THE GRAPH OF AN EQUATION

DEFINITION: The **graph of an equation** is the set of all points (x, y) in the rectangular coordinate system whose ordered pairs satisfy the equation.

EXAMPLE 1

Solution

Draw the graph of the equation $2x + 3y = 12$.

We begin by completing a table. It is convenient to use one point for which $x = 0$, a second point for which $y = 0$, and a third point as a check:

$$\begin{aligned} x = 0 &\rightarrow 2(0) + 3y = 12 \rightarrow y = 4 \\ y = 0 &\rightarrow 2x + 3(0) = 12 \rightarrow x = 6 \\ x = 3 &\rightarrow 2(3) + 3y = 12 \rightarrow y = 2 \end{aligned}$$

x	y	(x, y)
0	4	(0, 4)
6	0	(6, 0)
3	2	(3, 2)

Upon plotting the third point, we see that the three points are collinear; the graph of a linear equation must be a straight line, as shown in Figure 9.10.

Because the graph in Example 1 is a locus, every point on the line must also satisfy the given equation. It is easy to see that the point $(-3, 6)$ lies on the line shown in Figure 9.10. Notice that this ordered pair also satisfies the equation $2x + 3y = 12$; that is, $2(-3) + 3(6) = 12$ or $-6 + 18 = 12$.

For the equation in Example 1, the number 6 is known as the **x intercept** because $(6, 0)$ is the point at which the graph crosses the x axis; similarly, the number 4 is known as the **y intercept**. Most linear equations have two intercepts; these are generally represented by a (the x intercept) and b (the y intercept). To determine the x intercept, let $y = 0$ in the given equation and solve for x . The y intercept can be found similarly by choosing $x = 0$ and solving for y .

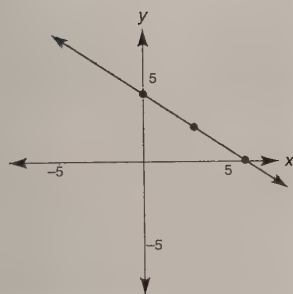


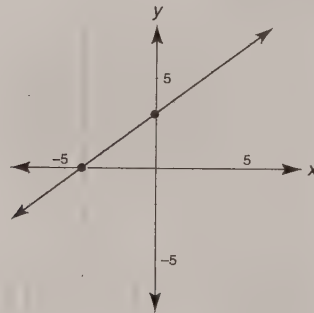
FIGURE 9.10

EXAMPLE 2

Find the x and y intercepts of the equation $3x - 4y = -12$, and use them to graph the equation.

Solution

The x intercept is found when $y = 0$; $3x - 4(0) = -12$, so $x = -4$. The x intercept is -4 , so $(-4, 0)$ is on the graph. The y intercept results when $x = 0$; $3(0) - 4y = -12$, so $y = 3$. The y intercept is 3 , so $(0, 3)$ is on the graph. Once the points $(-4, 0)$ and $(0, 3)$ are plotted, the graph can be completed by drawing the line through these points (see Figure 9.11).

**FIGURE 9.11**

As we shall see in Example 3, a linear equation may have only one intercept. Is it possible for a linear equation to have no intercepts at all?

EXAMPLE 3

Draw the graphs of the following equations:

a) $x = -2$

b) $y = 3$

Solution

First note that each equation is a linear equation:

$$x = -2 \text{ is equivalent to } (1 \cdot x) + (0 \cdot y) = -2$$

$$y = 3 \text{ is equivalent to } (0 \cdot x) + (1 \cdot y) = 3$$

- a) The equation $x = -2$ claims that the value of x is -2 regardless of the value of y ; this leads to the following table:

x	y	$\rightarrow$	(x, y)
-2	-2	$\rightarrow$	$(-2, -2)$
-2	0	$\rightarrow$	$(-2, 0)$
-2	5	$\rightarrow$	$(-2, 5)$

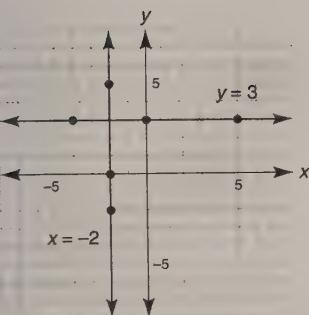


FIGURE 9.12

- b) The equation $y = 3$ claims that the value of y is 3 regardless of the value of x ; this leads to the following table:

x	y	$\rightarrow$	(x, y)
-4	3	$\rightarrow$	$(-4, 3)$
0	3	$\rightarrow$	$(0, 3)$
5	3	$\rightarrow$	$(5, 3)$

The graphs of the equations are shown in Figure 9.12.

NOTE: When an equation can be written in the form $x = a$ (for constant a), its graph is the vertical line containing the point $(a, 0)$. When an equation can be written in the form $y = b$ (for constant b), its graph is the horizontal line containing the point $(0, b)$. Most lines are oblique (or slanted); in such cases, it is convenient to describe the slope of the line.

THE SLOPE OF A LINE

DEFINITION: (Slope Formula) The slope of the line that contains the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{for } x_1 \neq x_2$$

NOTE: When $x_1 = x_2$, the line in question is vertical and we say that the slope is **undefined**.

Whereas the upper-case italic M means midpoint, we use the lower-case italic m for slope. Other terms used to describe the slope of a line include *pitch* and *grade*. A carpenter may say that a roofline has a $\frac{5}{12}$ pitch. [See Figure 9.13(a).] In constructing a stretch of roadway, an engineer may say that there is a grade of $\frac{3}{100}$ or 3 percent. [See Figure 9.13(b).]

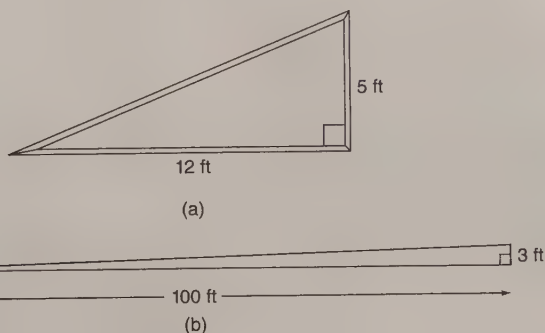
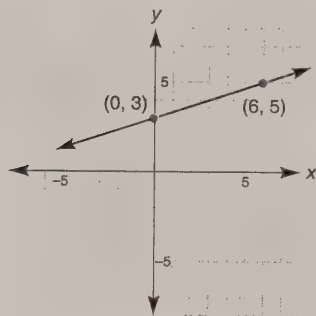
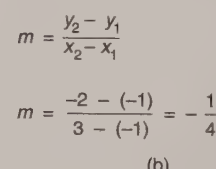


FIGURE 9.13

Whether in geometry, carpentry, or engineering, the term **slope** of a line refers to the ratio of the change along the vertical to the change along the horizontal, for any two points on the line in question. A line that “rises” from left to right has a *positive* slope and a line that “falls” from left to right has a *negative* slope. (See Figure 9.14.)


$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{5 - 3}{6 - 0} = \frac{2}{6} = \frac{1}{3}$$

(a)



Any horizontal line has slope 0; any vertical line has an undefined slope. Figure 9.15 shows an example of each of these types of lines.

Without graphing, find the slope of the line that contains:

- a) Using the Slope Formula and choosing $x_1 = 2$, $y_1 = 2$, $x_2 = 5$, and $y_2 = 3$, we have

$$m = \frac{3 - 2}{5 - 2} = \frac{1}{3}$$

NOTE: If drawn, this line will slant upward from left to right.

- b)** Let $x_1 = 1$, $y_1 = -1$, $x_2 = 1$, and $y_2 = 3$. Then we calculate

$$m = \frac{3 - (-1)}{1 - 1} = \frac{4}{0}$$

which is undefined.

In the Slope Formula, be sure to write the difference in values of y in the numerator; that is,

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

NOTE: If drawn, this line will be vertical because the x -coordinates are the same.

The slope of a line is unique; that is, the slope does not change when the following changes occur:

1. The order of the two points is reversed.
2. Different points on the line are selected.

The first situation is clear from the fact that $\frac{-a}{-b} = \frac{a}{b}$. The second situation is more difficult but depends on similar triangles.

For an explanation of point 2, consider Figure 9.16, in which points P_1 , P_2 , and P_3 are collinear. What we wish to show is that the slope of line ℓ is the same whether P_1 and P_2 , or P_2 and P_3 , are used in the Slope Formula. If horizontal and vertical segments are drawn as shown in Figure 9.16, we can show that triangles P_1P_2A and P_2P_3B are similar.

The similarity follows from the facts that $\angle 1 \cong \angle 2$ (because $\overline{P_1A} \parallel \overline{P_2B}$) and that $\angle A$ and $\angle B$ are right angles. Then $\frac{P_2A}{P_3B} = \frac{P_1A}{P_2B}$ because these are corresponding sides of similar triangles. By interchanging the means, we have $\frac{P_2A}{P_1A} = \frac{P_3B}{P_2B}$. But

$$\frac{P_2A}{P_1A} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{and} \quad \frac{P_3B}{P_2B} = \frac{y_3 - y_2}{x_3 - x_2}$$

$$\text{so} \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

Thus the slope is not changed by our having used either pair of points. For that matter, a fourth point could have been shown, in which case the slope of the line would not be changed if P_1 and P_2 , or P_3 and P_4 , were used; in every case, the slopes agree because of similar triangles.

In summary, if points P_1 , P_2 , and P_3 are collinear, then the slopes of $\overline{P_1P_2}$, $\overline{P_1P_3}$, and $\overline{P_2P_3}$ are the same. The converse of this statement is also true.

EXAMPLE 5

Are the points $A(2, -3)$, $B(5, 1)$, and $C(-4, -11)$ collinear?

Solution

Let $m_{\overline{AB}}$ and $m_{\overline{BC}}$ represent the slopes of $\overline{AB}$ and $\overline{BC}$, respectively. By the Slope Formula, we have

$$m_{\overline{AB}} = \frac{1 - (-3)}{5 - 2} = \frac{4}{3} \quad \text{and} \quad m_{\overline{BC}} = \frac{-11 - 1}{-4 - 5} = \frac{-12}{-9} = \frac{4}{3}$$

Because $m_{\overline{AB}} = m_{\overline{BC}}$, it follows that A , B , and C are collinear.

As we trace a line from one point to a second point, the Slope Formula tells us that

$$m = \frac{\text{change in } y}{\text{change in } x} \quad \text{or} \quad m = \frac{\text{vertical change}}{\text{horizontal change}}$$

This interpretation is used in Example 6.

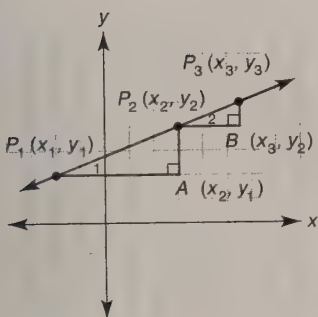


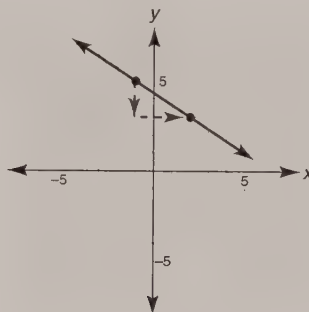
FIGURE 9.16

EXAMPLE 6

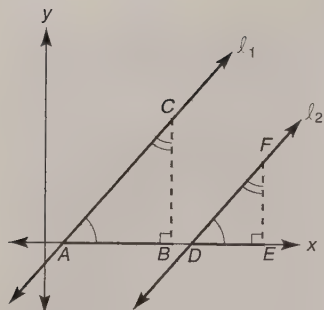
Draw the line through $(-1, 5)$ with slope $m = -\frac{2}{3}$.

Solution

First we plot the point $(-1, 5)$. The slope can be written as $m = \frac{-2}{3}$. Thus we let the change in y from the first to the second point be -2 while the change in x is 3 . From the first point $(-1, 5)$, we locate the second point by moving 2 units down and 3 units to the right. The line is then drawn as shown in Figure 9.17.

**FIGURE 9.17**

Two theorems are now stated without proof. However, drawings are provided, and these are to be used in the following exercises. Each proof depends on the fact that similar triangles are created through the use of the auxiliary segments included in the drawings.

**FIGURE 9.18**

THEOREM 9.2.1: If two nonvertical lines are parallel, then their slopes are equal.

NOTE: If $\ell_1 \parallel \ell_2$, then $m_1 = m_2$.

In Figure 9.18, note that $\overline{AC} \parallel \overline{DF}$. Also, $\overline{AB}$ and $\overline{DE}$ are horizontal, and $\overline{BC}$ and $\overline{FE}$ are auxiliary vertical segments. In the proof of Theorem 9.2.1, the goal is to show that $m_{\overline{AC}} = m_{\overline{DF}}$. The converse of Theorem 9.2.1 is also true; that is, if $m_1 = m_2$, then $\ell_1 \parallel \ell_2$.

THEOREM 9.2.2: If two lines (neither horizontal nor vertical) are perpendicular, then the product of their slopes is -1 .

NOTE: If $\ell_1 \perp \ell_2$, then $m_1 \cdot m_2 = -1$.

In Figure 9.19, auxiliary segments have again been added. The goal in proving Theorem 9.2.2 is to show that $m_{\overline{AC}} \cdot m_{\overline{CE}} = -1$. When the product of the slopes is -1 , we say that the slopes are **negative reciprocals**. The converse of Theorem 9.2.2 is also true; if $m_1 \cdot m_2 = -1$, then $\ell_1 \perp \ell_2$.

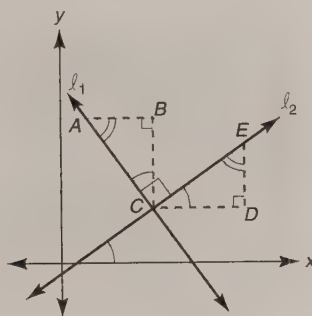


FIGURE 9.19

EXAMPLE 7

Given the points $A(-2, 3)$, $B(2, 1)$, $C(-1, 8)$, and $D(7, 3)$, are $\overline{AB}$ and $\overline{CD}$ parallel, perpendicular, or neither? (See Figure 9.20.)

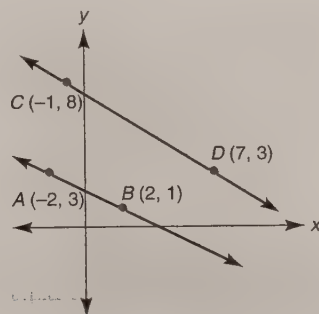


FIGURE 9.20

Solution

$$m_{\overline{AB}} = \frac{1 - 3}{2 - (-2)} = \frac{-2}{4} = -\frac{1}{2}$$

$$m_{\overline{CD}} = \frac{3 - 8}{7 - (-1)} = \frac{-5}{8} \text{ or } -\frac{5}{8}$$

Because $m_{\overline{AB}} \neq m_{\overline{CD}}$, $\overline{AB} \not\parallel \overline{CD}$. The slopes are not negative reciprocals, so $\overline{AB}$ is not perpendicular to $\overline{CD}$. Neither relationship holds for $\overline{AB}$ and $\overline{CD}$.

In Example 7, it would have been worthwhile to sketch the lines described. It is apparent from Figure 9.20 that no special relationship exists between the lines. A sketch may help show that a relationship does *not* exist, but sketching is not a dependable method for showing that lines *are* parallel or perpendicular.

EXAMPLE 8

Are the lines that are the graphs of $2x + 3y = 6$ and $3x - 2y = 12$ parallel, perpendicular, or neither?

Solution

Because $2x + 3y = 6$ contains the points $(3, 0)$ and $(0, 2)$, its slope is $\frac{2-0}{0-3} = -\frac{2}{3}$. The line $3x - 2y = 12$ contains $(0, -6)$ and $(4, 0)$; thus its slope is equal to $\frac{0-(-6)}{4-0} = \frac{6}{4}$ or $\frac{3}{2}$. Because the product of the slopes is $-\frac{2}{3} \cdot \frac{3}{2}$, or -1 , the lines described are perpendicular.

EXAMPLE 9

Determine the value of a for which the line through $(2, -3)$ and $(5, a)$ is perpendicular to the line $3x + 4y = 12$.

Solution

The line $3x + 4y = 12$ contains the points $(4, 0)$ and $(0, 3)$; this line has the slope

$$m = \frac{3-0}{0-4} = -\frac{3}{4}$$

For the two lines to be perpendicular, the second line must have slope $\frac{4}{3}$. Using the Slope Formula, we find that the second line has the slope

$$\frac{a - (-3)}{5 - 2}$$

$$\text{so } \frac{a + 3}{3} = \frac{4}{3}$$

Multiplying by 3, we obtain $a + 3 = 4$, so $a = 1$.

EXAMPLE 10

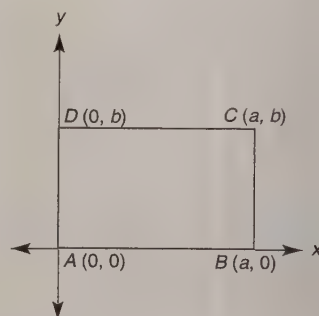
In Figure 9.21, show that the quadrilateral $ABCD$, with vertices $A(0, 0)$, $B(a, 0)$, $C(a, b)$, and $D(0, b)$, is a rectangle.

Solution

By applying the Slope Formula, we see that

$$m_{\overline{AB}} = \frac{0-0}{a-0} = 0$$

$$\text{and } m_{\overline{DC}} = \frac{b-b}{a-0} = 0$$



Then $\overline{AB}$ and $\overline{DC}$ are horizontal and therefore **FIGURE 9.21** are parallel to each other.

For $\overline{DA}$ and $\overline{CB}$, the slopes are undefined because the denominators in the Slope Formula both equal 0. Then $\overline{DA}$ and $\overline{CB}$ are both vertical and therefore parallel to each other.

Thus $ABCD$ is a parallelogram. With $\overline{AB}$ being horizontal and $\overline{DA}$ vertical, it follows that $\overline{DA} \perp \overline{AB}$. Therefore, $ABCD$ is a rectangle by definition.



9.2 EXERCISES

In Exercises 1 to 8, draw the graph of each equation. Name any intercepts.

1. $3x + 4y = 12$
2. $3x + 5y = 15$
3. $x - 2y = 5$
4. $x - 3y = 4$
5. $2x + 6 = 0$
6. $3y - 9 = 0$
7. $\frac{1}{2}x + y = 3$
(HINT: Multiply each term by 2; then graph the resulting equation.)
8. $\frac{2}{3}x - y = 1$

9. Find the slopes of the lines containing:

- a) (2, -3) and (4, 5)
- d) (-2.7, 5) and (-1.3, 5)
- b) (3, -2) and (3, 7)
- e) (a, b) and (c, d)
- c) (1, -1) and (2, -2)
- f) (a, 0) and (0, b)

10. Find the slopes of the lines containing:

- a) (3, -5) and (-1, 2)
- b) (-2, -3) and (-5, -7)
- c) $(2\sqrt{2}, -3\sqrt{6})$ and $(3\sqrt{2}, 5\sqrt{6})$
- d) $(\sqrt{2}, \sqrt{7})$ and $(\sqrt{2}, \sqrt{3})$
- e) (a, 0) and (a + b, c)
- f) (a, b) and (-b, -a)

11. Find x so that $\overline{AB}$ has slope m , where:

- a) A is (2, -3), B is (x, 5), and $m = 1$
- b) A is (x, -1), B is (3, 5), and $m = -0.5$

12. Find y so that $\overline{CD}$ has slope m , where:

- a) C is (2, -3), D is (4, y), and $m = \frac{3}{2}$
- b) C is (-1, -4), D is (3, y), and $m = -\frac{2}{3}$

13. Are these points collinear?

- a) A(-2, 5), B(0, 2), and C(4, -4)
- b) D(-1, -1), E(2, -2), and F(5, -5)

14. Are these points collinear?

- a) A(-1, -2), B(3, 2), and C(5, 5)
- b) D(a, c - d), E(b, c), and F(2b - a, c + d)

15. Parallel lines ℓ_1 and ℓ_2 have slopes m_1 and m_2 , respectively. Find m_2 if m_1 equals:

- a) $\frac{3}{4}$
- b) $-\frac{5}{3}$
- c) -2
- d) $\frac{a-b}{c}$

16. Perpendicular lines ℓ_1 and ℓ_2 have slopes m_1 and m_2 , respectively. Find m_2 if m_1 equals:

- a) 5
- b) $-\frac{5}{3}$
- c) $-\frac{1}{2}$
- d) $\frac{a-b}{c}$

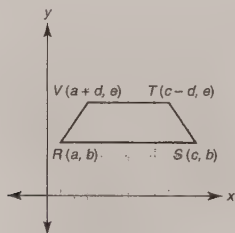
In Exercises 17 to 20, state whether the lines are parallel, perpendicular, the same, or none of these.

17. $2x + 3y = 6$ and $2x - 3y = 12$
18. $2x + 3y = 6$ and $4x + 6y = -12$
19. $2x + 3y = 6$ and $3x - 2y = 12$
20. $2x + 3y = 6$ and $4x + 6y = 12$
21. Find x such that the points A(x, 5), B(2, 3), and C(4, -5) are collinear.
22. Find a such that the points A(1, 3), B(4, 5), and C(a, a) are collinear.
23. Find x such that the line through (2, -3) and (3, 2) is perpendicular to the line through (-2, 4) and (x, -1).
24. Find x such that the line through (2, -3) and (3, 2) is parallel to the line through (-2, 4) and (x, -1).

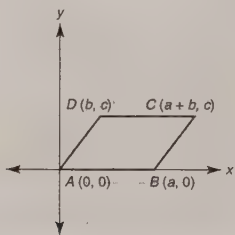
In Exercises 25 to 30, draw the line described.

25. Through (3, -2) and with $m = 2$
26. Through (-2, -5) and with $m = \frac{5}{7}$
27. With y intercept 5 and with $m = -\frac{3}{4}$
28. With x intercept -3 and with $m = 0.25$
29. Through (-2, 1) and parallel to the line $2x - y = 6$
30. Through (-2, 1) and perpendicular to the line that has intercepts $a = -2$ and $b = 3$
31. Use slopes to decide whether the triangle with vertices at (6, 5), (-3, 0), and (4, -2) is a right triangle.
32. If A(2, 2), B(7, 3), and C(4, x) are the vertices of a right triangle with right angle C, find the value of x .
33. ➤ If (2, 3), (5, -2), and (7, 2) are three vertices (not necessarily consecutive) of a parallelogram, find the possible locations of the fourth vertex.
34. Three vertices of rectangle ABCD are A(-5, 1), B(-2, -3), and C(6, y). Find the value of y and also the fourth vertex.

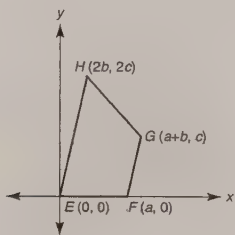
35. Show that quadrilateral $RSTV$ is an isosceles trapezoid.



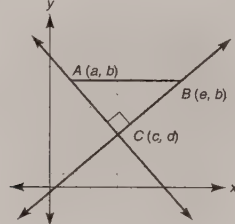
36. Show that quadrilateral $ABCD$ is a parallelogram.



37. Quadrilateral $EFGH$ has the vertices $E(0, 0)$, $F(a, 0)$, $G(a+b, c)$, and $H(2b, 2c)$. Verify that $EFGH$ is a trapezoid by showing that the slopes of two sides are equal.



38. Find an equation involving a , b , c , d , and e if $\overrightarrow{AC} \perp \overrightarrow{BC}$. (HINT: Use slopes.)



39. Prove that if two nonvertical lines are parallel, then their slopes are equal.

(HINT: See Figure 9.18.)

40. ➤ Prove that if two lines (neither horizontal nor vertical) are perpendicular, then the product of their slopes is -1 .

(HINT: See Figure 9.19. You need to show and use the fact that $\triangle ABC \sim \triangle EDC$.)



PREPARING TO DO ANALYTIC PROOFS

In this section, our goal is to lay the groundwork for analytic proofs of geometric theorems. An analytic proof requires the use of the coordinate system and the application of the formulas found in earlier sections of this chapter. Because of the need for these formulas, a summary follows. Be sure that you have these formulas memorized and know when and how to use them.

FORMULAS OF ANALYTIC GEOMETRY

Distance	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$
Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Special relationships for lines	$\ell_1 \parallel \ell_2 \leftrightarrow m_1 = m_2$ $\ell_1 \perp \ell_2 \leftrightarrow m_1 \cdot m_2 = -1$

NOTE: Neither ℓ_1 nor ℓ_2 is a vertical line in the preceding claims.

To see how the preceding list might be used in this and the next section, consider the following examples.

EXAMPLE 1

Suppose that you are to prove the following relationships:

- a) Two lines are parallel.
- b) Two lines are perpendicular.
- c) Two line segments are congruent.

Which formula(s) would you need to use? How would you complete your proof?

Solution

- a) Use the Slope Formula first, to find the slope of each line. Then show that the slopes are equal.
- b) Use the Slope Formula first, to find the slope of each line. Then show that $m_1 \cdot m_2 = -1$.
- c) Use the Distance Formula first, to find the length of each line segment. Then show that the resulting lengths are equal.

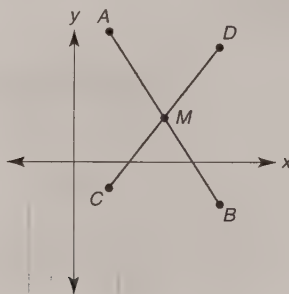
The following example has a proof that is subtle. A drawing is provided to help you understand the concept.

EXAMPLE 2

How can the Midpoint Formula be used to show that two line segments bisect each other?

Solution

If $\overline{AB}$ bisects $\overline{CD}$, and conversely, then M is the common midpoint of the two segments. (See Figure 9.22.) The Midpoint Formula is used to find the midpoint of each segment, and the results are then shown to be the same point. This establishes that each segment has been bisected by a point that is on the other segment.

**FIGURE 9.22****EXAMPLE 3**

Suppose that line ℓ_1 has slope $\frac{c}{d}$. Use this fact to identify the slopes of the following lines:

- a) ℓ_2 if $\ell_1 \parallel \ell_2$ b) ℓ_3 if $\ell_1 \perp \ell_3$

Solution

- a) $m_2 = \frac{c}{d}$ because $m_1 = m_2$ when $\ell_1 \parallel \ell_2$.
 b) $m_3 = -\frac{d}{c}$ because $m_1 \cdot m_3 = -1$ when $\ell_1 \perp \ell_3$.

EXAMPLE 4

What can you conclude if you know that the point (p, q) lies on the line $y = mx + b$?

Solution

Because (p, q) is on the line, it is also a solution for the equation $y = mx + b$. Therefore, $q = mp + b$.

To construct proofs of geometric theorems by analytic methods, we must use the hypothesis to determine the drawing. Unlike the drawings in Chapters 1–8, the figure must be placed in the coordinate system. Making the drawing requires careful placement of the figure and proper naming of the vertices, using coordinates of the rectangular system. The following guidelines should prove helpful in positioning the figure and in naming its vertices.

MAKING THE DRAWING FOR AN ANALYTIC PROOF

Some considerations for preparing the drawing:

1. Coordinates of the vertices must be general; for instance, you may use (a, b) as a vertex, but do *not* use $(2, 3)$.
2. Make the drawing satisfy the hypothesis without providing any additional qualities; if the theorem describes a rectangle, draw and label a rectangle but *not* a square.
3. For simplicity in your calculations, drop the figure into the rectangular coordinate system in such a manner that
 - a) As many 0 coordinates are used as possible.
 - b) The remaining coordinates represent positive numbers, because of your positioning of the figure in Quadrant I.

NOTE: In some cases, it is convenient to place a figure so that it has symmetry with respect to the y axis, in which case some negative coordinates are present.

4. When possible, use horizontal and vertical segments because you know their parallel and perpendicular relationships.
5. Use as few variable names in the coordinates as possible.

Now we consider Example 5, which clarifies the preceding list of suggestions. As you observe the drawing in each part of the example, imagine that $\triangle ABC$ has been cut out of a piece of cardboard and dropped into the coordinate system in the position indicated. Because we have freedom of placement, we want to choose the positioning that allows us the simplest possible solution.

EXAMPLE 5

Suppose that you are asked to make a drawing for the following theorem, which is to be proved analytically: "The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle." Explain why the placement of right $\triangle ABC$ in each part of Figure 9.23 is poorly done.

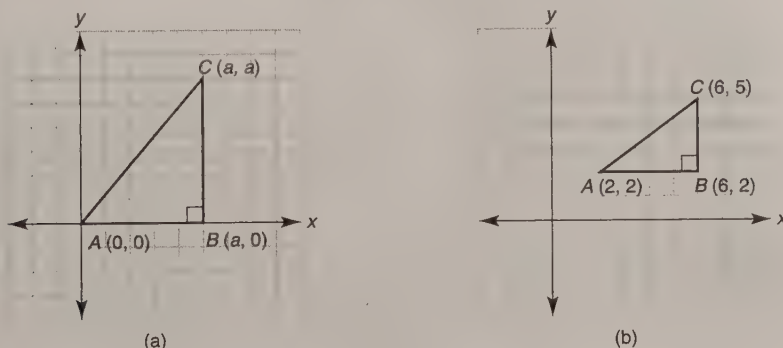
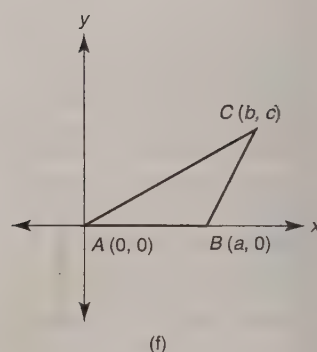
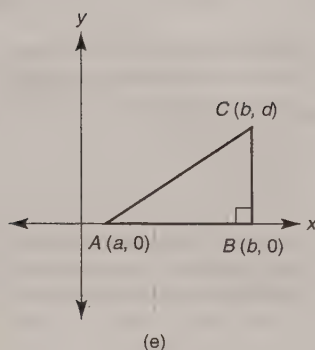
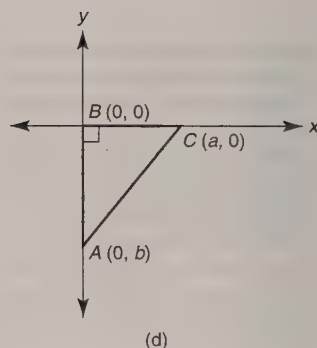
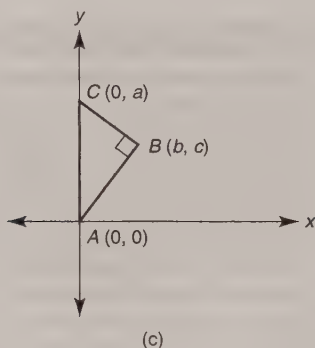
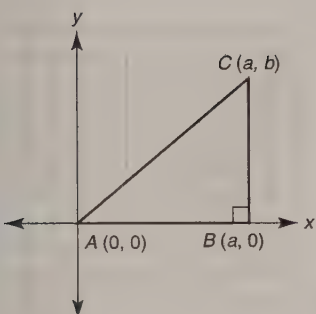


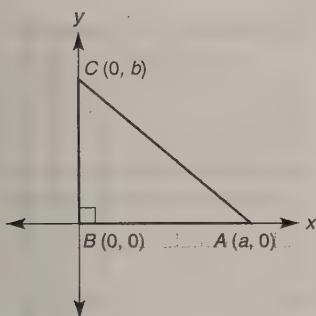
FIGURE 9.23

**FIGURE 9.23****Solution**

- a) The choice of vertices causes $AB = BC$, so the triangle is also an isosceles triangle. This contradicts point 2 of the list of suggestions.
- b) The coordinates are too specific! This contradicts point 1 of the list. A proof with these coordinates would *not* establish the general case.
- c) The drawing does not make use of horizontal and vertical lines to obtain the right angle. This violates point 4 of the list.
- d) This placement fails point 3 of the list, because b is a negative number. The length of $\overline{AB}$ would be $-b$, which could be confusing.
- e) This placement fails point 3 because we have not used as many 0 coordinates as we could have used. As we shall see, it also fails point 5.
- f) This placement fails point 2. The triangle is not a right triangle unless $a = b$.



(a)



(b)

FIGURE 9.24

In Example 5, we wanted to place $\triangle ABC$ so that we met as many of the conditions listed on page 433 as possible. Two convenient placements are given in Figure 9.24. The triangle in Figure 9.24(b) is slightly better than the one in 9.24(a) in that it uses four 0 coordinates rather than three. Another advantage of Figure 9.24(b) is that the placement forces angle B to be a right angle, because the x and y axes are perpendicular.

We now turn our attention to the conclusion of the theorem. A second list examines some considerations for proving statements analytically.

USING THE CONCLUSION TO DO AN ANALYTIC PROOF

Three considerations for using the conclusion as a guide:

1. If the conclusion is a conjunction " P and Q ," be sure to verify both parts of the conclusion.
2. The following pairings indicate how to prove statements of the type shown in the left column.

To prove the conclusion:

Use the:

- | | |
|--|---|
| a) Segments have equal lengths
(like $AB = CD$) | Distance Formula |
| b) Segments are parallel
(like $\overline{AB} \parallel \overline{CD}$) | Slope Formula
(need $m_{\overline{AB}} = m_{\overline{CD}}$) |
| c) Segments are perpendicular
(like $\overline{AB} \perp \overline{CD}$) | Slope Formula
(need $m_{\overline{AB}} \cdot m_{\overline{CD}} = -1$) |
| d) A segment is bisected | Distance Formula |
| e) Segments bisect each other | Midpoint Formula |
3. Anticipate the proof by thinking of the steps of the proof in reverse order; that is, reason "back" from the conclusion.

EXAMPLE 6

- Provide an ideal drawing for the following theorem: The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.
- By studying the theorem, name at least two of the formulas that will be used to complete the proof.

Solution

- We improve Figure 9.24(b) by giving the value $2a$ to the x coordinate of A and the value $2b$ to the y coordinate of C . (A factor of 2 makes it easier to calculate the midpoint M of $\overline{AC}$. See Figure 9.25 on the next page.)

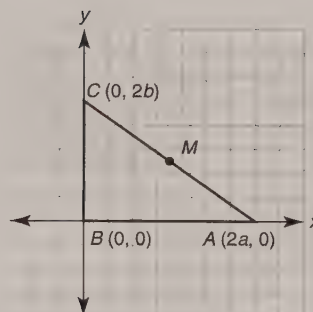


FIGURE 9.25

- b) The Midpoint Formula is applied to describe the midpoint of $\overline{AC}$. Using the formula,

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left(\frac{2a + 0}{2}, \frac{0 + 2b}{2} \right)$$

so the midpoint is (a, b) . The Distance Formula will also be needed because the theorem states that the distances from M to A , from M to B , and from M to C should all be equal.

The purpose of our next example is to demonstrate efficiency in the labeling of vertices.

EXAMPLE 7

If $MNPQ$ is a parallelogram in Figure 9.26, provide the coordinates of point P .

Solution

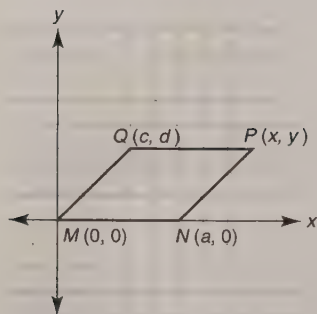


FIGURE 9.26

Consider $\square MNPQ$ in Figure 9.26. For the moment, we refer to point P as (x, y) .

Because $\overline{MN} \parallel \overline{QP}$, we have $m_{\overline{MN}} = m_{\overline{QP}}$. But $m_{\overline{MN}} = \frac{0-0}{a-0} = 0$ and $m_{\overline{QP}} = \frac{y-d}{x-c}$, so we are led to the equation

$$\frac{y-d}{x-c} = 0 \rightarrow y-d=0 \rightarrow y=d$$

Now P is described by (x, d) . Because $\overline{MQ} \parallel \overline{NP}$, we are also led to the equation equating slopes of these segments. But

$$m_{\overline{MQ}} = \frac{d-0}{c-0} = \frac{d}{c} \quad \text{and} \quad m_{\overline{NP}} = \frac{d-0}{x-a} = \frac{d}{x-a}$$

so
$$\frac{d}{c} = \frac{d}{x-a}$$

By using the Means-Extremes Property, we have

$$\begin{aligned} d(x - a) &= d \cdot c && \text{with } d \neq 0 \\ x - a &= c && \text{dividing by } d \\ x &= a + c && \text{adding } a \end{aligned}$$

Therefore, P is the point $(a + c, d)$.

In Example 7, we named the vertices of a parallelogram with the fewest possible letters. We now extend our result in Example 7 to allow for a rhombus—a parallelogram with two congruent adjacent sides.

EXAMPLE 8

Solution

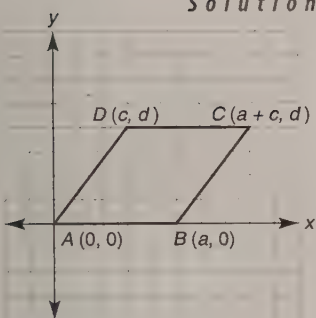


FIGURE 9.27

In Figure 9.27, find an equation that relates a , c , and d if $ABCD$ is a rhombus.

As we saw in Example 7, the coordinates of the vertices of $ABCD$ define a parallelogram. For emphasis, we note that $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$ because

$$m_{\overline{AB}} = m_{\overline{DC}} = 0 \quad \text{and} \quad m_{\overline{AD}} = m_{\overline{BC}} = \frac{d}{c}$$

For Figure 9.27 to represent a rhombus, it is necessary that $AB = AD$. Now $AB = a - 0 = a$ because $\overline{AB}$ is a horizontal segment. To find an expression for the length of $\overline{AD}$, we need to use the Distance Formula.

$$AD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AD = \sqrt{(c - 0)^2 + (d - 0)^2}$$

$$AD = \sqrt{c^2 + d^2}$$

Because $AB = AD$, we are led to $a = \sqrt{c^2 + d^2}$. Squaring, we have the desired relationship, $a^2 = c^2 + d^2$.

EXAMPLE 9

Solution

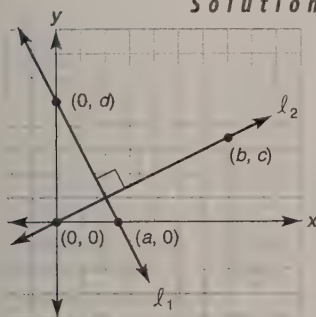


FIGURE 9.28

If $\ell_1 \perp \ell_2$ in Figure 9.28, find a relationship among the variables a , b , c , and d .

First we find the slopes of lines ℓ_1 and ℓ_2 . For ℓ_1 , we have

$$m_1 = \frac{0 - d}{a - 0} = -\frac{d}{a}$$

For ℓ_2 , we have

$$m_2 = \frac{c - 0}{b - 0} = \frac{c}{b}$$

With $\ell_1 \perp \ell_2$, it follows that $m_1 \cdot m_2 = -1$. Substituting the slopes found above into the equation $m_1 \cdot m_2 = -1$, we have

$$-\frac{d}{a} \cdot \frac{c}{b} = -1 \quad \text{so} \quad -\frac{dc}{ab} = -1$$

Equivalently, $\frac{dc}{ab} = 1$ and $dc = ab$.

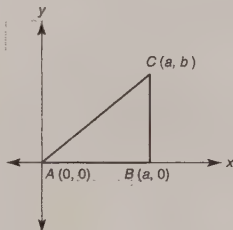


9.3 EXERCISES

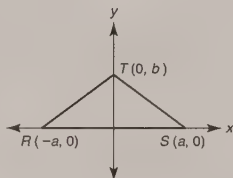
- Find an expression for:
 - The distance between $(a, 0)$ and $(0, a)$
 - The slope of the segment joining (a, b) and (c, d)
- Find the coordinates of the midpoint of the segment that joins the points
 - $(a, 0)$ and $(0, b)$
 - $(2a, 0)$ and $(0, 2b)$
- Find the slope of the line containing the points
 - $(a, 0)$ and $(0, a)$
 - $(a, 0)$ and $(0, b)$
- Find the slope of the line that is:
 - Parallel to the line containing $(a, 0)$ and $(0, b)$
 - Perpendicular to the line through $(a, 0)$ and $(0, b)$

In Exercises 5 to 10, the real numbers a, b, c , and d are positive.

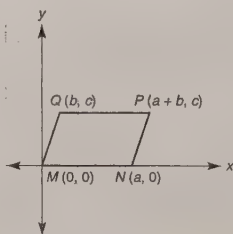
- Consider the triangle with vertices at $A(0, 0)$, $B(a, 0)$, and $C(a, b)$. Explain why $\triangle ABC$ is a right triangle.



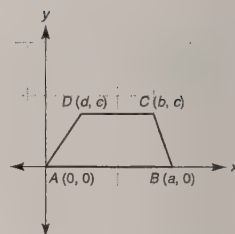
- Consider the triangle with vertices at $R(-a, 0)$, $S(a, 0)$, and $T(0, b)$. Explain why $\triangle RST$ is an isosceles triangle.



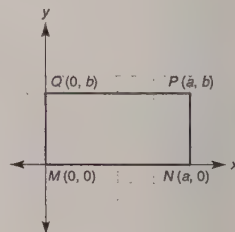
- Consider the quadrilateral with vertices at $M(0, 0)$, $N(a, 0)$, $P(a + b, c)$, and $Q(b, c)$. Explain why $MNPQ$ is a parallelogram.



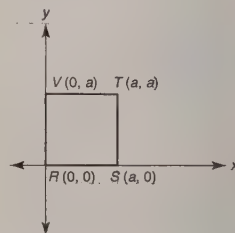
- Consider the quadrilateral with vertices at $A(0, 0)$, $B(a, 0)$, $C(b, c)$, and $D(d, c)$. Explain why $ABCD$ is a trapezoid.



- Consider the quadrilateral with vertices at $M(0, 0)$, $N(a, 0)$, $P(a, b)$, and $Q(0, b)$. Explain why $MNPQ$ is a rectangle.

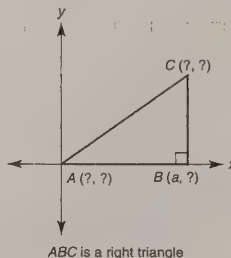


- Consider the quadrilateral with vertices at $R(0, 0)$, $S(a, 0)$, $T(a, a)$, and $V(0, a)$. Explain why $RSTV$ is a square.

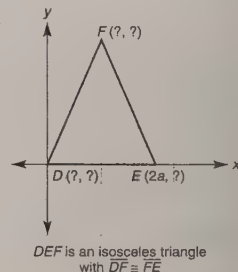


In Exercises 11 to 16, supply the missing coordinates for the vertices, using as few variables as possible.

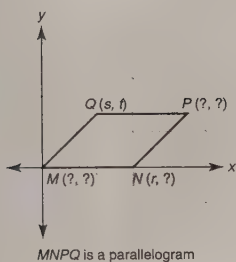
- 11.



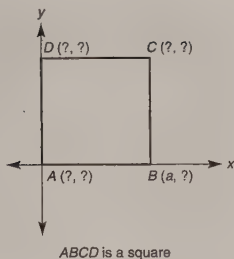
- 12.



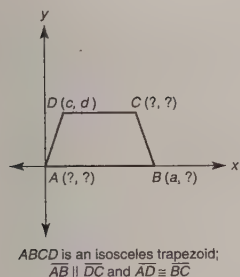
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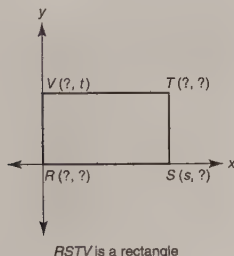
14.



15.

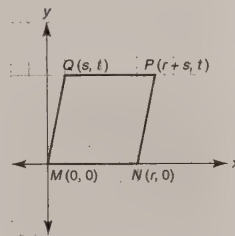


16.

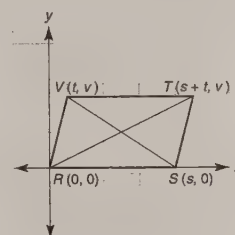


In Exercises 23 to 28, find the equation (relationship) requested, and eliminate fractions or radicals from the equation.

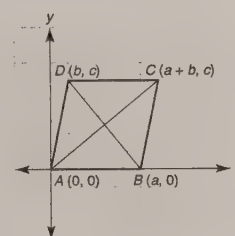
23. If $\square MNPQ$ is a rhombus, state an equation that relates r , s , and t .



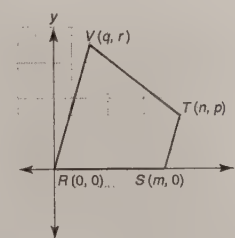
24. For $\square RSTV$, suppose that $RT = VS$. State an equation that relates s , t , and v .



25. For $\square ABCD$, suppose that diagonals $\overline{AC}$ and $\overline{DB}$ are $\perp$. State an equation that relates a , b , and c .



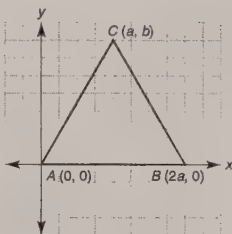
26. For quadrilateral $RSTV$, suppose that $\overline{RV} \parallel \overline{ST}$. State an equation that relates m , n , p , q , and r .



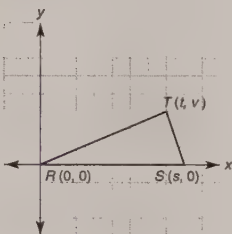
In Exercises 17 to 22, draw an ideally placed figure in the coordinate system; then name the coordinates of each vertex of the figure.

17. a) A square
b) A square (midpoints of sides are needed)
18. a) A rectangle
b) A rectangle (midpoints of sides are needed)
19. a) A parallelogram
b) A parallelogram (midpoints of sides are needed)
20. a) A triangle
b) A triangle (midpoints of sides are needed)
21. a) An isosceles triangle
b) An isosceles triangle (midpoints of sides are needed)
22. a) A trapezoid
b) A trapezoid (midpoints of sides are needed)

27. Suppose that $\triangle ABC$ is an equilateral triangle. State an equation that relates variables a and b .

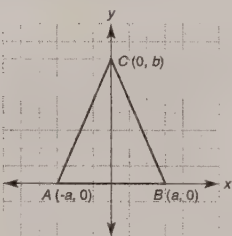


28. Suppose that $\triangle RST$ is an isosceles triangle, with $\overline{RS} \cong \overline{RT}$. State an equation that relates s , t , and v .



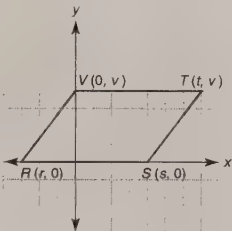
29. The drawing shows isosceles $\triangle ABC$ with $\overline{AC} \cong \overline{BC}$.

- What type of number is a ?
- What type of number is $-a$?
- Find an expression for the length of $\overline{AB}$.



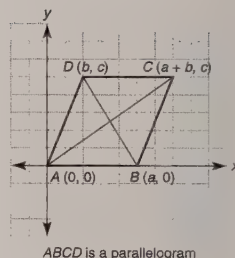
30. The drawing shows parallelogram $RSTV$.

- What type of number is r ?
- Find an expression for RS .
- Describe the coordinate t in terms of the other variables.



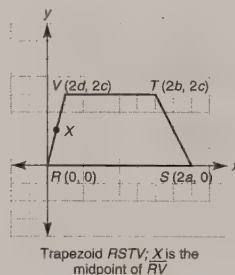
31. Which formula would you use to establish each of the following claims?

- $\overline{AC} \perp \overline{DB}$
- $AC = DB$
- $\overline{DB}$ and $\overline{AC}$ bisect each other
- $\overline{AD} \parallel \overline{BC}$



32. Which formula would you use to establish each of the following claims?

- The coordinates of X are (d, c)
- $m_{\overline{VT}} = 0$
- $\overline{VT} \parallel \overline{RS}$
- The length of $\overline{RV}$ is $2\sqrt{d^2 + c^2}$



In Exercises 33 to 36, draw and label a well-placed figure in the coordinate system for each theorem. Do not attempt to prove the theorem!

- The line segment joining the midpoints of the two non-parallel sides of a trapezoid is parallel to the bases of the trapezoid.
- If the midpoints of the sides of a quadrilateral are joined in order, the resulting quadrilateral is a parallelogram.
- The diagonals of a rectangle are equal in length.
- The diagonals of a rhombus are perpendicular to each other.



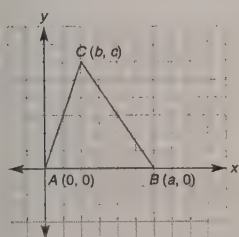
ANALYTIC PROOFS

When we use algebra along with the rectangular coordinate system to prove a geometric theorem, the proof is termed **analytic**. The analytic (algebraic) approach relies heavily on the placement of the figure in the coordinate system and on the application of the Distance Formula, the Midpoint Formula, or the Slope Formula (at the appropriate time). In order to contrast analytic proof with **synthetic** proof (the two-column or paragraph proofs used in earlier chapters), we repeat in this section some of our earlier theorems.

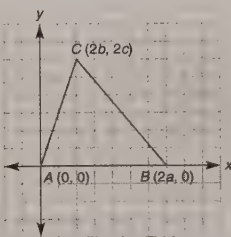
In Section 9.3, we saw how to place triangles having special qualities in the coordinate system. We review this information in Table 9.1, and then, in Example 1, we consider the proof of a theorem involving triangles. In Table 9.1, you will find that the figure determined by any positive choices of a , b , and c matches the type of triangle described.

TABLE 9.1

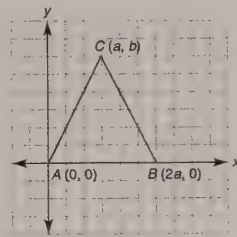
Analytic Proof: Suggestions for Placement of the Triangle



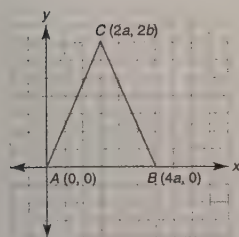
General Triangle



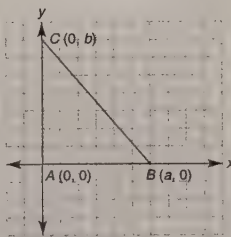
General Triangle
(Midpoints)



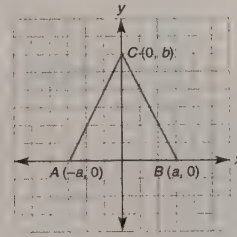
Isosceles Triangle



Isosceles Triangle
(Midpoints)



Right Triangle



Equilateral Triangle
(where $2a = \sqrt{a^2 + b^2}$,
so $3a^2 = b^2$)

EXAMPLE 1

Prove the following theorem by the analytic method (see Figure 9.29).

THEOREM 9.4.1: The line segment determined by the mid-points of two sides of a triangle is parallel to the third side.

Plan: Use the Slope Formula; if $m_{\overline{MN}} = m_{\overline{AC}}$, then $\overline{MN} \parallel \overline{AC}$.

Proof

As shown in Figure 9.29, $\triangle ABC$ has vertices at $A(0, 0)$, $B(2a, 0)$, and $C(2b, 2c)$. With M the midpoint of $\overline{BC}$,

$$M = \left(\frac{2a + 2b}{2}, \frac{0 + 2c}{2} \right) \text{ which simplifies to } (a + b, c)$$

With N the midpoint of $\overline{AB}$,

$$N = \left(\frac{0 + 2a}{2}, \frac{0 + 0}{2} \right) \text{ which simplifies to } (a, 0)$$

Next we apply the Slope Formula to determine $m_{\overline{MN}}$ and $m_{\overline{AC}}$. Now $m_{\overline{MN}} = \frac{c - 0}{(a + b) - a} = \frac{c}{b}$; also, $m_{\overline{AC}} = \frac{2c - 0}{2b - 0} = \frac{2c}{2b} = \frac{c}{b}$. Because $m_{\overline{MN}} = m_{\overline{AC}}$, we see that $\overline{MN} \parallel \overline{AC}$.

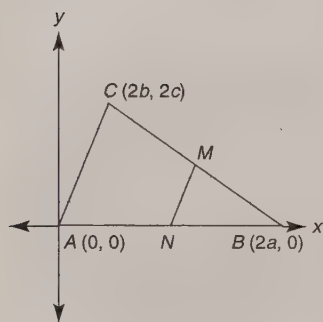


FIGURE 9.29

In Table 9.2, we review possible placements of various quadrilaterals in the rectangular coordinate system. Again, the variables named represent positive numbers. (Table 9.2 follows Example 2.)

As we did in Example 1, we include a “plan” for Example 2. Although no plan is shown for Example 3 or Example 4, one is necessary before the proof can be written.

EXAMPLE 2

Prove the following theorem by the analytic method. (See Figure 9.30.)

THEOREM 9.4.2: The diagonals of a parallelogram bisect each other.

Plan: Use the Midpoint Formula to show that the two diagonals have a common midpoint. Use a factor of 2 in the coordinates.

Proof

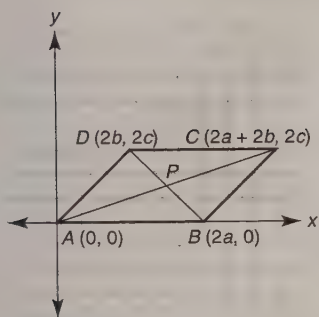
In Figure 9.30, quadrilateral $ABCD$ is a parallelogram. The diagonals intersect at point P . By the Midpoint Formula, we have

$$\begin{aligned} M_{\overline{AC}} &= \left(\frac{0 + (2a + 2b)}{2}, \frac{0 + 2c}{2} \right) \\ &= (a + b, c) \end{aligned}$$

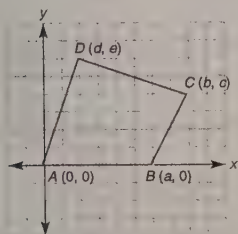
Also, the midpoint of $\overline{DB}$ is

$$\begin{aligned} M_{\overline{DB}} &= \left(\frac{2a + 2b}{2}, \frac{0 + 2c}{2} \right) \\ &= (a + b, c) \end{aligned}$$

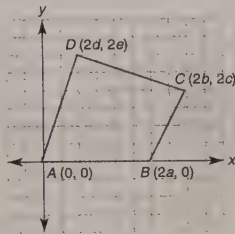
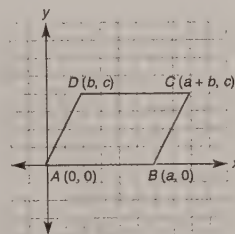
Thus $(a + b, c)$ is the common midpoint of the two diagonals and must be the point of intersection of $\overline{AC}$ and $\overline{DB}$. Then $\overline{AC}$ and $\overline{DB}$ must bisect each other at point P .

**FIGURE 9.30****TABLE 9.2**

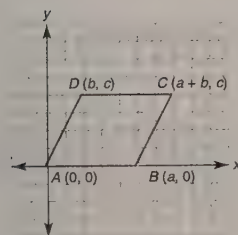
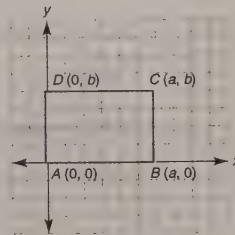
Analytic Proof: Suggestions for Placement of the Quadrilateral



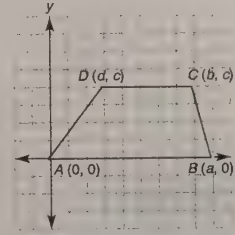
General Quadrilateral

General Quadrilateral
(Midpoints)

Parallelogram

Rhombus
(where $a = \sqrt{b^2 + c^2}$
so $a^2 = b^2 + c^2$)

Rectangle



Trapezoid

The proof of Theorem 9.4.2 is not unique! In Section 9.5, we could prove Theorem 9.4.2 by using a three-step proof:

1. Find the equations of the two lines.
2. Determine the point of intersection of these lines.
3. Show that this point of intersection is the common midpoint.

But the phrase *bisect each other* in the theorem implied the use of the Midpoint Formula. Our approach to Example 2 was far easier and just as correct. The use of the Midpoint Formula is generally the best approach when the phrase *bisect each other* appears in the statement of a theorem.

We now outline the method of analytic proof.

COMPLETING AN ANALYTIC PROOF

1. Read the theorem carefully to distinguish the hypothesis and the conclusion. The hypothesis characterizes the figure to use.
2. Use the hypothesis (and nothing more) to determine a convenient placement of the figure in the rectangular coordinate system. Then label the figure. See Tables 9.1 and 9.2.
3. If any special quality is provided by the hypothesis, be sure to state this early in the proof. (For example, a rhombus should be described as a parallelogram that has two congruent adjacent sides.)
4. Study the conclusion, and devise a plan to prove this claim; this may involve reasoning back from the conclusion step by step until the hypothesis is reached.
5. Write the proof, being careful to order each statement properly and to justify each statement.

EXAMPLE 3

Prove Theorem 9.4.3 by the analytic method. (See Figure 9.31 on the following page.)

THEOREM 9.4.3: The diagonals of a rhombus are perpendicular.

Proof

In Figure 9.31, $ABCD$ has the coordinates of a parallelogram. Because $\square ABCD$ is a rhombus, $AB = AD$. Then $a = \sqrt{b^2 + c^2}$ by the Distance Formula, and squaring gives $a^2 = b^2 + c^2$. The Slope Formula leads to

$$m_{\overline{AC}} = \frac{c - 0}{(a + b) - 0} \quad \text{and} \quad m_{\overline{DB}} = \frac{0 - c}{a - b}$$

$$\text{so} \quad m_{\overline{AC}} = \frac{c}{a + b} \quad \text{and} \quad m_{\overline{DB}} = \frac{-c}{a - b}$$

REMINDER

We prove that lines are perpendicular by showing that the product of their slopes is -1 .

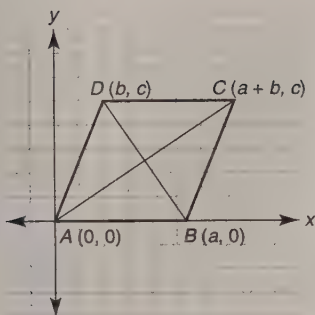


FIGURE 9.31

Then the product of the slopes of the diagonals is

$$\begin{aligned}
 m_{\overline{AC}} \cdot m_{\overline{DB}} &= \frac{c}{a+b} \cdot \frac{-c}{a-b} \\
 &= \frac{-c^2}{a^2 - b^2} \\
 &= \frac{-c^2}{(b^2 + c^2) - b^2} \quad (\text{replaced } a^2 \text{ by } b^2 + c^2) \\
 &= \frac{-c^2}{c^2} = -1
 \end{aligned}$$

Then $\overline{AC} \perp \overline{DB}$ because the product of their slopes equals -1 .

In Example 3, we had to use the condition that two adjacent sides of the rhombus were congruent in order to complete the proof. Had that condition been omitted, the product of slopes could not have been shown to equal -1 . In general, the diagonals of a parallelogram are *not* perpendicular.

In our next example, we consider the proof of the converse of an earlier theorem. Although it is easy to complete an analytic proof of the statement “The diagonals of a rectangle are equal in length,” the proof of the converse is not as straightforward.

EXAMPLE 4

Prove Theorem 9.4.4 by the analytic method. (See Figure 9.32.)

THEOREM 9.4.4: If the diagonals of a parallelogram are equal in length, then the parallelogram is a rectangle.

Proof

In parallelogram $ABCD$ in Figure 9.32, $AC = DB$. Applying the Distance Formula,

$$AC = \sqrt{[(a+b)-0]^2 + (c-0)^2}$$

and

$$DB = \sqrt{(a-b)^2 + (0-c)^2}$$

Then it follows that

$$\sqrt{(a+b)^2 + c^2} = \sqrt{(a-b)^2 + (-c)^2}$$

$$(a+b)^2 + c^2 = (a-b)^2 + (-c)^2 \quad \text{squaring}$$

$$a^2 + 2ab + b^2 + c^2 = a^2 - 2ab + b^2 + c^2 \quad \text{simplifying}$$

$$4ab = 0$$

$$a \cdot b = 0 \quad \text{dividing by 4}$$

Thus

$$a = 0 \quad \text{or} \quad b = 0$$

Because $a \neq 0$ (otherwise points A and B would coincide), it is necessary that $b = 0$, so point D is on the y axis. The resulting coordinates of the figure are $A(0, 0)$, $B(a, 0)$, $C(a, c)$, and $D(0, c)$. Then $ABCD$ must be a rectangle with a right angle at A because $\overline{AB}$ is horizontal and $\overline{AD}$ is vertical.

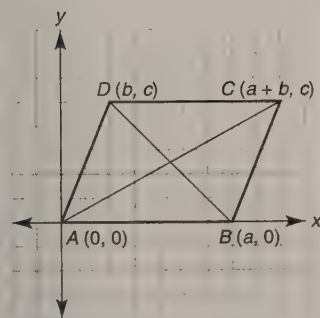


FIGURE 9.32

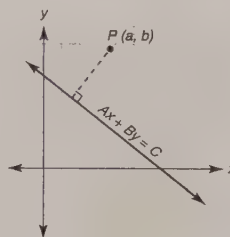


9.4 EXERCISES

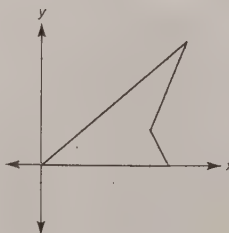
In Exercises 1 to 17, complete an analytic proof for each theorem.

- The diagonals of a rectangle are equal in length.
- The opposite sides of a parallelogram are equal in length.
- The diagonals of a square are perpendicular bisectors of each other.
- The diagonals of an isosceles trapezoid are equal in length.
- The median from the vertex of an isosceles triangle to the base is perpendicular to the base.
- The medians to the congruent sides of an isosceles triangle are equal in length.
- The segments that join the midpoints of the consecutive sides of a quadrilateral form a parallelogram.
- The segments that join the midpoints of the opposite sides of a quadrilateral bisect each other.
- The segments that join the midpoints of the consecutive sides of a rectangle form a rhombus.
- The segments that join the midpoints of the consecutive sides of a rhombus form a rectangle.
- The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.
- The median of a trapezoid is parallel to the bases of the trapezoid and has a length equal to one-half the sum of the lengths of the two bases.
- The segment that joins the midpoints of two sides of a triangle is parallel to the third side and has a length equal to one-half the length of the third side.
- The perpendicular bisector of the base of an isosceles triangle contains the vertex of the triangle.
- If the midpoint of one side of a rectangle is joined to the endpoints of the opposite side, then an isosceles triangle is formed.
- If the median to one side of a triangle is also an altitude of the triangle, then the triangle is isosceles.

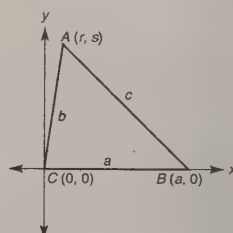
- If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.
- Use the analytic method to decide what type of quadrilateral is formed when the midpoints of the consecutive sides of a parallelogram are joined by line segments.
- Use the analytic method to decide what type of triangle is formed when the midpoints of the sides of an isosceles triangle are joined by line segments.
- Describe the steps of the procedure that enables us to find the distance from a point $P(a, b)$ to the line $Ax + By = C$.



- Would the theorem of Exercise 7 remain true for a concave quadrilateral like the one shown?



Exercise 21



Exercise 22

- Complete an analytic proof of the following theorem: In a triangle that has sides of lengths a , b , and c , if $c^2 = a^2 + b^2$, then the triangle is a right triangle.



EQUATIONS OF LINES

In Section 9.2, we saw that equations such as $2x + 3y = 6$ and $4x - 12y = 60$ have graphs that are lines. To graph the equation of the general form $Ax + By = C$, that equation is often replaced with an equivalent equation of the form $y = mx + b$. For instance, $2x + 3y = 6$ can be transformed into $y = -\frac{2}{3}x + 2$; such equations are known as *equivalent* because their ordered-pair solutions (and graphs) are identical. In particular, we must express a linear equation in the form $y = mx + b$ in order to plot it on a graphing calculator.

EXAMPLE 1

Write the equation $4x - 12y = 60$ in the form $y = mx + b$.

Solution

Given $4x - 12y = 60$, we subtract $4x$ from each side of the equation to obtain $-12y = -4x + 60$. Dividing by -12 yields

$$\frac{-12y}{-12} = \frac{-4x}{-12} + \frac{60}{-12}$$

Then $y = \frac{1}{3}x - 5$.

SLOPE-INTERCEPT FORM OF A LINE

We now turn our attention to a method for finding the equation of a line. In the following technique, the equation can be found if the slope and the y -intercept of the line are known. The form $y = mx + b$ is known as the Slope-Intercept Form of a line.

THEOREM 9.5.1: (Slope-Intercept Form of a Line) The line whose slope is m and whose y -intercept is b has the equation $y = mx + b$.

Proof

Consider the line whose slope is m (see Figure 9.33). Using the Slope Formula

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

we designate (x, y) as P_2 and $(0, b)$ as P_1 . Then

$$m = \frac{y - b}{x - 0} \text{ or } m = \frac{y - b}{x}$$

Multiplying by x , we have $mx = y - b$. Then $mx + b = y$ so that $y = mx + b$.

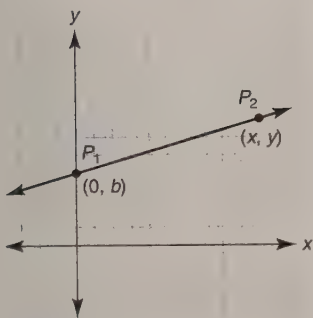


FIGURE 9.33

EXAMPLE 2

Find the general equation $Ax + By = C$ for the line with slope $m = -\frac{2}{3}$ and y intercept -2 .

Solution With $y = mx + b$, we have

$$y = -\frac{2}{3}x - 2$$

Multiplying by 3, we obtain

$$3y = -2x - 6 \quad \text{so that} \quad 2x + 3y = -6$$

NOTE: An equivalent and correct solution is $-2x - 3y = 6$.

It is often easier to graph an equation if it is in the form $y = mx + b$. When an equation can be changed to this form, we know that its graph is a line that has slope m and contains $(0, b)$.

EXAMPLE 3

Draw the graph of $\frac{1}{2}x + y = 3$.

Solution

Solving for y , we have $y = -\frac{1}{2}x + 3$. Then $m = -\frac{1}{2}$ and the y intercept is 3. We first plot the point $(0, 3)$. Because $m = -\frac{1}{2}$ or $\frac{-1}{2}$, the vertical change -1 corresponds to a horizontal change of $+2$. Thus the second point is located 1 unit down from and 2 units to the right of the first point. The line is drawn in Figure 9.34.

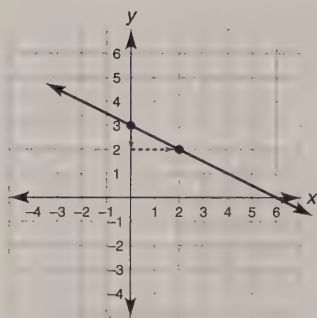


FIGURE 9.34

POINT-SLOPE FORM OF A LINE

If a point other than the y intercept of the line is known, we generally do not use the Slope-Intercept Form to find the equation of the line. Instead, the Point-Slope Form of the equation of a line is used. This form is also used when the coordinates of two points of the line are known; in that case, the value of m is found by the Slope Formula. The form $y - y_1 = m(x - x_1)$ is known as the Point-Slope Form of a line.

THEOREM 9.5.2: (Point-Slope Form of a Line) The line that has slope m and contains the point (x_1, y_1) has the equation

$$y - y_1 = m(x - x_1)$$

Proof

Let P_1 be the given point (x_1, y_1) on the line, and let P_2 be (x, y) , which represents any other point on the line. (See Figure 9.35.) Using the Slope Formula, we have

$$m = \frac{y - y_1}{x - x_1}$$

Multiplying the equation by $(x - x_1)$ yields

$$m(x - x_1) = y - y_1$$

It follows that

$$y - y_1 = m(x - x_1)$$

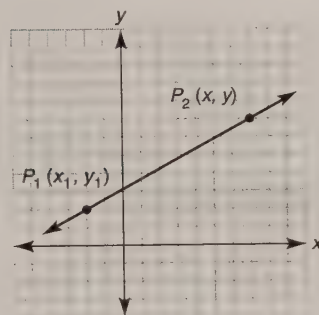


FIGURE 9.35

EXAMPLE 4

Find the general equation, $Ax + By = C$, for the line that has the slope $m = 2$ and contains the point $(-1, 3)$.

Solution

Applying the Point-Slope Form, the line in Figure 9.36 has the equation

$$\begin{aligned} y - 3 &= 2[x - (-1)] \\ y - 3 &= 2(x + 1) \\ y - 3 &= 2x + 2 \\ -2x + y &= 5 \end{aligned}$$

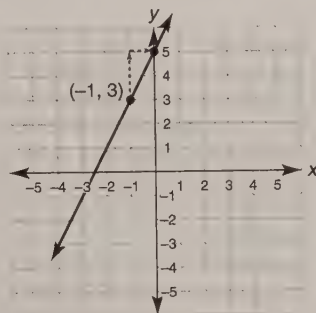


FIGURE 9.36

NOTE: An equivalent answer for Example 4 is the equation $2x - y = -5$. The form $y = 2x + 5$ emphasizes that the slope is $m = 2$ and the y -intercept is $(0, 5)$.

EXAMPLE 5

Find an equation for the line containing the points $(-1, 2)$ and $(4, 1)$.

Solution

To use the Point-Slope Form, we need to know the slope of the line (see Figure 9.37). When we choose $P_1(-1, 2)$ and $P_2(4, 1)$, the Slope Formula reads

$$m = \frac{1 - 2}{4 - (-1)} = \frac{-1}{5} = -\frac{1}{5}$$

Therefore,
$$y - 2 = -\frac{1}{5}[x - (-1)]$$

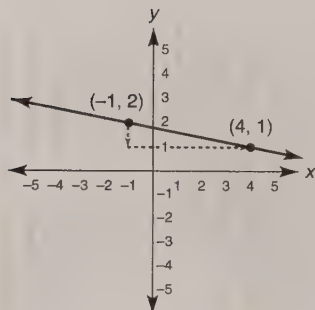


FIGURE 9.37

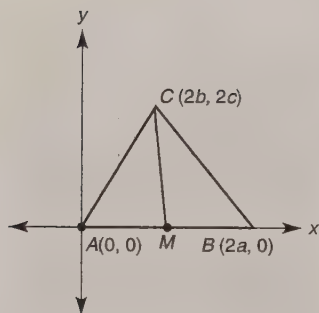
EXAMPLE 6**Solution**

FIGURE 9.38

Then $y - 2 = -\frac{1}{5}[x + 1]$

and $y - 2 = -\frac{1}{5}x - \frac{1}{5}$

Multiplying the equation by 5, we obtain

$$5y - 10 = -1x - 1 \text{ so that } x + 5y = 9$$

NOTE: Other correct forms of the answer are $-x - 5y = -9$ and $y = -\frac{1}{5}x + \frac{9}{5}$. In any correct form, the given points must satisfy the equation.

In Example 6, we use the Point-Slope Form to find an equation for a median of a triangle.

For $\triangle ABC$, the vertices are $A(0, 0)$, $B(2a, 0)$, and $C(2b, 2c)$. Find the equation of median $\overline{CM}$ in the form $y = mx + b$. See Figure 9.38.

For $\overline{CM}$ to be a median of $\triangle ABC$, M must be the midpoint of $\overline{AB}$. Then

$$M = \left(\frac{0 + 2a}{2}, \frac{0 + 0}{2} \right) = (a, 0)$$

To determine an equation for $\overline{CM}$, we also need to know its slope. With $M(a, 0)$ and $C(2b, 2c)$ on $\overline{CM}$, the slope is $m_{\overline{CM}} = \frac{2c - 0}{2b - a}$ or $\frac{2c}{2b - a}$. With $M = (a, 0)$ as the point on the line, $y - y_1 = m(x - x_1)$ becomes

$$y - 0 = \frac{2c}{2b - a}(x - a) \quad \text{or} \quad y = \frac{2c}{2b - a}x - \frac{2ac}{2b - a}$$

SOLVING SYSTEMS OF EQUATIONS

In earlier chapters, we solved systems of equations such as

$$\begin{aligned} x + 2y &= 6 \\ 2x - y &= 7 \end{aligned}$$

by using the Addition Property or the Subtraction Property of Equality. We review the method in Example 7. The solution for the system is an ordered pair; in fact, the point is the intersection of the graphs of the given equations.

EXAMPLE 7

Solve the following system by using algebra:

$$\begin{cases} x + 2y = 6 \\ 2x - y = 7 \end{cases}$$

Solution

When we multiply the second equation by 2, the system becomes

$$\begin{cases} x + 2y = 6 \\ 4x - 2y = 14 \end{cases}$$

Adding these equations yields $5x = 20$ so that $x = 4$. Substituting $x = 4$ into the first equation, we get $4 + 2y = 6$, so $2y = 2$. Then $y = 1$. The solution is the ordered pair $(4, 1)$.

Another method for solving a system of equations is geometric and requires graphing. Solving by graphing amounts to finding the point of intersection of the linear graphs. That point is the ordered pair that is the common solution (when one exists) for the two equations. Notice that Example 8 repeats the system of Example 7.

EXAMPLE 8

Solve the following system by graphing:

$$\begin{cases} x + 2y = 6 \\ 2x - y = 7 \end{cases}$$

Solution

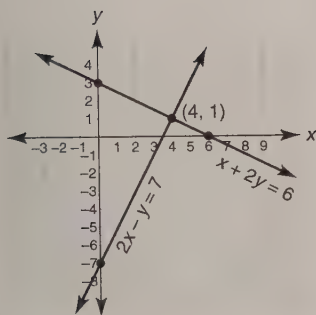


FIGURE 9.39

Each equation is changed to the form $y = mx + b$ so that the slope and the y-intercept are used in graphing:

$$x + 2y = 6 \rightarrow 2y = -1x + 6 \rightarrow y = -\frac{1}{2}x + 3$$

$$2x - y = 7 \rightarrow -y = -2x + 7 \rightarrow y = 2x - 7$$

The graph of $y = -\frac{1}{2}x + 3$ is a line with y-intercept 3 and slope $m = -\frac{1}{2}$. The graph of $y = 2x - 7$ is a line with y-intercept -7 and slope $m = 2$.

The graphs are drawn in the same coordinate system. See Figure 9.39. The point of intersection $(4, 1)$ is the common solution for each of the given equations and thus is the solution of the system.

NOTE: To check the result of Examples 7 and 8, we show that $(4, 1)$ satisfies both of the given equations:

$$x + 2y = 6 \rightarrow 4 + 2(1) = 6 \text{ is true}$$

$$2x - y = 7 \rightarrow 2(4) - 1 = 7 \text{ is true}$$

For the solution to be verified, both statements resulting from substitution into the given equations must be true. If either or both statements are false, we do not have the solution of the system.

Advantages of the method of solving a system of equations by graphing include the following:

1. It is easy to understand why a system such as

$$\begin{cases} x + 2y = 6 \\ 2x - y = 7 \end{cases} \quad \text{can be replaced by} \quad \begin{cases} x + 2y = 6 \\ 4x - 2y = 14 \end{cases}$$

when we are solving by addition or subtraction. The graphs of $2x - y = 7$ and $4x - 2y = 14$ are the same line; each can be changed to the form $y = 2x - 7$.

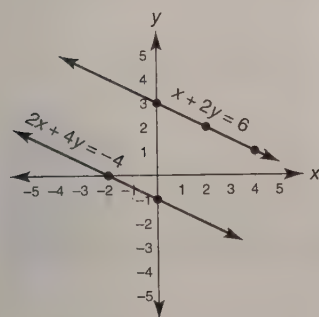


FIGURE 9.40

2. It is easy to understand why a system such as

$$\begin{cases} x + 2y = 6 \\ 2x + 4y = -4 \end{cases}$$

has no solution. In Figure 9.40, the graphs are parallel lines.

The first equation is equivalent to $y = -\frac{1}{2}x + 3$ and the second equation can be changed to $y = -\frac{1}{2}x - 1$. Both lines have slope $m = -\frac{1}{2}$ and therefore are parallel.

Algebraic substitution can also be used to solve a system of equations. In our approach, we write each equation in the form $y = mx + b$ and then equate the expressions for y . Once the x coordinate of the solution is known, we substitute this value of x into either equation to find the value of y .

EXAMPLE 9

Use substitution to solve: $\begin{cases} x + 2y = 6 \\ 2x - y = 7 \end{cases}$

Solution

Solving for y , we have

$$\begin{aligned} x + 2y = 6 &\rightarrow 2y = -x + 6 \rightarrow y = -\frac{1}{2}x + 3 \\ 2x - y = 7 &\rightarrow -y = -2x + 7 \rightarrow y = 2x - 7 \end{aligned}$$

Equating the expressions for y yields $-\frac{1}{2}x + 3 = 2x - 7$. Then $-2\frac{1}{2}x = -10$, or $-2.5x = -10$. Dividing by -2.5 , we obtain $x = 4$. Substitution of 4 for x in the equation $y = 2x - 7$ leads to $y = 2(4) - 7$, so $y = 1$. The solution is the ordered pair $(4, 1)$.

NOTE: Substitution of $x = 4$ into the equation $y = -\frac{1}{2}x + 3$ would lead to the same value of y , namely $y = 1$. Thus one can substitute into either equation.

The method illustrated in Example 9 is also used in our final example. In the proof of Theorem 9.5.3, we use equations of lines to determine the centroid of a triangle.

EXAMPLE 10

Formulate a plan to complete the proof of Theorem 9.5.3. See Figure 9.41.

THEOREM 9.5.3: The three medians of a triangle are concurrent at a point that is two-thirds the distance from any vertex to the midpoint of the opposite side.

Solution

The proof can be completed as follows:

1. Find the coordinates of the two midpoints X and Y . See Figures 9.41(a) and 9.41(b). Note that

$$X = (a + b, c) \quad \text{and} \quad Y = (b, c)$$

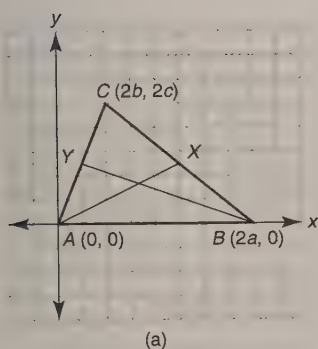
2. Find the equations of the lines containing $\overline{AX}$ and $\overline{BY}$. The equations for $\overline{AX}$ and $\overline{BY}$ are $y = \frac{c}{a+b}x$ and $y = \frac{-c}{2a-b}x + \frac{2ac}{2a-b}$, respectively.
3. Find the point of intersection Z of $\overline{AX}$ and $\overline{BY}$. Solving the system provides the solution

$$Z = \left(\frac{2}{3}(a+b), \frac{2}{3}c \right)$$

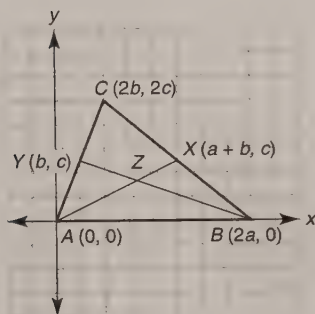
4. It can now be shown that $AZ = \frac{2}{3} \cdot AX$ and $BZ = \frac{2}{3} \cdot BY$. See Figure 9.41(b), in which we can show that

$$AZ = \frac{2}{3}\sqrt{(a+b)^2 + c^2} \quad \text{and} \quad AX = \sqrt{(a+b)^2 + c^2}$$

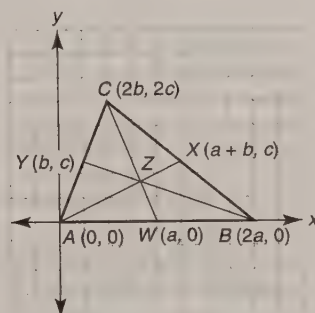
5. It can also be shown that point Z lies on the third median $\overline{CW}$, whose equation is $y = \frac{2c}{2b-a}(x-a)$. See Figure 9.41(c).
6. We can also show that $CZ = \frac{2}{3} \cdot CW$, which completes the proof.



(a)



(b)



(c)

FIGURE 9.41

9.5 EXERCISES

In Exercises 1 to 4, use division to write an equation of the form $Ax + By = C$ that is equivalent to the one provided. Then write the given equation in the form $y = mx + b$.

1. $8x + 16y = 48$

2. $15x - 35y = 105$

3. $-6x + 18y = -240$

4. $27x - 36y = 108$

In Exercises 5 to 8, draw the graph of each equation by using the method of Example 3.

5. $y = 2x - 3$

6. $y = -2x + 5$

7. $\frac{2}{5}x + y = 6$

8. $3x - 2y = 12$

In Exercises 9 to 24, find the equation of the line described. Leave the solution in the form $Ax + By = C$.

9. The line has slope $m = -\frac{2}{3}$ and contains $(0, 5)$.
10. The line has slope $m = -3$ and contains $(0, -2)$.
11. The line contains $(2, 4)$ and $(0, 6)$.
12. The line contains $(-2, 5)$ and $(2, -1)$.
13. The line contains $(0, -1)$ and $(3, 1)$.
14. The line contains $(-2, 0)$ and $(4, 3)$.
15. The line contains $(0, b)$ and $(a, 0)$.
16. The line contains (b, c) and has slope d .
17. The line has intercepts $a = 2$ and $b = -2$.
18. The line has intercepts $a = -3$ and $b = 5$.
19. The line goes through $(-1, 5)$ and is parallel to the line $5x + 2y = 10$.
20. The line goes through $(0, 3)$ and is parallel to the line $3x + y = 7$.
21. The line goes through $(0, -4)$ and is perpendicular to the line $y = \frac{3}{4}x - 5$.
22. The line goes through $(2, -3)$ and is perpendicular to the line $2x - 3y = 6$.
23. The line is the perpendicular bisector of the line segment that joins $(3, 5)$ and $(5, -1)$.
24. The line is the perpendicular bisector of the line segment that joins $(-4, 5)$ and $(1, 1)$.

In Exercises 25 and 26, find the equation of the line in the form $y = mx + b$.

25. The line contains (g, h) and is perpendicular to the line $y = \frac{a}{b}x + c$.
26. The line contains (g, h) and is parallel to the line $y = \frac{a}{b}x + c$.

In Exercises 27 to 32, use graphing to find the point of intersection of the two lines. Use Example 8 as a guide.

27. $y = \frac{1}{2}x - 3$ and $y = \frac{1}{3}x - 2$
28. $y = 2x + 3$ and $y = 3x$
29. $2x + y = 6$ and $3x - y = 19$
30. $\frac{1}{2}x + y = -3$ and $\frac{3}{4}x - y = 8$
31. $4x + 3y = 18$ and $x - 2y = 10$
32. $2x + 3y = 3$ and $3x - 2y = 24$

In Exercises 33 to 38, use algebra to find the point of intersection of the two lines whose equations are provided. Use Example 7 as a guide.

33. $2x + y = 8$ and $3x - y = 7$
34. $2x + 3y = 7$ and $x + 3y = 2$
35. $2x + y = 11$ and $3x + 2y = 16$
36. $x + y = 1$ and $4x - 2y = 1$
37. $2x + 3y = 4$ and $3x - 4y = 23$
38. $5x - 2y = -13$ and $3x + 5y = 17$

In Exercises 39 to 42, use substitution to solve the system. Use Example 9 as a guide.

39. $y = \frac{1}{2}x - 3$ and $y = \frac{1}{3}x - 2$
40. $y = 2x + 3$ and $y = 3x$
41. $y = a$ and $y = bx + c$
42. $x = d$ and $y = fx + g$
43. For $\triangle ABC$, the vertices are $A(0, 0)$, $B(a, 0)$, and $C(b, c)$. In terms of a , b , and c , find the coordinates of the orthocenter of $\triangle ABC$. (The orthocenter is the point of concurrence for the altitudes of a triangle.)
44. For isosceles $\triangle PNQ$, the vertices are $P(-2a, 0)$, $N(2a, 0)$, and $Q(0, 2b)$. In terms of a and b , find the coordinates of the circumcenter of $\triangle PNQ$. (The circumcenter is the point of concurrence for the perpendicular bisectors of the sides of a triangle.)

In Exercises 45 and 46, complete an analytic proof for each theorem.

45. The altitudes of a triangle are concurrent.
46. The perpendicular bisectors of the sides of a triangle are concurrent.

For Exercises 47 and 48, assume that $A \neq 0$ and $B \neq 0$.

47. *Prove:* For $C \neq D$, the lines $Ax + By = C$ and $Ax + By = D$ are parallel.
48. *Prove:* The lines $Ax + By = C$ and $Bx - Ay = D$ are perpendicular.

A Look Beyond: The Banach-Tarski Paradox

In the 1920s, two Polish mathematicians proposed a mathematical dilemma to their colleagues. Known as the Banach-Tarski paradox, their proposal has puzzled students of geometry for decades. What was most baffling was that the proposal indicated that matter could be created through rearrangement of the pieces of a figure! The following steps outline the Banach-Tarski paradox.

First consider the square whose sides are each of length 8. [See Figure 9.42(a).] By counting squares or by applying a formula, it is clear that the 8-by-8 square must have an area of 64 square units. We now subdivide the square (as shown) to form two right triangles and two trapezoids. Note the dimensions indicated on each piece of the square in Figure 9.42(b).

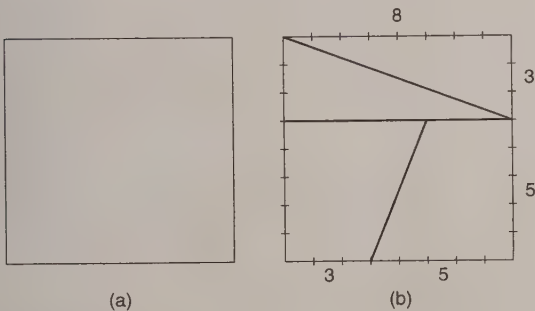


FIGURE 9.42

The parts of the square are now rearranged to form a rectangle (see Figure 9.43) whose dimensions are 13 and 5. This rectangle clearly has an area that measures 65 square units, 1 square unit more than the given square! How is it possible that the second figure has an area greater than the first?

The puzzle is real, but you may also realize that something is wrong. This paradox can be explained by considering the slopes of lines. The triangles, which have legs of lengths 3 and 8, determine a hypotenuse whose slope is $-\frac{3}{8}$. Although the side of the trapezoid appears to be collinear with the hypotenuse, it actually has a slope of $-\frac{2}{5}$. It was easy to accept that the segments were collinear because the slopes are nearly equal; in fact, $-\frac{3}{8} = -0.375$ and

$-\frac{2}{5} = -0.400$. In Figure 9.44 (which is somewhat exaggerated), a very thin parallelogram appears in the space between the original segments of the cut-up square. One may quickly conclude that the area of that parallelogram is 1 square unit, and the paradox has been resolved once more!

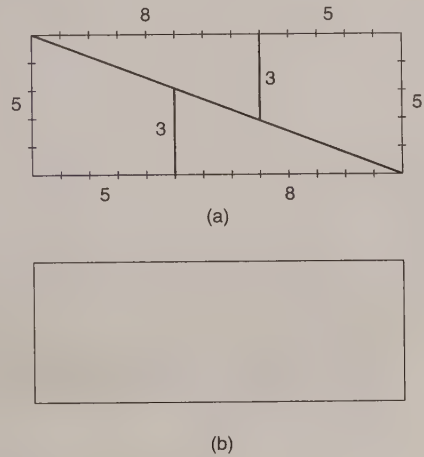


FIGURE 9.43

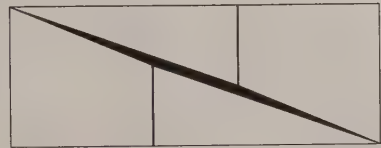


FIGURE 9.44



Summary

■ A Look Back at Chapter 9

Our goal in this chapter was to relate algebra and geometry. This relationship is called analytic geometry or coordinate geometry. Formulas for the length of a line segment, the midpoint of a line segment, and the slope of a line were developed. We found the equation for a line and used it for graphing. Analytic proofs were provided for a number of theorems of geometry.

■ A Look Ahead to Chapter 10

In the next chapter, we will again deal with the right triangle. Three trigonometric ratios (sine, cosine, and tangent) will be defined for an acute angle of the right triangle in terms of its sides. An area formula for triangles will be derived using the sine ratio. We will also prove the Law of Sines and the Law of Cosines for acute triangles. Another unit for measuring an angle, called the radian, will be introduced.

■ Important Terms and Concepts of Chapter 9

- 9.1 Analytic Geometry
 - Cartesian Coordinate System
 - Rectangular Coordinate System

- x Axis, y Axis, Quadrants, Origin
- x Coordinate, y Coordinate
- Ordered Pair
- Distance Formula
- Linear Equation
- Midpoint Formula

- 9.2 Graphs of Equations
 - x Intercept, y Intercept
 - Slope, Slope Formula
 - Negative Reciprocal

- 9.3 Preparation for Analytic Proofs

- 9.4 Analytic Proofs
 - Synthetic Proofs

- 9.5 Equations of Lines
 - Slope-Intercept Form of a Line
 - Point-Slope Form of a Line
 - Systems of Equations

■ A Look Beyond: The Banach-Tarski Paradox

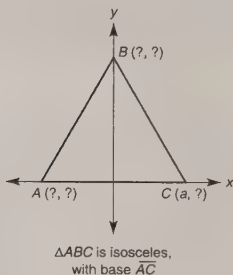


Review Exercises

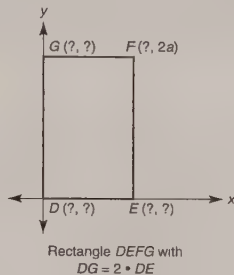
- Find the distance between each pair of points:
 - (6, 4) and (6, -3)
 - (-5, 2) and (7, -3)
 - (1, 4) and (-5, 4)
 - ($x - 3, y + 2$) and ($x, y - 2$)
 - Find the distance between each pair of points:
 - (2, -3) and (2, 5)
 - (-4, 1) and (4, 5)
 - (3, -2) and (-7, -2)
 - ($x - 2, y - 3$) and ($x + 4, y + 5$)
 - Find the midpoint of the line segment that joins each pair of points in Exercise 1.
 - Find the midpoint of the line segment that joins each pair of points in Exercise 2.
 - Find the slope of the line joining each pair of points in Exercise 1.
 - Find the slope of the line joining each pair of points in Exercise 2.
 - (2, 1) is the midpoint of $\overline{AB}$, in which A has coordinates (8, 10). Find the coordinates of B .
 - The y axis is the perpendicular bisector of $\overline{RS}$. Find the coordinates of R if S is the point (-3, 7).
 - If A has coordinates (2, 1) and B has coordinates ($x, 3$), find x such that the slope of $\overline{AB}$ is -3.
 - If R has coordinates (-5, 2) and S has coordinates (2, y), find y such that the slope of $\overline{RS}$ is $-\frac{6}{7}$.
 - Without graphing, determine whether the pairs of lines are parallel, perpendicular, the same, or none of these:
 - $x + 3y = 6$ and $3x - y = -7$
 - $2x - y = -3$ and $y = 2x - 14$
 - $y + 2 = -3(x - 5)$ and $2y = 6x + 11$
 - $0.5x + y = 0$ and $2x - y = 10$
 - Determine whether the points (-6, 5), (1, 7), and (16, 10) are collinear.
 - Find x such that (-2, 3), ($x, 6$), and (8, 8) are collinear.
 - Draw the graph of $3x + 7y = 21$, and name the x and y intercepts.
 - Draw the graph of $4x - 3y = 9$ by changing the equation to Slope-Intercept Form.
 - Draw the graph of $y + 2 = \frac{-2}{3}(x - 1)$.
 - Write the equation for:
 - The line through (2, 3) and (-3, 6)
 - The line through (-2, -1) and parallel to the line through (6, -3) and (8, -9)
 - The line through (3, -2) and perpendicular to $x + 2y = 4$
 - The line through (-3, 5) and parallel to the x axis
 - Show that the triangle whose vertices are $A(-2, -3)$, $B(4, 5)$, and $C(-4, 1)$ is a right triangle.
 - Show that the triangle whose vertices are $A(3, 6)$, $B(-6, 4)$, and $C(1, -2)$ is an isosceles triangle.
 - Show that the quadrilateral whose vertices are $R(-5, -3)$, $S(1, -11)$, $T(7, -6)$, and $V(1, 2)$ is a parallelogram.
- In Exercises 21 and 22, find the intersection of the graphs of the two equations by graphing.*
- | | |
|--------------------|-----------------|
| 21. $4x - 3y = -3$ | 22. $y = x + 3$ |
| $x + 2y = 13$ | $y = 4x$ |
- In Exercises 23 and 24, solve the systems of equations in Exercises 21 and 22 by using algebraic methods.*
- Refer to Exercise 21.
 - Refer to Exercise 22.
 - Three of four vertices of a parallelogram are (0, -2), (6, 8), and (10, 1). Find the possibilities for the coordinates of the remaining vertex.
 - $A(3, 1)$, $B(5, 9)$, and $C(11, 3)$ are the vertices of $\triangle ABC$.
 - Find the length of the median from B to $\overline{AC}$.
 - Find the slope of the altitude from B to $\overline{AC}$.
 - Find the slope of a line through B parallel to $\overline{AC}$.

In Exercises 27 to 30, supply the missing coordinates for the vertices, using as few variables as possible.

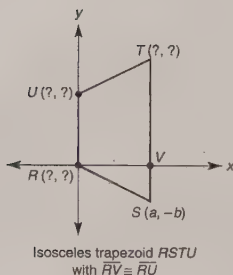
27.



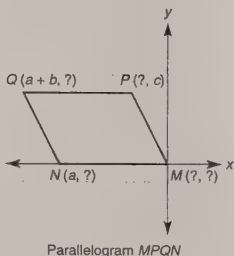
28.



29.



30.



31. $A(2a, 2b)$, $B(2c, 2d)$, and $C(0, 2e)$ are the vertices of $\triangle ABC$.

- Find the length of the median from C to $\overline{AB}$.
- Find the slope of the altitude from B to $\overline{AC}$.
- Find the equation of the altitude from B to $\overline{AC}$.

Prove the statements in Exercises 32 to 36 by using analytic geometry.

- The segments that join the midpoints of consecutive sides of a parallelogram form another parallelogram.
- If the diagonals of a rectangle are perpendicular, then the rectangle is a square.
- If the diagonals of a trapezoid are equal in length, then the trapezoid is an isosceles trapezoid.
- If two medians of a triangle are equal in length, then the triangle is isosceles.
- The segments joining the midpoints of consecutive sides of an isosceles trapezoid form a rhombus.

CHAPTER 3

3.1 EXERCISES

1. $\angle A$; $\overline{AB}$; No; No 3. $m\angle A = 72^\circ$ 5. SAS
 7. $\triangle AED \cong \triangle FDE$ 9. SSS 11. AAS
 13. ASA 15. ASA 17. SSS
 19. (a) $\angle A \cong \angle A$ (b) ASA 21. $\overline{AD} \cong \overline{EC}$
 23. $\overline{MO} \cong \overline{MO}$ 25. 1. Given 2. $\overline{AC} \cong \overline{AC}$ 3. SSS
 33. Yes; SAS or SSS 35. No
 37. (a) $\triangle CBE, \triangle ADE, \triangle CDE$ (b) $\triangle ADC$ (c) $\triangle CBD$

3.1 SELECTED PROOFS

27. PROOF

Statements	Reasons
1. $\overline{PQ}$ bisects $\angle MPN$	1. Given
2. $\angle 1 \cong \angle 2$	2. If a ray bisects an $\angle$, it forms two $\cong \angle$ s
3. $\overline{MP} \cong \overline{NP}$	3. Given
4. $\overline{PQ} \cong \overline{PQ}$	4. Identity
5. $\triangle MPQ \cong \triangle NPQ$	5. SAS

31. PROOF

Statements	Reasons
1. $\angle VRS \cong \angle TSR$ and $\overline{RV} \cong \overline{TS}$	1. Given
2. $\overline{RS} \cong \overline{RS}$	2. Identity
3. $\triangle RST \cong \triangle SRV$	3. SAS

3.2 EXERCISES

9. $m\angle 2 = 48^\circ$; $m\angle 3 = 48^\circ$; $m\angle 5 = 42^\circ$; $m\angle 6 = 42^\circ$
 11. 1. Given
 2. If two lines are $\perp$, then they form right $\angle$ s
 3. Identity 4. $\triangle HJK \cong \triangle HJL$ 5. $\overline{KJ} \cong \overline{JL}$
 17. $c = 5$ 19. $b = 8$ 21. $c = \sqrt{41}$
 31. (a) 8 (b) 37° (c) 53° 33. 751 feet

3.2 SELECTED PROOFS

1. PROOF

Statements	Reasons
1. $\angle 1$ and $\angle 2$ are right $\angle$ s $\overline{CA} \cong \overline{DA}$	1. Given
2. $\overline{AB} \cong \overline{AB}$	2. Identity
3. $\triangle ABC \cong \triangle ABD$	3. HL

5. PROOF

Statements	Reasons
1. $\angle R$ and $\angle V$ are right $\angle$ s $\angle 1 \cong \angle 2$	1. Given
2. $\angle R \cong \angle V$	2. All right $\angle$ s are $\cong$
3. $\overline{ST} \cong \overline{ST}$	3. Identity
4. $\triangle RST \cong \triangle VST$	4. AAS

13. PROOF

Statements	Reasons
1. $\angle$ s P and R are right $\angle$ s	1. Given
2. $\angle P \cong \angle R$	2. All right $\angle$ s are $\cong$
3. M is the midpoint of $\overline{PR}$	3. Given
4. $\overline{PM} \cong \overline{MR}$	4. The midpoint of a segment forms two $\cong$ segments
5. $\angle NMP \cong \angle QMR$	5. If two lines intersect, the vertical angles formed are $\cong$
6. $\triangle NPM \cong \triangle QRM$	6. ASA
7. $\angle N \cong \angle Q$	7. CPCTC

23. PROOF

Statements	Reasons
1. $\overline{DF} \cong \overline{DG}$ and $\overline{FE} \cong \overline{EG}$	1. Given
2. $\overline{DE} \cong \overline{DE}$	2. Identity
3. $\triangle FDE \cong \triangle GDE$	3. SSS
4. $\angle FDE \cong \angle GDE$	4. CPCTC
5. $\overline{DE}$ bisects $\angle FDG$	5. If a ray divides an $\angle$ into two $\cong \angle$ s, then the ray bisects the angle

27. PROOF

Statements	Reasons
1. $\angle 1 \cong \angle 2$ and $\overline{MN} \cong \overline{QP}$	1. Given
2. $\overline{MP} \cong \overline{MP}$	2. Identity
3. $\triangle NMP \cong \triangle QPM$	3. SAS
4. $\angle 3 \cong \angle 4$	4. CPCTC
5. $\overline{MQ} \parallel \overline{NP}$	5. If two lines are cut by a transversal so that the alt. int. $\angle$ s are $\cong$, then the lines are $\parallel$

3.3 EXERCISES

1. Isosceles 3. $\overline{VT} \cong \overline{VU}$ 5. $m\angle U = 69^\circ$
 7. $m\angle V = 36^\circ$ 9. Underdetermined
 11. Overdetermined 13. Determined 15. 55°
 17. $m\angle 2 = 68^\circ$; $m\angle 1 = 44^\circ$ 19. $m\angle 5 = 124^\circ$
 21. $m\angle A = 52^\circ$; $m\angle B = 64^\circ$; $m\angle C = 64^\circ$
 23. 26 25. 12 27. Yes
 29. 1. Given 2. $\angle 3 \cong \angle 2$ 3. $\angle 1 \cong \angle 2$
 4. If two $\angle$ s of a $\triangle$ are $\cong$, then the opposite sides are $\cong$
 35. $m\angle A = 40^\circ$ 37. 75°

3.3 SELECTED PROOF

31. PROOF

Statements	Reasons
1. $\angle 1 \cong \angle 3$	1. Given
2. $\overline{RU} \cong \overline{VU}$	2. Given
3. $\angle R \cong \angle V$	3. If two sides of a $\triangle$ are $\cong$, then the $\angle$ s opposite these sides are also $\cong$
4. $\triangle RUS \cong \triangle VUT$	4. ASA
5. $\overline{SU} \cong \overline{TU}$	5. CPCTC
6. $\triangle STU$ is isosceles	6. If a $\triangle$ has two $\cong$ sides, it is an isosceles $\triangle$

3.4 EXERCISES

19. Construct a 90° angle; bisect it to form two 45° $\angle$ s. Bisect one of the 45° angles to get a 22.5° $\angle$.
 29. 120° 31. 150° 37. D is on the bisector of $\angle A$.

3.5 EXERCISES

1. False 3. True 5. True 7. False 9. True
 11. (a) Not possible ($100^\circ + 100^\circ + 60^\circ \neq 180^\circ$)
 (b) Possible ($45^\circ + 45^\circ + 90^\circ = 180^\circ$)
 13. (a) Possible
 (b) Not possible ($8 + 9 = 17$)
 (c) Not possible ($8 + 9 < 18$)
 15. 4 cm
 17. 72° (two such angles)
 36° (one angle only)
 19. Nashville
 21. 1. $m\angle ABC > m\angle DBE$ and $m\angle CBD > m\angle EBF$
 3. Angle-Addition Postulate 4. $m\angle ABD > m\angle DBF$
 25. $BC < EF$ 27. $2 < x < 10$
 29. $x + 2 < y < 5x + 12$
 31. Proof: Assume $PM = PN$. Then $\triangle MPN$ is isosceles. But that contradicts the hypothesis; thus our assumption must be wrong, and $PM \neq PN$.

3.5 SELECTED PROOF

23. PROOF

Statements	Reasons
1. Quad. $RSTU$ with diagonal $\overline{US}$; $\angle R$ and $\angle TUS$ are right $\angle$ s	1. Given
2. $TS > US$	2. The shortest distance from a point to a line is the $\perp$ distance
3. $US > UR$	3. Same as (2)
4. $TS > UR$	4. Transitive Prop. of Inequality

REVIEW EXERCISES

15. (a) $\overline{PR}$ (b) $\overline{PQ}$ 16. $\overline{BC}$, $\overline{AC}$, $\overline{AB}$
 17. $\angle R$, $\angle Q$, $\angle P$ 18. $\overline{DA}$ 19. (b) 20. 5, 35
 21. 20° 22. 115° 23. $m\angle C = 64^\circ$ 24. Isosceles
 25. The triangle is also equilateral. 26. 60°

REVIEW EXERCISES SELECTED PROOFS

1. PROOF

Statements	Reasons
1. $\angle AEB \cong \angle DEC$	1. Given
2. $\overline{AE} \cong \overline{ED}$	2. Given
3. $\angle A \cong \angle D$	3. If two sides of a $\triangle$ are $\cong$, then the $\angle$ s opposite these sides are also $\cong$
4. $\triangle AEB \cong \triangle DEC$	4. ASA

5. PROOF

Statements	Reasons
1. $\overline{AB} \cong \overline{DE}$ and $\overline{AB} \parallel \overline{DE}$	1. Given
2. $\angle A \cong \angle D$	2. If two $\parallel$ lines are cut by a transversal, then the alt. int. $\angle$ s are $\cong$
3. $\overline{AC} \cong \overline{DF}$	3. Given
4. $\triangle BAC \cong \triangle EDF$	4. SAS
5. $\angle BCA \cong \angle EFD$	5. CPCTC
6. $\overline{BC} \parallel \overline{FE}$	6. If two lines are cut by a transversal so that alt. int. $\angle$ s are $\cong$, then the lines are $\parallel$

9. PROOF

Statements	Reasons
1. $\overline{YZ}$ is the base of an isosceles triangle	1. Given
2. $\angle Y \cong \angle Z$	2. Base $\angle$ s of an isosceles $\triangle$ are $\cong$
3. $\overrightarrow{XA} \parallel \overrightarrow{YZ}$	3. Given
4. $\angle 1 \cong \angle Y$	4. If two $\parallel$ lines are cut by a transversal, then the corresponding $\angle$ s are $\cong$
5. $\angle 2 \cong \angle Z$	5. If two $\parallel$ lines are cut by a transversal, then the alt. int. $\angle$ s are $\cong$
6. $\angle 1 \cong \angle 2$	6. Transitive Prop. for Congruence

13. PROOF

Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$	1. Given
2. $\angle BAD \cong \angle CDA$	2. Given
3. $\overline{AD} \cong \overline{AD}$	3. Identity
4. $\triangle BAD \cong \triangle CDA$	4. SAS
5. $\angle CAD \cong \angle BDA$	5. CPCTC
6. $\overline{AE} \cong \overline{ED}$	6. If two $\angle$ s of a $\triangle$ are $\cong$, then the sides opposite these $\angle$ s are also $\cong$
7. $\triangle AED$ is isosceles	7. If a $\triangle$ has two $\cong$ sides, then it is an isosceles $\triangle$

CHAPTER 5

5.1 EXERCISES

1. (a) $\frac{4}{5}$ (b) $\frac{4}{5}$ (c) $\frac{2}{3}$ (d) Incommensurable
3. (a) $\frac{5}{8}$ (b) $\frac{1}{3}$ (c) $\frac{4}{3}$ (d) Incommensurable
5. (a) 3 (b) 8 7. (a) 6 (b) 4
9. (a) $\pm 2\sqrt{7} \approx \pm 5.29$ (b) $\pm 3\sqrt{2} \approx \pm 4.24$
11. (a) 4 (b) $-\frac{5}{6}$ or 3
13. (a) $\frac{3 \pm \sqrt{33}}{4} \approx 2.19$ or -0.69
 (b) $\frac{7 \pm \sqrt{89}}{4} \approx 4.11$ or -0.61

Chapter 5
continued on next page.

15. 6.3 m/sec 17. $10\frac{1}{2}$ 19. ≈ 24 outlets
 21. (a) $4\sqrt{3} \approx 6.93$ (b) $4\frac{1}{2}$
 23. Secretary's salary is \$18,500; salesperson's salary is \$27,750; vice-president's salary is \$46,250
 25. 40° and 50° 27. 30.48 cm 29. $2\frac{4}{7} \approx 2.57$
 31. $a = 12$; $b = 16$ 33. 4 in. by $4\frac{2}{3}$ in.
 35. (a) $\frac{5+5\sqrt{5}}{2}$ (b) 8.1

5.2 EXERCISES

1. (a) $\triangle ABC \sim \triangle XTN$ (b) $\triangle ACB \sim \triangle NXT$
 3. Yes; yes. Spheres have the same shape; one is generally an enlargement of the other (could be congruent).
 5. SSS~ 7. SAS~ 9. SAS~
 11. (a) 82° (b) 42° (c) $10\frac{1}{2}$ (d) 8
 13. (a) Yes (b) Yes (c) Yes
 15. 1. $\overline{MN} \perp \overline{NP}$ and $\overline{QR} \perp \overline{RP}$ 2. If two lines are $\perp$, then they form a rt. $\angle$ 3. $\angle N \cong \angle QRP$ 4. Identity
 5. $\triangle MNP \sim \triangle QRP$; AA
 17. 1. $\angle H \cong \angle F$ 2. If two $\angle$ s are vertical $\angle$ s, then they are $\cong$ 3. $\triangle HJK \sim \triangle FGK$; AA
 27. $4\frac{1}{2}$ 29. 16 31. 12
 33. $10 + 2\sqrt{5}$ or $10 - 2\sqrt{5}$; ≈ 14.47 or 5.53
 35. 3 ft 9 in. 37. 74 ft 39. 150 feet

5.2 SELECTED PROOFS

19. PROOF

Statements	Reasons
1. $\overline{AB} \parallel \overline{DF}$ and $\overline{BD} \parallel \overline{FG}$	1. Given
2. $\angle A \cong \angle FEG$ and $\angle BCA \cong \angle G$	2. If two $\parallel$ lines are cut by a transversal, then the corresponding $\angle$ s are $\cong$
3. $\triangle ABC \sim \triangle EFG$	3. AA

23. *Proof:* By hypothesis, $\overline{RS} \parallel \overline{UV}$. $\angle R \cong \angle V$ and $\angle S \cong \angle U$ because they are alternate interior angles.
 $\triangle RTS \sim \triangle VTU$ by AA. It follows that $\frac{RT}{VT} = \frac{RS}{VU}$, because corresponding sides of similar $\triangle$ s are proportional.

25. PROOF

Statements	Reasons
1. $\triangle DEF \sim \triangle MNP$ $\overline{DG}$ and $\overline{MQ}$ are altitudes	1. Given

(continued)

$$2. \overline{DG} \perp \overline{EF} \text{ and } \overline{MQ} \perp \overline{NP}$$

$$3. \angle DGE \text{ and } \angle MQN \text{ are rt. } \angle \text{s}$$

$$4. \angle DGE \cong \angle MQN$$

$$5. \angle E \cong \angle N$$

$$6. \triangle DGE \sim \triangle MQN$$

$$7. \frac{DG}{MQ} = \frac{DE}{MN}$$

2. An altitude is a segment drawn from a vertex $\perp$ to the opposite side

3. $\perp$ lines form a rt. $\angle$

4. Right $\angle$ s are $\cong$

5. If two $\triangle$ s are $\sim$, then the corresponding $\angle$ s are $\cong$

6. AA

7. Corresponding sides of $\sim \triangle$ s are proportional

5.3 EXERCISES

1. $\triangle RST \sim \triangle RVS \sim \triangle SVT$ 3. 4.5
 5. (a) 10 (b) $\sqrt{34} \approx 5.83$ 7. (a) 8 (b) 4
 9. (a) Right (b) Acute (c) Right (d) No $\triangle$
 11. 15 ft 13. $6\sqrt{5} \approx 13.4$ m 15. 12 cm
 17. The base is 8; the altitude is 6; the diagonal is 10.
 19. $6\sqrt{7} \approx 15.87$ in 21. 12 in 23. 4 25. $9\frac{3}{13}$ in
 27. $5\sqrt{5} \approx 11.18$ 33. 60°
 35. $TS = 13$; $RT = 13\sqrt{2} \approx 18.38$

5.4 EXERCISES

1. $YZ = 8$ and $XY = 8\sqrt{2} \approx 11.31$
 3. $XZ = 10$; $YZ = 10$
 5. $DF = 5\sqrt{3} \approx 8.66$ and $FE = 10$
 7. $DE = 12$; $FE = 24$
 9. $HL = 6$; $HK = 12$; $MK = 6$
 11. $AC = 6$ and $AB = 6\sqrt{2} \approx 8.49$
 13. $RS = 6$ and $RT = 6\sqrt{3} \approx 10.39$
 15. $DB = 5\sqrt{6} \approx 12.25$ 17. $6\sqrt{3} + 6 \approx 16.39$
 19. 45° 21. 60° ; 146 ft further
 23. $DC = 2\sqrt{3} \approx 3.46$
 $DB = 4\sqrt{3} \approx 6.93$
 25. $6\sqrt{3} \approx 10.39$ 27. $4\sqrt{3} \approx 6.93$
 29. $6 + 6\sqrt{3} \approx 16.39$

5.5 EXERCISES

1. 30 oz of ingredient A; 24 oz of ingredient B;
 36 oz of ingredient C
 3. (a) Yes (b) Yes 5. $EF = 4\frac{1}{6}$, $FG = 3\frac{1}{3}$, $GH = 2\frac{1}{2}$
 7. $x = 5\frac{1}{3}$, $DE = 5\frac{1}{3}$, $EF = 6\frac{2}{3}$ 9. $EC = 16\frac{4}{5}$
 11. $a = 5$; $AD = 4$ 13. (a) No (b) Yes
 15. 9 17. $4\sqrt{6} \approx 9.80$ 19. $\frac{AC}{CE} = \frac{AD}{DE}$, $\frac{DC}{CB} = \frac{DE}{EB}$
 21. $SV = 2\sqrt{3} \approx 3.46$; $VT = 4\sqrt{3} \approx 6.93$

23. $x = \frac{1 + \sqrt{73}}{2}$ or $x = \frac{1 - \sqrt{73}}{2}$; reject both because each will give a negative number for the length of a side.
 25. 1. Given 2. Means-Extremes Property
 3. Addition Property of Equality 4. Distributive Property
 5. Means-Extremes Property 6. Substitution
 31. $\frac{-1 + \sqrt{5}}{2} \approx 0.62$

5.5 SELECTED PROOF

27. PROOF

Statements	Reasons
1. $\triangle RST$ with M the midpoint of $\overline{RS}$ $\overline{MN} \parallel \overline{ST}$	1. Given
2. $RM = MS$	2. The midpoint of a segment divides the segment into two segments of equal measure
3. $\frac{RM}{MS} = \frac{RN}{NT}$	3. If a line is $\parallel$ to one side of a $\triangle$ and intersects the other two sides, then it divides these sides proportionally
4. $\frac{MS}{MS} = 1 = \frac{RN}{NT}$	4. Substitution
5. $RN = NT$	5. Means-Extremes Property
6. N is the midpoint of $\overline{RT}$	6. If a point divides a segment into two segments of equal measure, then the point is a midpoint

REVIEW EXERCISES

1. False 2. True 3. False 4. True 5. True
 6. False 7. True
 8. (a) $\pm 3\sqrt{2} \approx \pm 4.24$ (b) 26 (c) -1 (d) 2
 (e) 7 or -1 (f) $-\frac{9}{5}$ or 4 (g) 6 or -1 (h) -6 or 3
 9. \$3.78 10. Six packages 11. \$79.20
 12. The lengths of the sides are 8, 12, 20, and 28. 13. 18
 14. 20 and $22\frac{1}{2}$ 15. 150° 18. $x = 5$; $m\angle F = 97^\circ$
 19. $AB = 6$ and $BC = 12$ 20. 3 21. $4\frac{1}{2}$ 22. $6\frac{1}{4}$
 23. $5\frac{3}{5}$ 24. 10 25. 6 26. $EO = 1\frac{1}{5}$; $EK = 9$
 29. (a) $8\frac{1}{3}$ (b) 21 (c) $2\sqrt{3} \approx 3.46$ (d) 3
 30. (a) 16 (b) 40 (c) $2\sqrt{5} \approx 4.47$ (d) 4
 31. (a) 30° (b) 24 (c) 20 (d) 16
 32. $AE = 20$; $EF = 15$; $AF = 25$ 33. $4\sqrt{2} \approx 5.66$ in.

34. $3\sqrt{2} \approx 4.24$ cm 35. 25 cm 36. $5\sqrt{3} \approx 8.66$ in.
 37. $4\sqrt{3} \approx 6.93$ in. 38. 12 cm
 39. (a) $y = 9$; $x = 9\sqrt{2} \approx 12.73$ (b) $y = 6$; $x = 4\frac{1}{2}$
 (c) $y = 3$; $x = 12$ (d) $x = 2\sqrt{14} \approx 7.48$; $y = 13$
 40. 11 km
 41. (a) Acute (b) No $\triangle$ (c) Obtuse (d) Right
 (e) No $\triangle$ (f) Acute (g) Obtuse (h) Obtuse

REVIEW EXERCISES SELECTED PROOFS

16. PROOF

Statements	Reasons
1. $ABCD$ is a parallelogram; $\overline{DB}$ intersects $\overline{AE}$ at point F	1. Given
2. $\overline{DC} \parallel \overline{AB}$	2. Opposite sides of a parallelogram are $\parallel$
3. $\angle CDB \cong \angle ABD$	3. If two $\parallel$ lines are cut by a transversal, then the alt. int. $\angle$ s are $\cong$
4. $\angle DEF \cong \angle BAF$	4. Same as (3)
5. $\triangle DFE \sim \triangle BFA$	5. AA
6. $\frac{AF}{EF} = \frac{AB}{DE}$	6. Corresponding sides of $\sim \triangle$ s are proportional

17. PROOF

Statements	Reasons
1. $\angle ADC \cong \angle 2$	1. If two lines intersect, then the vertical $\angle$ s formed are $\cong$
2. $\angle 1 \cong \angle 2$	2. Given
3. $\angle ADC \cong \angle 1$	3. Transitive Prop. for Congruence
4. $\angle A \cong \angle A$	4. Identity
5. $\triangle BAE \sim \triangle CAD$	5. AA
6. $\frac{AB}{AC} = \frac{BE}{CD}$	6. Corresponding sides of $\sim \triangle$ s are proportional

CHAPTER 9

9.1 EXERCISES

3. (a) 4 (b) 8 (c) 5 (d) 9
 5. $b = 3.5$ or $b = 10.5$
 7. (a) 5 (b) 10 (c) $2\sqrt{5}$ (d) $\sqrt{a^2 + b^2}$
 9. (a) $(2, \frac{-3}{2})$ (b) (1, 1) (c) (4, 0) (d) $(\frac{a}{2}, \frac{b}{2})$
 11. (a) (-3, 4) (b) (0, -2) (c) $(-a, 0)$
 (d) $(-b, -c)$ 13. (a) $(4, -\frac{5}{2})$ (b) (0, 4)
 (c) $(\frac{7}{2}, -1)$ (d) (a, b) 15. (a) (5, -1)
 (b) (0, -5) (c) (2, -a) (d) $(b, -c)$
 17. (a) (-3, -4) (b) (-2, 0) (c) $(-a, 0)$
 (d) $(-b, c)$ 19. (a) $x = 4$ (b) $y = b$ (c) $x = 2$
 (d) $y = 3$ 21. (2.5, -13.7) 23. (2, 3); 16
 25. (a) Isosceles (b) Equilateral
 (c) Isosceles right triangle
 27. $x + y = 6$ 29. $(a, a\sqrt{3})$ or $(a, -a\sqrt{3})$
 31. (0, $1 + 3\sqrt{3}$) and (0, $1 - 3\sqrt{3}$) 33. 17 35. 9
 37. (a) 135π units³ (b) 75π units³
 39. (a) 96π units³ (b) 144π units³
 41. (a) 90π units² (b) 90π units²

9.2 EXERCISES

1. (4, 0) and (0, 3) 3. (5, 0) and $(0, -\frac{5}{2})$ 5. (-3, 0)
 7. (6, 0) and (0, 3) 9. (a) 4 (b) Undefined (c) -1
 (d) 0 (e) $\frac{d-b}{c-a}$ (f) $-\frac{b}{a}$ 11. (a) 10
 (b) 15 13. (a) Collinear (b) Non-collinear
 15. (a) $\frac{3}{4}$ (b) $-\frac{5}{3}$ (c) -2 (d) $\frac{a-b}{c}$
 17. None of these 19. Perpendicular 21. $\frac{3}{2}$
 23. 23 31. Right triangle 33. (4, 7); (0, -1); (10, -3)
 37. $m_{EH} = \frac{2c-0}{2b-0} = \frac{2c}{2b} = \frac{c}{b}$
 $m_{FG} = \frac{c-0}{(a+b)-a} = \frac{c}{b}$

Due to equal slopes, $\overline{EH} \parallel \overline{FG}$. Thus $EFGH$ is a trapezoid.

9.2 SELECTED PROOF

35. $m_{VT} = \frac{e-e}{(c-d)-(a+d)} = \frac{0}{c-a-2d} = 0$
 $m_{RS} = \frac{b-b}{c-a} = \frac{0}{c-a} = 0$
 $\therefore \overline{VT} \parallel \overline{RS}$
 $RV = \sqrt{[(a+d)-a]^2 + (e-b)^2}$
 $= \sqrt{d^2 + (e-b)^2}$

$$\begin{aligned}
 &= \sqrt{d^2 + e^2 - 2be + b^2} \\
 ST &= \sqrt{[c-(c-d)]^2 + (b-e)^2} \\
 &= \sqrt{(d)^2 + (b-e)^2} \\
 &= \sqrt{d^2 + b^2 - 2be + e^2} \\
 \therefore RV &= ST
 \end{aligned}$$

$RSTV$ is an isosceles trapezoid.

9.3 EXERCISES

1. (a) $a\sqrt{2}$ if $a > 0$ (b) $\frac{d-b}{c-a}$
 3. (a) -1 (b) $-\frac{b}{a}$
 5. $\overline{AB}$ is horizontal and $\overline{BC}$ is vertical; $\therefore \overline{AB} \perp \overline{BC}$. Hence, $\angle B$ is a right $\angle$ and $\triangle ABC$ is a right triangle.
 7. $m_{QM} = \frac{c-0}{b-0} = \frac{c}{b}$
 $m_{PN} = \frac{c-0}{(a+b)-a} = \frac{c}{b}$
 $\therefore \overline{QM} \parallel \overline{PN}$
 $m_{QP} = \frac{c-c}{(a+b)-b} = \frac{0}{a} = 0$
 $m_{MN} = \frac{0-0}{a-0} = \frac{0}{a} = 0$
 $\therefore \overline{QP} \parallel \overline{MN}$
 Because both pairs of opposite sides are parallel, $MNPQ$ is a parallelogram.
 9. $m_{MN} = 0$ and $m_{QP} = 0$; $\therefore \overline{MN} \parallel \overline{QP}$. $\overline{QM}$ and $\overline{PN}$ are both vertical; $\therefore \overline{QM} \parallel \overline{PN}$. Hence, $MQPN$ is a parallelogram. Because $\overline{QM}$ is vertical and $\overline{MN}$ is horizontal, $\angle QMN$ is a right angle. Because parallelogram $MQPN$ has a right $\angle$, it is also a rectangle.

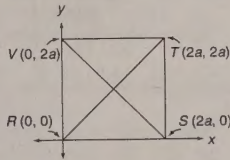
11. $A = (0, 0)$; $B = (a, 0)$; $C = (a, b)$
 13. $M = (0, 0)$; $N = (r, 0)$; $P = (r + s, t)$
 15. $A = (0, 0)$; $B = (a, 0)$; $C = (a - c, d)$
 17. (a) Square
 $A = (0, 0)$; $B = (a, 0)$; $C = (a, a)$; $D = (0, a)$
 (b) Square (with midpoints of sides)
 $A = (0, 0)$; $B = (2a, 0)$; $C = (2a, 2a)$; $D = (0, 2a)$
 19. (a) Parallelogram
 $A = (0, 0)$; $B = (a, 0)$; $C = (a + b, c)$; $D = (b, c)$
 (NOTE: D chosen before C)
 (b) Parallelogram (with midpoints of sides)
 $A = (0, 0)$; $B = (2a, 0)$; $C = (2a + 2b, 2c)$; $D = (2b, 2c)$
 21. (a) Isosceles triangle
 $R = (0, 0)$; $S = (2a, 0)$; $T = (a, b)$
 (b) Isosceles triangle (with midpoints)
 $R = (0, 0)$; $S = (4a, 0)$; $T = (2a, 2b)$
 23. $r^2 = s^2 + t^2$ 25. $c^2 = a^2 - b^2$ 27. $b^2 = 3a^2$
 29. (a) Positive (b) Negative (c) $2a$
 31. (a) Slope Formula (b) Distance Formula
 (c) Midpoint Formula (d) Slope Formula

9.4 EXERCISES

21. True. The quadrilateral that results is a parallelogram.

9.4 SELECTED PROOFS

3. The diagonals of a square are perpendicular bisectors of each other.



Proof: Let square $RSTV$ have the vertices shown. Then the midpoints of the diagonals are $M_{\overline{RT}} = (a, a)$ and $M_{\overline{VS}} = (a, a)$. Also, $m_{\overline{RT}} = 1$ and $m_{\overline{VS}} = -1$. Because the two diagonals share the midpoint (a, a) and the product of their slopes is -1 , they are perpendicular bisectors of each other.

7. The segments that join the midpoints of the consecutive sides of a quadrilateral form a parallelogram.

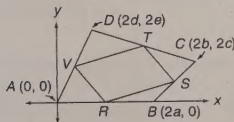
Proof: The midpoints of the sides of quadrilateral $ABCD$ are

$$R = \left(\frac{0 + 2a}{2}, \frac{0 + 0}{2} \right) = (a, 0)$$

$$S = \left(\frac{2a + 2b}{2}, \frac{0 + 2c}{2} \right) = (a + b, c)$$

$$T = \left(\frac{2d + 2b}{2}, \frac{2e + 2c}{2} \right) = (d + b, e + c)$$

$$V = \left(\frac{0 + 2d}{2}, \frac{0 + 2e}{2} \right) = (d, e)$$



Now we determine slopes as follows:

$$m_{\overline{RS}} = \frac{c - 0}{(a + b) - a} = \frac{c}{b}$$

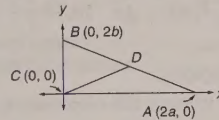
$$m_{\overline{ST}} = \frac{(e + c) - c}{(d + b) - (a + b)} = \frac{e}{d - a}$$

$$m_{\overline{TV}} = \frac{(e + c) - e}{(d + b) - d} = \frac{c}{b}$$

$$m_{\overline{VR}} = \frac{e - 0}{d - a} = \frac{e}{d - a}$$

Because $m_{\overline{RS}} = m_{\overline{TV}}$, $\overline{RS} \parallel \overline{TV}$. Also $m_{\overline{ST}} = m_{\overline{VR}}$ so $\overline{ST} \parallel \overline{VR}$. Then $RSTV$ is a parallelogram.

11. The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.



Proof: Let right $\triangle ABC$ have vertices as shown. Then D , the midpoint of the hypotenuse, is given by

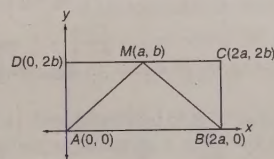
$$D = \left(\frac{0 + 2a}{2}, \frac{2b + 0}{2} \right) = (a, b)$$

$$\begin{aligned} \text{Now } BD &= DA = \sqrt{(2a - a)^2 + (0 - b)^2} \\ &= \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} \end{aligned}$$

$$\begin{aligned} \text{Also, } CD &= \sqrt{(a - 0)^2 + (b - 0)^2} \\ &= \sqrt{a^2 + b^2} \end{aligned}$$

Then D is equidistant from A , B , and C .

15. If the midpoint of one side of a rectangle is joined to the endpoints of the opposite sides, an isosceles triangle is formed.



Proof: Let rectangle $ABCD$ have endpoints as shown above. With M the midpoint of $\overline{DC}$,

$$M = \left(\frac{0 + 2a}{2}, \frac{2b + 2b}{2} \right) = (a, b)$$

$$MA = \sqrt{(a - 0)^2 + (b - 0)^2}$$

$$MA = \sqrt{a^2 + b^2}$$

$$MB = \sqrt{(a - 2a)^2 + (b - 0)^2}$$

$$MB = \sqrt{a^2 + b^2}$$

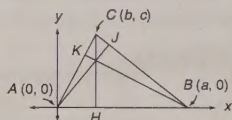
Because $MA = MB$, $\triangle AMB$ is isosceles.

9.5 EXERCISES

1. $x + 2y = 6$; $y = -\frac{1}{2}x + 3$
3. $-x + 3y = -40$; $y = \frac{1}{3}x - \frac{40}{3}$
11. $x + y = 6$
13. $-2x + 3y = -3$
15. $bx + ay = ab$
17. $-x + y = -2$
19. $5x + 2y = 5$
21. $4x + 3y = -12$
23. $-x + 3y = 2$
25. $y = -\frac{b}{a}x + \frac{bg + ha}{a}$
27. $(6, 0)$
29. $(5, -4)$
31. $(6, -2)$
33. $(3, 2)$
35. $(6, -1)$
37. $(5, -2)$
39. $(6, 0)$
41. $(\frac{a-c}{b}, a)$
43. $(b, \frac{ab-b^2}{c})$

9.5 SELECTED PROOFS

45. The altitudes of a triangle are concurrent.



Proof: For $\triangle ABC$, let $\overline{CH}$, $\overline{AJ}$, and $\overline{BK}$ name the altitudes. Because $\overline{AB}$ is horizontal ($m_{\overline{AB}} = 0$), $\overline{CH}$ is vertical and has the equation $x = b$.

Because $m_{\overline{BC}} = \frac{c-0}{b-a} = \frac{c}{b-a}$, the slope of altitude $\overline{AJ}$ is $m_{\overline{AJ}} = -\frac{b-a}{c} = -\frac{a-b}{c}$. Since $\overline{AJ}$ contains $(0, 0)$, its equation is $y = \frac{a-b}{c}x$.

The intersection of altitudes $\overline{CH}$ ($x = b$) and $\overline{AJ}$ ($y = \frac{a-b}{c}x$) is at $x = b$ so that $y = \frac{a-b}{c} \cdot b = \frac{b(a-b)}{c} = \frac{ab-b^2}{c}$. That is, $\overline{CH}$ and $\overline{AJ}$ intersect at $(b, \frac{ab-b^2}{c})$. The remaining altitude is $\overline{BK}$. Since $m_{\overline{AC}} = \frac{c-0}{b-0} = \frac{c}{b}$, $m_{\overline{BK}} = -\frac{b}{c}$. Because $\overline{BK}$ contains $(a, 0)$, its equation is $y - 0 = -\frac{b}{c}(x - a)$ or $y = -\frac{b}{c}(x - a)$.

For the three altitudes to be concurrent, $(b, \frac{ab-b^2}{c})$ must lie on the line $y = -\frac{b}{c}(x - a)$. Substitution leads to

$$\begin{aligned} \frac{ab-b^2}{c} &= -\frac{b}{c}(b-a) \\ \frac{ab-b^2}{c} &= -\frac{b(b-a)}{c} \\ \frac{ab-b^2}{c} &= -\frac{b^2+ab}{c}, \text{ which is true.} \end{aligned}$$

Then the three altitudes are concurrent.

47. For $B \neq 0$, the equation $Ax + By = C$ is equivalent to

$$By = -Ax + C, \text{ or } y = -\frac{A}{B}x + \frac{C}{B}$$

For $B \neq 0$, the equation $Ax + By = D$ is equivalent to $y = -\frac{A}{B}x + \frac{D}{B}$. Because $m_1 = m_2 = -\frac{A}{B}$, the graphs (lines) are parallel.

(NOTE: If $B = 0$, both graphs (lines) are vertical and parallel.)

REVIEW EXERCISES

1. (a) 7 (b) 6 (c) 13 (d) 5
2. (a) 8 (b) 10 (c) $4\sqrt{5}$ (d) 10
3. (a) $(6, \frac{1}{2})$ (b) $(-2, 4)$ (c) $(1, -\frac{1}{2})$ (d) $(\frac{2x-3}{2}, y)$
4. (a) (2) (b) $(-2, -2)$ (c) $(0, 3)$ (d) $(x+1, +1)$
5. (a) Undefined (b) 0 (c) $-\frac{5}{12}$ (d) $-\frac{4}{3}$
6. (a) Undefined (b) 0 (c) $\frac{1}{2}$ (d) $\frac{4}{3}$
7. $(-4, -8)$
8. $(3, 7)$
9. $x = \frac{4}{3}$
10. $y = -4$
11. (a) Perpendicular (b) Parallel (c) Neither (d) Perpendicular
12. Not collinear
13. $x = 4$
14. $(7, 0)$ and $(0, 3)$
17. (a) $3x + 5y = 21$ (b) $3x + y = -7$ (c) $-2x + y = -8$ (d) $y = 5$
18. $m_{\overline{AB}} = \frac{4}{3}$; $m_{\overline{BC}} = \frac{1}{2}$; $m_{\overline{AC}} = -2$. Because $m_{\overline{AC}} \cdot m_{\overline{BC}} = -1$, $\overline{AC} \perp \overline{BC}$ and $\angle C$ is a rt. $\angle$.
19. $AB = \sqrt{85}$; $BC = \sqrt{85}$. Because $AB = BC$, the triangle is isosceles.
20. $m_{\overline{RS}} = -\frac{4}{3}$; $m_{\overline{ST}} = \frac{5}{6}$; $m_{\overline{TV}} = -\frac{4}{3}$; $m_{\overline{RV}} = \frac{5}{6}$. Therefore $\overline{RS} \parallel \overline{TV}$ and $\overline{RV} \parallel \overline{ST}$ and $RSTV$ is a parallelogram.
21. $(3, 5)$
22. $(1, 4)$
23. $(3, 5)$
24. $(1, 4)$
25. $(16, 11)$, $(4, -9)$, $(-4, 5)$
26. (a) $\sqrt{53}$ (b) -4 (c) $\frac{1}{4}$
27. $A = (-a, 0)$; $B = (0, b)$; $C = (a, 0)$
28. $D = (0, 0)$; $E = (a, 0)$; $F = (a, 2a)$; $G = (0, 2a)$
29. $R = (0, 0)$; $U = (0, a)$; $T = (a, a+b)$
30. $M = (0, 0)$; $N = (a, 0)$; $Q = (a+b, c)$; $P = (b, c)$
31. (a) $\sqrt{(a+c)^2 + (b+d-2e)^2}$ (b) $-\frac{a}{b-e}$ or $\frac{a}{e-b}$ (c) $y - 2d = \frac{a}{e-b}(x - 2c)$

Summary of Changes in Content and Emphasis in 5–8 Mathematics*

Increased Attention

Problem Solving

- Pursuing open-ended problems and extended problem-solving projects
- Investigating and formulating questions from problem situations
- Representing situations verbally, numerically, graphically, geometrically, or symbolically

Communication

- Discussing, writing, reading, and listening to mathematical ideas

Reasoning

- Reasoning in spatial contexts
- Reasoning with proportions
- Reasoning from graphs
- Reasoning inductively and deductively

Connections

- Connecting mathematics to other subjects and to the world outside the classroom
- Connecting topics within mathematics
- Applying mathematics

Number/Operations/Computation

- Developing number sense
- Developing operation sense
- Creating algorithms and procedures
- Using estimation both in solving problems and in checking the reasonableness of results
- Exploring relationships among representations of, and operations on, whole numbers, fractions, decimals, integers, and rational numbers
- Developing an understanding of ratio, proportion, and percent

Patterns and Functions

- Identifying and using functional relationships
- Developing and using tables, graphs, and rules to describe situations
- Interpreting among different mathematical representations

Algebra

- Developing an understanding of variables, expressions, and equations
- Using a variety of methods to solve linear equations and informally investigate inequalities and nonlinear equations

Statistics

- Using statistical methods to describe, analyze, evaluate, and make decisions

Probability

- Creating experimental and theoretical models of situations involving probabilities

Geometry

- Developing an understanding of geometric objects and relationships
- Using geometry in solving problems

Measurement

- Estimating and using measurement to solve problems

Instructional Practices

- Actively involving students individually and in groups in exploring, conjecturing, analyzing, and applying mathematics in both a mathematical and a real-world context
- Using appropriate technology for computation and exploration
- Using concrete materials
- Being a facilitator of learning
- Assessing learning as an integral part of instruction

Decreased Attention

Problem Solving

- Practicing routine, one-step problems
- Practicing problems categorized by types (e.g., coin problems, age problems)

Communication

- Doing fill-in-the-blank worksheets
- Answering questions that require only yes, no, or a number as responses

Reasoning

- Relying on outside authority (teacher or an answer key)

Connections

- Learning isolated topics
- Developing skills out of context

Number/Operations/Computation

- Memorizing rules and algorithms
- Practicing tedious paper-and-pencil computations
- Finding exact forms of answers
- Memorizing procedures, such as cross-multiplication, without understanding
- Practicing rounding numbers out of context

Patterns and Functions

- Topics seldom in the current curriculum

Algebra

- Manipulating symbols
- Memorizing procedures and drilling on equation solving

Statistics

- Memorizing formulas

Probability

- Memorizing formulas

Geometry

- Memorizing geometric vocabulary
- Memorizing facts and relationships

Measurement

- Memorizing and manipulating formulas
- Converting within and between measurement systems

Instructional Practices

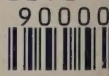
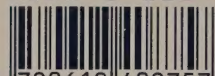
- Teaching computations out of context
- Drilling on paper-and-pencil algorithms
- Teaching topics in isolation
- Stressing memorization
- Being the dispenser of knowledge
- Testing for the sole purpose of assigning grades



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